An Exact Connection between two Solvable SDEs and a Non Linear Utility Stochastic PDEs

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- The security market consists of one riskless asset S^0, $dS^0_t = S^0_t r_t dt$, and d continuous risky assets $(S^i)_{1 \leq i \leq d}$ defined on a filtred Brownian space $(\Omega, \mathcal{F}_{t\geq 0}, \mathbb{P})$:

$$dS^i_t / S^i_t = b^i_t dt + \sigma^i_t dW_t, \qquad 1 \leq i \leq d$$

where W is a N-dimentional brownien motion with $N > d$.

- Risk premium vector, η_t with $\quad b(t) - r(t)\mathbf{1} = \sigma_t \eta_t$

Def A positive wealth process is defined as a pair (x, π) : $x > 0$ is the initial value of the portfolio. $\pi = (\pi^i)_{1 \leq i \leq d}$ is the (predictable) proportion of each asset held in the portfolio, assumed to be S-integrable process

- Thanks to AOA in the market, denoting $\mathcal{R}^\sigma$ the range of σ and $\kappa = \sigma\pi$ wealth process X^κ is driven by

$$\frac{dX^\kappa_t}{X^\kappa_t} = r_t dt + \kappa_t.(dW_t + \eta_t dt), \kappa_t \in \mathcal{R}^\sigma_t$$

Consistent Dynamic Utility

Let $\mathscr{X}$ be the family of positive portfolios.

Definition : An $\mathscr{X}$-Consistent progressive utility $U(t,x)$ process is a positive adapted random field s.t.

- **Concavity assumption :** for $t \geq 0$, : $x \mapsto U(t,x)$ is an increasing concave function, (in short utility function) .
- **Consistency with the class of test portfolios :** For any admissible wealth process $X \in \mathscr{X}$, $\mathbb{E}(U(t,X_t)) < +\infty$ and

$$\mathbb{E}(U(t,X_t)/\mathcal{F}_s) \leq U(s,X_s), \ \forall s \leq t.$$

- **Existence of optimal** For any initial wealth $x > 0$, there exists an optimal wealth process (benchmark) $X^* \in \mathscr{X}$ $(X^*_0 = x)$,

$$U(s,X^*_s) = \mathbb{E}(U(t,X^*_t)/\mathcal{F}_s) \ \forall s \leq t.$$

- **In short** for any admissible wealth $X \in \mathscr{X}$, $U(t,X_t)$ is a supermartingale, and a martingale for the optimal-benchmark wealth X^*.

Progressive Utility of Itô's Type

Let U be a dynamic utility (concave, increasing) ,

$$dU(t,x) = \beta(t,x)dt + \gamma(t,x)dW_t$$

such that $U(t, X_t^{\kappa})$ is a supermartingale for $\kappa \in \mathcal{R}^{\sigma}$ and a martingale for the optimal one

Open questions

- What about the drift β and the volatility γ of the utility ?
- Under which assumptions on (β, γ) can one be sure that solutions are concave and increasing ?

Main difficulties come from the forward definition

Itô-Ventzel's Formula (Kunita)

- Let ϕ and ψ be Itô-Ventzel's regular one-dimensional stochastic flows

$$d\phi(t,x) = \mu(t,x)dt + \gamma(t,x)dW_t, \quad d\psi(t,x) = \alpha(t,x)dt + \nu(t,x)dW_t.$$

- The compound random field $\phi o \psi(t,x) = \phi(t,\psi(t,x))$ is a regular semimartingale

Itô-Ventzel's Formula

$$\begin{aligned} d(\phi o \psi)(t,x) &= \mu(t,\psi(t,x))dt + \gamma(t,\psi(t,x))dW_t \\ &+ \phi_x(t,\psi(t,x))d\psi(t,x) + \frac{1}{2}\phi_{xx}(t,x)(t,\psi(t,x))||\nu(t,x)||^2 dt \\ &+ \langle \gamma_x(t,\psi(t,x)), \nu(t,x)\rangle dt. \end{aligned}$$

The volatility of $\phi o \psi$ is given by $\nu^{\phi o \psi}(t,x) = \gamma(t,\psi(t,x)) + \phi_x(t,\psi(t,x))\nu(t,x)$.

