Compter des chemins confinés dans un quadrant : une approche unifiée *via* les problèmes frontière

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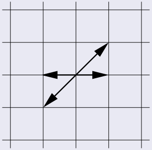
Université Pierre et Marie Curie

Colloque “Jeunes Probabilistes et Statisticiens”
Jeudi 6 mai 2010

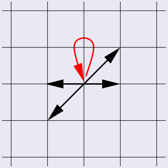
Counting the number of paths confined in some regions of the plane

Let $\mathcal{S}$ be a step set



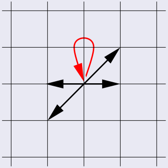
and let $q(i, j, k)$ be the number of paths confined in some regions of the plane with increments in $\mathcal{S}$, starting from (i_0, j_0) and ending in (i, j) at time k.

Example



$$q(0,0,0) = 1$$

Example



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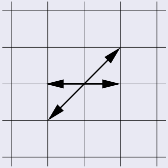
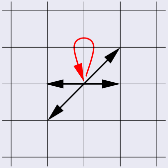


$q(0, 0, 1) = 0$

Example



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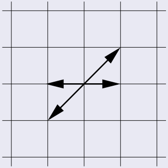

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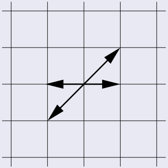
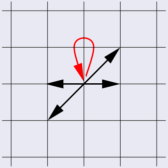


$q(0, 0, 2) = ?$

Example



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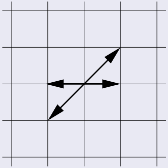

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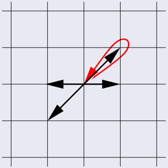

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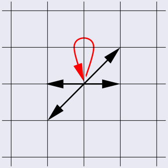

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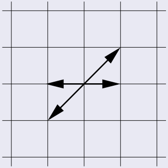

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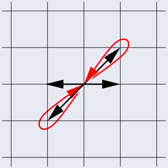

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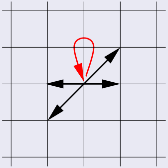

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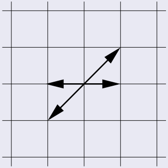

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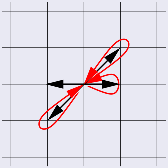

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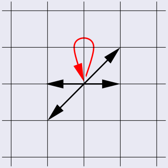

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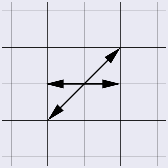
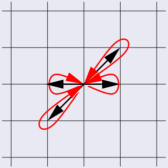
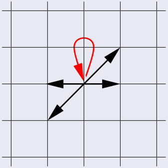


$q(0,0,1)=0$

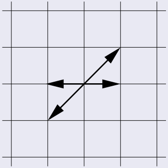


$q(0,0,2)=4$

Example



$q(0, 0, 0) = 1$



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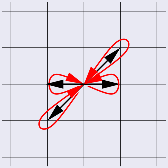
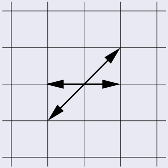


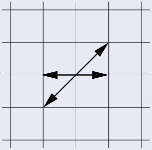
$q(0, 0, 2) = 4$



$q(0, 0, 3) = 0$

Counting the number of paths confined in some regions of the plane

Let $\mathcal{S}$ be a step set



and let $q(i,j,k)$ be the number of paths confined in some regions of the plane with increments in $\mathcal{S}$, starting from (i_0, j_0) and ending in (i,j) at time k. Let

$$Q(x,y,z) = \sum_{i,j,k} q(i,j,k) x^i y^j z^k.$$

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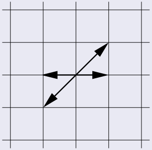

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- Finding explicitly $Q(x,y,z)$.

Counting the number of paths confined in some regions of the plane

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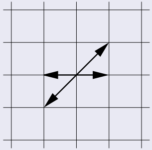

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- Finding explicitly $Q(x,y,z)$.
- Dependence of $Q(x,y,z)$ w.r.t. $\mathcal{S}$, *e.g.* its nature.

