Lower bounds for stock price probability distributions in stochastic volatility models

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Outline

- Introduction : financial modeling, option pricing under stochastic volatility.
  - Underlying processes are solutions of SDEs.
- Asymptotic (in space) estimates of probability laws useful for option modeling and pricing problems.
- Our main tool: Itô processes around deterministic curves.

The Option Pricing problem

- Risky asset modeled by a continuous-time martingale $(S_t, t \geq 0)$ (the "underlying").
- Specifications of S through SDEs, e.g.

$$dS_t = \sigma(t, S_t)dW_t.$$

The option (at some $T > 0$):

$$\phi(S_T), \quad \text{ex.} \quad \phi(s) = (s - K)^+ \text{ (option "Call")}$$

The price of the option:

$$\mathbb{E}[\phi(S_T)]$$

- Historical exemple: Black-Scholes model

$$S_t(\sigma) = \sigma S_t dW_t, \quad \sigma > 0$$

$$\mathbb{E}[(S_T(\sigma) - K)^+] = C_{BS}(K, T, \sigma).$$

Explicit formulae for C_{BS}.

Stochastic volatility models

The model:

$$dS_t = f(V_t)S_t dW_t^1$$
$$dV_t = \beta(t, V_t)dt + \gamma(t, V_t)dW_t^2$$
$$d < W^1, W^2 >_t = \rho$$

ex. $f(v) = \sqrt{v}$. $(S_t, t \geq 0)$ satisfies

$$S_t = \int_0^t f(V_s)S_s dW_s^1$$

$(S_t; t \geq 0)$ a positive local martingale $\longrightarrow$ always an integrable supermartingale (apply Fatou's lemma to $S_{t\wedge\tau_n}$).

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$(S_t; t \geq 0)$ a positive local martingale $\longrightarrow$ always an integrable supermartingale (apply Fatou's lemma to $S_{t \wedge \tau_n}$).

But, complications may arise:

- Moments of order $p > 1$ may become infinite (Andersen & Piterbarg (07), Keller-Ressel (09)) $\longrightarrow$ Manipulate variances with care.
- S not a *true* martingale (Lions & Musiela (05), Jourdain (05), ...)

Local-stochastic volatility (LSV) models

The model:

$$dS_t = f(V_t)S_t\eta(t, S_t)dW_t^1$$
$$dV_t = \beta(t, V_t)dt + \gamma(t, V_t)dW_t^2$$
$$d < W^1, W^2 >_t = \rho$$

$(S_t; t \geq 0)$ satisfies

$$S_t = \int_0^t f(V_s)S_s\eta(s, X_s)dW_s^1$$

$(S_t; t \geq 0)$ a positive local martingale $\longrightarrow$ always an integrable supermartingale.

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Explosion of $\mathbb{E}[S_t^p]$

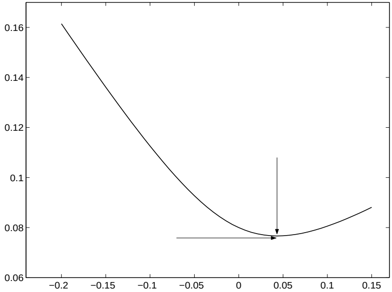
Related to :

- Well-posedness of the model (manipulate variances and do Monte Carlo simulation)
- Shape of the Implied Volatility curve

The implied volatility: the unique $\sigma(T,k)$ such that

$$\mathbb{E}[(S_T - S_0 e^k)^+] = C_{BS}(S_0 e^k, T, \sigma(T,k)).$$

Exemple of $\sigma(T,\cdot)^2$ for fixed T:



Explosion of $\mathbb{E}[S_t^p]$ and shape of implied volatility

The critical exponents:

$$p_T^*(S) = \sup\{p : \mathbb{E}[S_T^p] < \infty\}, \qquad q_T^*(S) = \sup\{q : \mathbb{E}[S_T^{-q}] < \infty\}.$$

In some particular models, can be computed explicitly as functions of model parameters.

The Moment Formula (Lee (04)):

$$\limsup_{k\to\infty} \frac{T\sigma(T,k)^2}{k} = g(p_T^*(S)-1), \quad \limsup_{k\to-\infty} \frac{T\sigma(T,k)^2}{k} = g(q_T^*(S)), \tag{1}$$

for a known function g: $g(x) = 2 - 4\left(\sqrt{x^2+x} - x\right), g(\infty) = 0$.