Lemma (Drift Constraint)

Let U be a progressive utility of class $\mathcal{C}^{(2)}$ in the sense of Kunita with local characteristics (β,γ). Then, for any admissible portfolio X^κ,

$$\begin{aligned} dU(t,X^\kappa_t) &= \Big(U_x(t,X^\kappa_t)X^\kappa_t\,\kappa_t+\gamma(t,X^\kappa_t)\Big).dW_t \\ &+ \Big(\beta(t,X^\kappa_t)+U_x(t,X^\kappa_t)r_tX^\kappa_t+\frac{1}{2}U_{xx}(t,X^\kappa_t)\mathcal{Q}(t,X^\kappa_t,X^\kappa_t\kappa_t)\Big)dt, \\ \text{where } \mathcal{Q}(t,x,\kappa) &:= \|x\kappa\|^2+2x\kappa.\Big(\frac{U_x(t,x)\eta^\sigma_t+\gamma_x(t,x)}{U_{xx}(t,x)}\Big). \end{aligned}$$

Let γ^σ_x be the orthogonal projection of γ_x on $\mathcal{R}^\sigma$. Let $\mathcal{Q}^*(t,x)=\inf_{\kappa\in\mathcal{R}^\sigma}\mathcal{Q}(t,x,\kappa)$; the minimum of this quadratic form is achieved at the optimal policy κ^* given by

$$x\kappa^*_t(x)=-\tfrac{1}{U_{xx}(t,x)}\big(U_x(t,x)\eta^\sigma_t+\gamma^\sigma_x(t,x)\big)$$

$$x^2\mathcal{Q}^*(t,x)=-\tfrac{1}{U_{xx}(t,x)^2}||U_x(t,x)\eta^\sigma_t+\gamma^\sigma_x(t,x))||^2=-||x\kappa^*_t(x)||^2.$$

Theorem (Verification Theorem :)

Let U be a Itô-Ventzel regular progressive utility (concave, increasing) with decomposition $dU(t,x) = \beta(t,x)dt + \gamma(t,x)dW_t$. We assume that :

Hyp *The drift β is related to the volatility :*

$$\beta(t,x) = -U_x(t,x)r_t x + \frac{1}{2U_{xx}(t,x)}||U_x(t,x)\eta_t^\sigma + \gamma_x^\sigma(t,x)||^2$$

$\Rightarrow$ *Then U satisfies*

$$dU(t,x) = \big(- U_x(t,x)r_t x + \frac{1}{2U_{xx}(t,x)}||(U_x(t,x)\eta_t^\sigma + \gamma_x^\sigma(t,x))||^2\big)dt + \gamma(t,x)dW_t$$

$\Rightarrow$ *The optimal policy $\kappa^*(t,x)$ is $\kappa^*(t,x) = -\frac{1}{xU_{xx}(t,x)}\big(U_x(t,x)\eta_t^\sigma + \gamma_x^\sigma(t,x)\big)$*

$\Rightarrow$ *The volatility $\gamma(t,x)$ verifies*

$$U_x(t,x)\eta_t + \gamma_x(t,x) = -xU_{xx}(t,x)\kappa^*(t,x) + \gamma_x^\perp(t,x) : \gamma_x^\perp(t,x) \in \mathcal{R}_t^{\sigma,\perp}$$

Theorem

Under previous hypothesis,

- Assume that $\kappa^*(t,x)$ is sufficiently smooth so that the equation

$$\frac{dX_t^*}{X_t^*} = (r_t dt + \kappa^*(t, X_t^*).(dW_t + \eta_t dt)$$

has a (unique ? strong ?) positive solution for any initial wealth x>0.

$\Rightarrow$ Then, if $U(t, X_t^*)$ is a martingale the progressive increasing utility U is an $\mathcal{X}$-consistent utility, with optimal wealth X_t^*.

Inverse flows

Proposition

Let ϕ be a strictly monotone Itô-Ventzel regular flow with inverse process $\xi(t,y)=\phi(t,.)^{-1}(y)$. Assume $d\phi(t,x)=\mu(t,x)dt+\gamma(t,x)dW_t$,

i) *The inverse flow $\xi(t,y)$ has as dynamics, with $\nu^{\xi}(t,y)=-\xi_y\gamma(t,\xi)$*

$$d\xi(t,y)=\nu^{\xi}(t,y)dW_t+\Big(\frac{1}{2}\partial_y\big(\frac{\|\nu^{\xi}(t,y)\|^2}{\xi_y}\big)-\mu(t,\xi)\xi_y(t,y)\Big)dt$$

ii) *If $\phi=\Phi_x(t,x)$ with $d\Phi(t,x)=M(t,x)dt+C(t,x)dW_t$, then $\xi=\Xi_y(t,y)$*

$$d\Xi(t,y)=-C(t,\xi)dW_t-M(t,\xi)dt+\frac{1}{2}\frac{\|C_x(t,\xi)\|^2}{\Phi_{xx}(t,\xi)}dt$$