Rational functions $Q(x, y, z)$

$$Q(x, y, z) = \frac{P(x, y, z)}{R(x, y, z)}.$$

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Holonomic functions $Q(x,y,z)$

$$\sum_{i,j,k} P_{i,j,k}(x,y,z) \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial y^j} \frac{\partial^k}{\partial z^k} Q(x,y,z) = 0.$$

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Inclusions

{Rational functions} ⊂ {Algebraic functions} ⊂ {Holonomic functions}.

Number of paths in the *plane*

$Q(x, y, z)$ is rational.

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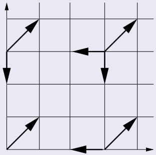

Kreweras walk
algebraic

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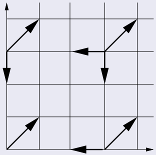

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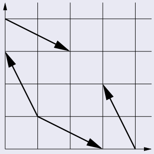
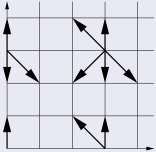


Knight walk
non-holonomic

Walks with small steps in the quarter plane starting at $(0,0)$

$\mathcal{S} \subset \{-1,0,1\}^2 \setminus \{(0,0)\}$



There are 2^8 such problems.

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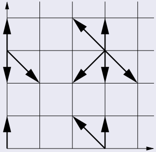

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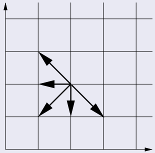
Some of these 2^8 models are :



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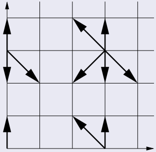

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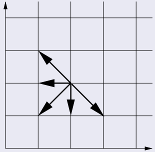

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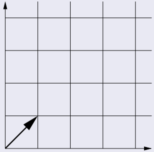

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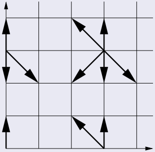

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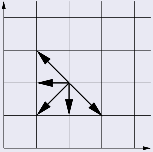

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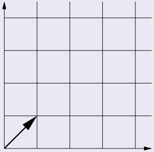

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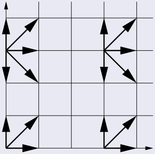

intrinsic to the half plane,

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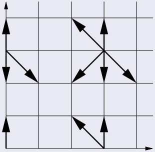

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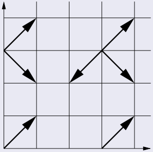

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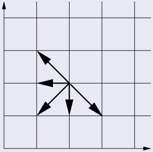

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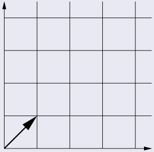

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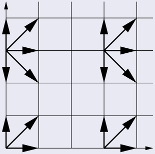

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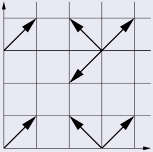

symmetrical.

Walks with small steps in the quarter plane starting at $(0,0)$

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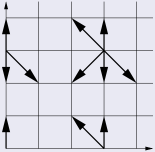

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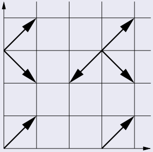

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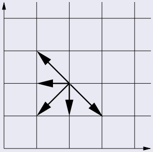

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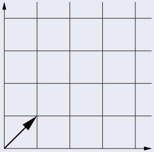

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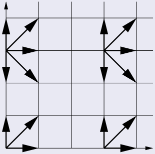

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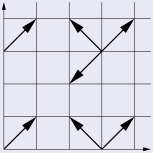

symmetrical.

Finally, it remains 79 problems !

The functional equation

The kernel :

$$K(x,y,z) = xyz\left[\sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z\right].$$

The functional equation :

$$K(x,y,z)Q(x,y,z) =$$
$$K(x,0,z)Q(x,0,z) + K(0,y,z)Q(0,y,z) - K(0,0,z)Q(0,0,z) - xy.$$

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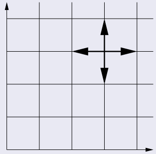
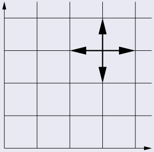
The group of the walk : an example



The jump generating function $x + \frac{1}{x} + y + \frac{1}{y}$

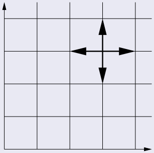
The group of the walk : an example



The jump generating function $x + \frac{1}{x} + y + \frac{1}{y}$ is left unchanged by

$$\psi(x,y) = \left(x, \frac{1}{y}\right), \qquad \phi(x,y) = \left(\frac{1}{x}, y\right),$$