Explosion of $\mathbb{E}[S_t^p]$ and shape of implied volatility

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Useful for:

- Model selection: if market implied volatility has "wings", moments in the model *must* explode.
- Model calibration : in (1), compare the left hand side (market data) to the right one (model) $\longrightarrow$ gives a *guess* of model parameters.

Approach to moment explosion: lower bounds for the law of S_T

Consider $X_t = \log(S_t/S_0)$, so that $\mathbb{E}[S_T^p] = S_0^p \mathbb{E}[e^{pX_T}]$.

We look for lower bounds in the range:

$$\mathbb{P}(X_T > y) \geq C_* e^{-c^* y}$$

because this implies $\mathbb{E}[e^{pX_T}] = \infty$ for all $p > c^*$ (apply Markov's inequality).

Our main tool : an estimation along the whole trajectory

$$\mathbb{P}(X_T > y) \geq \mathbb{P}(|X_T - 2y| < y) \geq \mathbb{P}(|X_t - x_t| < R_t, \forall t \leq T)$$

with the proper curves x and R ($x_0 = X_0 = 0$; $x_T = 2y$; $R_t > 0$; $R_T = y$).

Estimates for Itô processes around deterministic curves

Consider for a moment:

- a stochastic process $Y_t \in \mathbb{R}^n$;
- a deterministic differentiable curve $y_t \in \mathbb{R}^n$;
- a time depending radius $R_t > 0$

Assume Y satisfies

$$Y_t = x_0 + \int_0^t b(s, Y_s)ds + \sum_{j=1}^d \int_0^t a_j(s, Y_s)dW_s^j, \quad t \leq \tau_R,$$

with $\tau_R = \inf\{t : |Y_t - y_t| \geq R_t\}$ (temps de sortie du “tube”).

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with $\tau_R = \inf\{t : |Y_t - y_t| \geq R_t\}$ (temps de sortie du "tube").
The coefficients a and b are:

- Locally bounded: $|b(t, Y_{t\wedge\tau_R})| + \sum_j |a_j(t, Y_{t\wedge\tau_R})| \leq c_t$
- Locally Lipschitz: $\mathbb{E}\Big[|\sigma_j(s, Y_s) - \sigma_j(t, Y_t)|^2 1_{\{\tau_R \geq s\}}\Big] \leq L_t(s-t)$;
- Locally elliptic: $\lambda_t I_n \leq \sigma\sigma^*(t, Y_{t\wedge\tau_R}) \leq \gamma_t I_n$,

The lower bound

Consider the rate function:

$$F(t) = \frac{|y_t'|^2}{\lambda_t} + (c_t^2 + L_t^2)\Big(\frac{1}{\lambda_t} + \frac{1}{R_t^2}\Big)$$

The lower bound: (Bally, Fernandez &Meda (09), under technical conditions on $|y_t'|$, R_t, c_t, L_t, λ_t, γ_t)

$$\mathbb{P}(|Y_t - y_t| \leq R_t, \forall t \leq T) \geq \exp\Big(-Q\Big(1 + \int_0^T F(t)dt\Big)\Big) \qquad (2)$$

with Q a universal constant.

What we do: take (2) and optimize over the curves y_t, R_t.

In our model, the process Y is the couple (X, V) (we need to estimate V to estimate X).

Application to LSV models

A class of models widely employed ($f(v) = \sqrt{v}$):

$$\begin{aligned} dX_t &= -\frac{1}{2}\eta^2(t, X_t)V_t dt + \eta(t, X_t)\sqrt{V_t}dW_t^1; \quad X_0 = 0 \\ dV_t &= \beta(t, V_t)dt + \gamma(t, V_t)\sqrt{V_t}dW_t^2; \quad V_0 > 0. \end{aligned} \tag{3}$$

- Hypotheses : η, γ Lipschitz, bounded and bounded away from zero; β measurable with sub-linear growth.

Consider any couple $(X_t, V_t; t \leq T)$ that satisfies (3) (no discussion about existence and uniqueness is needed).

MAIN RESULT: there exist c_T, y_T such that

$$\mathbb{P}\Big(|(X_t, V_t) - (\tilde{x}_t, \tilde{v}_t)| \leq \tilde{R}_t, \forall t \leq T\Big) \geq \exp\big(-c_T|y|\big), \quad \forall\, |y| > y_T$$

$$\tilde{v}_t = V_0\left(\sqrt{\frac{|y| + V_0}{V_0}}\phi(t)\right)^2; \;\; \tilde{x}_t = sign(y)(\tilde{v}_t - V_0); \;\; \tilde{R}_t = \frac{1}{2}\sqrt{\tilde{v}_t};$$

$$\phi(t) = \sinh(t/2)/\sinh(T/2).$$

Corollaries

The bound for the "tube":

$$\mathbb{P}\Big(|(X_t, V_t) - (\tilde{x}_t, \tilde{v}_t)| \le \tilde{R}_t, \forall t \le T\Big) \ge \exp\big(-c_T |y|\big).$$

The constant c_T explicitly depends on model parameters.