Convex conjugate SPDE I

Dual SPDE

Let $\tilde{U}(t,y) \overset{def}{=} \inf_{x\in Q_+^*}\big(U(t,x) - x\,y\big)$ be the conjugate of U, with Itô-Ventzel regularity, denote $\tilde{\gamma}(t,y) = \gamma(t, -\tilde{U}_y(t,y))$ then

$$d\tilde{U}(t,y) = \Big[\frac{1}{2\tilde{U}_{yy}(t,y)}\big(\|\tilde{\gamma}_y(t,y)\|^2 - \|\tilde{\gamma}_y^\sigma(t,y) + y\tilde{U}_{yy}(t,y)\eta_t^\sigma\|^2\big) + y\tilde{U}_y(t,y)r_t\Big]dt$$
$$+\tilde{\gamma}(t,y).dW_t$$

In particular the drift $\tilde{\beta}$ can be written us the solution of the following optimization program, achieved at $\nu^*(t,y) = \frac{-\tilde{\gamma}_y^\perp(t,y)}{y\tilde{U}_{yy}(t,y)} = \frac{\gamma_x^\perp\big(t,-\tilde{U}_y(t,y)\big)}{y}$.

$$\tilde{\beta}(t,y) = y\tilde{U}_y(t,y)r_t - \frac{1}{2}y^2\tilde{U}_{yy}(t,y)\inf_{\nu_t\in\mathcal{R}^{\sigma,\perp}}\{\|\nu_t-\eta_t^\sigma\|^2 + 2(\nu_t-\eta_t^\sigma).(\frac{\tilde{\gamma}_y(t,y)}{y\tilde{U}_{yy}(t,y)})\}$$

Theorem

- *The convex conjugate $\tilde{U}(t,y)$ is consistent with the family $\mathscr{Y}$ of state density prices Y^ν, defined from*

$$\frac{dY_t}{Y_t} = -r_t dt + (\nu_t - \eta_t) dW_t, \quad \nu_t \in \mathcal{R}_t^{\sigma,\perp}$$

That is : $\tilde{U}(t, Y^\nu)$ is a submartingale for any $Y^\nu \in \mathscr{Y}$, and a martingale for some process $Y^{\nu^*} (:= Y^*)$.

Where the dual optimal parameter $\nu^*(t,y)$ is given by

$$\nu^*(t,y) = \frac{-\tilde{\gamma}_y^\perp(t,y)}{y\tilde{U}_{yy}(t,y)} = \frac{\gamma_x^\perp(t, -\tilde{U}_y(t,y))}{y}.$$

Moreover, $Y_t^*(y) = U_x(t, X_t^*(-\tilde{u}_y(y)))$.

Remark : If $X_t^*(x)$ is strictly monotone in x, by taking the inverse $\mathcal{X}(t,x)$, we obtain $U_x(t,x) = \mathcal{Y}(t, \mathcal{X}(t,x))$ with $\mathcal{Y}(t,x) := Y_t^*(u_x(x))$. Integrating, we get $U(t,x)$.

$\mathscr{X}$-Consistent Utilities with given optimal portfolio

Approach by Stochastic Flows

Monotony assumption

Let $X_t^*(x)$ be a wealth process assumed to be continuous and increasing in x from 0 to $+\infty$.

- true in a lot examples,
- may be a consequence of no arbitrage opportunity.
- from flows point of view, it is implied by coefficient regularity.
- implies the mononotony and continuity of $y \mapsto Y_t^*(y)$.

Hyp Moreover, $X_t^*(x)$ is assumed to be a Itô-Ventzel stochastic flow

$$\frac{dX_t^*(x)}{X_t^*(x)} = r_t dt + \kappa^*(t,x).(dW_t + \eta_t dt)$$

- Denote by $\mathcal{X}(t,z)$ the inverse flow of $X_t^*(z)$.