The group of the walk : an example



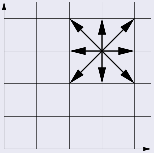
The jump generating function $x + \frac{1}{x} + y + \frac{1}{y}$ is left unchanged by

$$\psi(x,y) = \left(x, \frac{1}{y}\right), \qquad \phi(x,y) = \left(\frac{1}{x}, y\right),$$

and thus by any element of the group

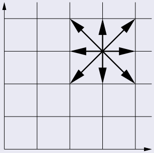
$$\langle \psi, \phi \rangle = \left\{ (x,y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}.$$

The group of the walk : the general case



The function $\displaystyle\sum_{(i,j)\in\mathcal{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y)x^i = \sum_{j=-1}^{1} A_j(x)y^j$

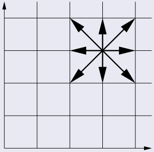
The group of the walk : the general case



The function $\sum_{(i,j)\in\mathcal{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y)x^i = \sum_{j=-1}^{1} A_j(x)y^j$ is left unchanged by

$$\psi(x,y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)}\frac{1}{y}\right), \qquad \phi(x,y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)}\frac{1}{x}, y\right),$$

The group of the walk : the general case



The function $\sum_{(i,j)\in\mathcal{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y)x^i = \sum_{j=-1}^{1} A_j(x)y^j$ is left unchanged by

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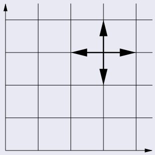
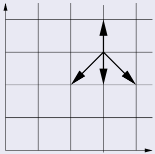
and thus by any element of the group

$$\langle \psi, \phi \rangle.$$

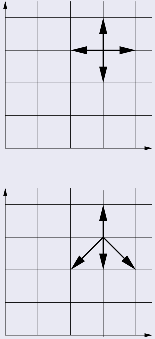
Some examples



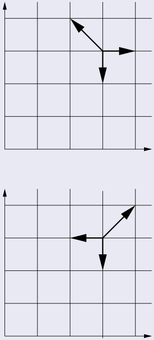


Order 4,

Some examples

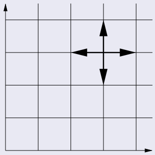


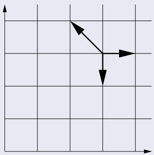
Order 4,

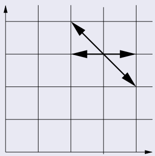


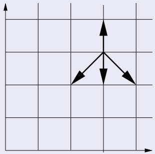
order 6,

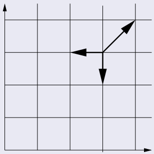
Some examples

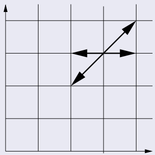






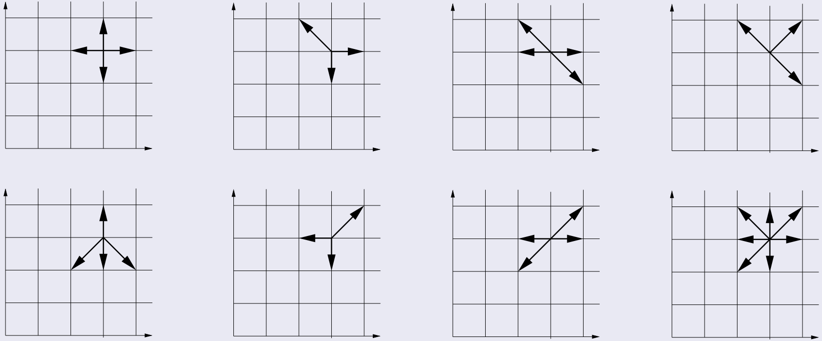






Order 4, order 6, order 8,

Some examples



Order 4, order 6, order 8, order ∞.