The same estimate holds for the terminal cdf:

$$\mathbb{P}(|X_T| > y) \ge \exp\big(-c_T y\big).$$

Consequences are:

- The critical exponents are finite:

$$p^*_T(S) \vee q^*_T(S) \le c_T.$$

- In the considered class of models, the implied volatility always has "wings":

$$\limsup_{k \to \pm\infty} \frac{T\sigma(T,k)^2}{k} \ge c_T, \quad \forall T > 0.$$

Remarks

- "Small balls" estimates : We also show that

$$\mathbb{P}\Big(|(X_T, V_T) - (y, |y| + V_0)| \leq R^{(j)}(y)\Big) \geq \exp\big(-(j+1)d_T|y|\big). \quad (4)$$

with $R^{(0)}(y) = \sqrt{|y|}$, $R^{(1)}(y) = 1$, $R^{(2)}(y) = \frac{1}{\sqrt{|y|}}$...

- Density estimates : Using (4) and the integration by parts formula of Malliavin calculus, we also prove that X_T admits a density p_{X_T} such that

$$p_{X_T}(y) \geq \frac{1}{M} \exp(-e_T|y|) \quad (5)$$

for all y with $|y| > M$.

Ideas of proof

Recall the model:

$$
\begin{aligned}
dX_t &= -\frac{1}{2}\eta^2(t, X_t)V_t dt + \eta(t, X_t)\sqrt{V_t} dW_t^1, \quad X_0 = 0 \\
dV_t &= \beta(t, V_t)dt + \gamma(t, V_t)\sqrt{V_t} dW_t^1 \quad V_0 > 0.
\end{aligned}
\tag{6}
$$

Step 1. Growth of coefficients in (6) mainly determined by V_t.
The local bounds, local Lipschitz constants, local ellipticity constants in the "tube" around any (x_t, v_t), provided that v_t stays away from zero and $R_t < v_t$, are

$$
c_t = c v_t; \qquad L_t = L v_t;
$$

$$
\gamma_t = \gamma v_t; \qquad \lambda_t = \lambda v_t.
$$

The bound $\mathbb{P}(\tau_R > T) \geq \exp\left(-Q\left(1 + \int_0^T F(t)dt\right)\right)$ holds with

$$
F(t) = \frac{(x_t')^2 + (v_t')^2}{v_t} + 2(v_t^2 + v_t)\left(\frac{1}{v_t} + \frac{1}{R_t^2}\right);
$$

Ideas of proof

Step 2. Choice of curves and radius. Starting from the expression of F:

$$F =: F_{x,v,R}(t) = +\frac{(x_t')^2 + (v_t')^2}{v_t} + 2(v_t^2 + Lv_t)\Big(\frac{1}{v_t} + \frac{1}{R_t^2}\Big),$$

under the constraints $(x_0, v_0) = (0, V_0)$, $x_T = y$, we choose:

$$R_t = \frac{1}{2}\sqrt{v_t}; \qquad (x_t')^2 = (v_t')^2, \quad t \in [0, T]$$

so that $v_t = |x_t| + V_0$.

Step 3. Lagrangian minimization The rate function reduces to

$$F_v = \mathcal{L}(v_t, v_t'), \qquad \mathcal{L}(v_t, v_t') = \frac{(v_t')^2}{v_t} + v_t$$

Ideas of proof

Step 4. Explicit computations Euler-Lagrange equations for $\mathcal{L}$ are explicitly solved under the constraints $(x_0, v_0) = (0, V_0)$, $(x_T, v_T) = (y, |y| + V_0)$, giving

$$\tilde{v}_t = V_0\left(\sqrt{\frac{|y| + V_0}{V_0}}\phi(t)\right)^2; \;\; \tilde{x}_t = sign(y)(\tilde{v}_t - V_0); \;\; \tilde{R}_t = \frac{1}{2}\sqrt{\tilde{v}_t};$$

$$\phi(t) = \sinh(t/2)/\sinh(T/2).$$

- Explicit computation of $\int_0^T F_v(t)dt$ gives the lower bound.

Selected References

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