Proposition

Let $\mathcal{X}(t,z)$ be the inverse flow of $X^*(t,z)$, satisfying $X^* Y^0$ is a martingale. Let u be an utility function such that $x \mapsto u_{xx}(x) X^*_t(x)$ is integrable near to infinity. Define the processes U and $\tilde{U}$ by,

$$U(t,x) = Y^0_t \int_0^x u_x(\mathcal{X}(t,z))dz, \quad \tilde{U}(t,y) = \int_y^{+\infty} X^*_t(-\tilde{u}_y(\frac{z}{Y^0_t}))dz. \qquad (1)$$

U is a progressive utility, whose the convex conjugate is $\tilde{U}$, and satisfies the dynamics

$$\begin{aligned} dU(t,x) &= \Big(-U(t,x)r_t + \frac{1}{2U_{xx}(t,x)} ||\gamma^\sigma_x(t,x) + U_x(t,x)\eta^\sigma_t||^2 \Big) dt + \gamma(t,x).dW_t \\ \gamma_x(t,x) &= -U_{xx}(t,x)x\kappa^*(t,x) - U_x(t,x)\eta^\sigma_t \end{aligned}$$

$\tilde{U}(t, yY^0_t)$ and $U(t, X^*_t)$ are martingale processes and U is a $\mathcal{X}$-consistent stochastic utility, with optimal wealth X^*.

Theorem (General Characterization)

Let $(X^*_t(x) \in \mathscr{X})$ and $(Y^*_t(y)) \in \mathscr{Y}$ be two increasing flows, regular s.t. $(X^*_x(t,x)Y^*_t(y))$ is a martingale. Let u be an utility function, put $\mathcal{Y}(t,x) = Y^*_t(u_x(x))$, $\mathcal{X}(t,z) = (X^*_t(.))^{-1}$. and define the processes U and $\tilde{U}$ by

$$U(t,x) = \int_0^x \mathcal{Y}(t,\mathcal{X}(t,z))dz, \quad \tilde{U}(t,y) = \int_y^{+\infty} X^*_t((\mathcal{Y})^{-1}(t,z))dz.$$

Then, U is a progressive utility, whose the convex conjugate is $\tilde{U}$, and the dynamics

$$dU(t,x) = \Big(-xU_x(t,x)r_t + \frac{1}{2U_{xx}(t,x)}||\gamma^\sigma_x(t,x) + U_x(t,x)\eta^\sigma_t||^2\Big)dt + \gamma(t,x).dW_t,$$

*with $\gamma(t,x) = -U(t,x)\eta^\sigma_t - \int_0^x \Big(zU_{xx}(t,z)\kappa^*(t,z) - \nu^*_t(U_x(t,z))\Big)dz$.
The associated optimal portfolio and the optimal dual process are X^* and Y^*.
Moreover $U(t,X^*_t)$ is a martingale, so that U is an $\mathscr{X}$-consistent stochastic utility.*

Theorem (Converse point of view)

Consider a utility stochastic PDE with initial condition $u(.)$,

$$dU(t,x) = \Big(-xU_x(t,x)r_t + \frac{1}{2U_{xx}(t,x)}||\gamma_x^\sigma(t,x) + U_x(t,x)\eta_t^\sigma||^2\Big)dt + \gamma(t,x).dW_t.$$

Put $-xU_{xx}(t,x)\kappa^*(t,x) = \gamma_x^\sigma(t,x) + U_x(t,x)\eta_t^\sigma$ *and* $\nu_t^*(U_x(t,x)) = \gamma_x^\perp(t,x)$. *Assume that the SDE's*

$$\frac{dX_t^*(x)}{X_t^*(x)} = r_t dt + \kappa^*(t, X_t^*(x)).(dW_t + \eta_t^\sigma dt), \quad \frac{dY_t^*(y)}{Y_t^*(y)} = -r_t dt + \big(\nu_t^*(Y_t^*(y)) - \eta_t^\sigma\big).dW_t$$

admit solutions which are increasing and regular flows in the sense of Kunita. Let $\mathcal{Y}(t,x) = Y_t^*(u_x(x))$, $\mathcal{X}(t,z) = (X_t^*(.))^{-1}$ *and assume* $x \mapsto \mathcal{Y}(t,\mathcal{X}(t,z))$ *is integrable near to zero. Then there exists an increasing and concave solution U of the SPDE given by*

$$U(t,x) = \int_0^x \mathcal{Y}(t,\mathcal{X}(t,z))dz$$

Moreover, if X^* and Y^* are unique then U is the unique solution of the SPDE.

The main assumption is that the optimal portfolio is increasing in x, because we have the same characterization in more abstract form (minimal regularities assumption), based on the properties of the optimum.

- "An Exact Connection between two Solvable SDEs and a Non Linear Utility Stochastic PDEs" (2010) with Nicole El Karoui.
- "Stochastic Utilities With a Given Optimal Portfolio : Approach by Stochastic Flows" (2010) with Nicole El Karoui.
- "Random Risk Aversion and Explicit Consistent Utilities Construction" (2010) with Nicole El Karoui.

Thank you for your attention