79 problems : finite and infinite groups

79 problems

79 problems : finite and infinite groups

79 problems → 23 admit a finite group

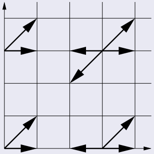
79 problems → 56 have an infinite group

79 problems : finite and infinite groups

79 problems

23 admit a finite group

22 are solved



1 is not solved

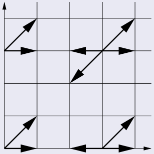
56 have an infinite group

79 problems : finite and infinite groups

79 problems

23 admit a finite group

- 22 are solved
- 1 is not solved



56 have an infinite group

- 2 are solved
- 54 are not solved

The functional equation

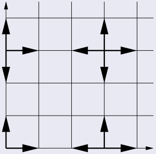
The kernel :

$$K(x,y,z) = xyz \left[\sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z \right].$$

The functional equation :

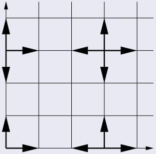
$$K(x,y,z)Q(x,y,z) = $$
$$K(x,0,z)Q(x,0,z) + K(0,y,z)Q(0,y,z) - K(0,0,z)Q(0,0,z) - xy.$$

An example of resolution in the case of a finite group



The jump function $\sum_{(i,j)\in\mathcal{S}} x^i y^j = x + \frac{1}{x} + y + \frac{1}{y}$ is left unchanged by the elements (x, y), $(\frac{1}{x}, y)$, $(\frac{1}{x}, \frac{1}{y})$, $(x, \frac{1}{y})$. This is also the case of $J(x, y, z) = \sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z$.

An example of resolution in the case of a finite group



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$$J(x,y,z)xyzQ(x,y,z) = zxQ(x,0,z) + zyQ(0,y,z) - xy$$

$$-J(x,y,z)\frac{1}{x}yzQ(\tfrac{1}{x},y,z) = -z\frac{1}{x}Q(\tfrac{1}{x},0,z) - zyQ(0,y,z) + \frac{1}{x}y$$

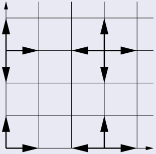
$$J(x,y,z)\frac{1}{x}\frac{1}{y}zQ(\tfrac{1}{x},\tfrac{1}{y},z) = z\frac{1}{x}Q(\tfrac{1}{x},0,z) + z\frac{1}{y}Q(0,\tfrac{1}{y},z) - \frac{1}{x}\frac{1}{y}$$

$$-J(x,y,z)x\frac{1}{y}zQ(x,\tfrac{1}{y},z) = -zxQ(x,0,z) - z\frac{1}{y}Q(0,\tfrac{1}{y},z) + x\frac{1}{y}$$

We obtain :

$$\sum_{\theta\in\langle\psi,\phi\rangle} (-1)^\theta \theta\left[xyzQ(x,y,z)\right] = \frac{-xy + \frac{1}{x}y - \frac{1}{x}\frac{1}{y} + x\frac{1}{y}}{J(x,y,z)}$$

An example of resolution in the case of a finite group



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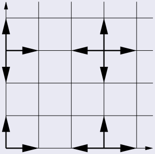
An example of resolution in the case of a finite group



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\begin{aligned}
J(x,y,z)xyzQ(x,y,z) &= \quad zxQ(x,0,z) + zyQ(0,y,z) - xy \\
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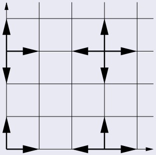
J(x,y,z)\tfrac{1}{x}\tfrac{1}{y}zQ(\tfrac{1}{x},\tfrac{1}{y},z) &= \quad z\tfrac{1}{x}Q(\tfrac{1}{x},0,z) + z\tfrac{1}{y}Q(0,\tfrac{1}{y},z) - \tfrac{1}{x}\tfrac{1}{y} \\
-J(x,y,z)x\tfrac{1}{y}zQ(x,\tfrac{1}{y},z) &= -zxQ(x,0,z) - z\tfrac{1}{y}Q(0,\tfrac{1}{y},z) + x\tfrac{1}{y}
\end{aligned}
$$

We obtain :

$$
\sum_{\theta\in\langle\psi,\phi\rangle} (-1)^\theta \theta[xyzQ(x,y,z)] = \frac{-xy + \frac{1}{x}y - \frac{1}{x}\frac{1}{y} + x\frac{1}{y}}{J(x,y,z)}
$$

An example of resolution in the case of a finite group



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-J(x,y,z)x\tfrac{1}{y}zQ(x,\tfrac{1}{y},z) &= -zxQ(x,0,z) - z\tfrac{1}{y}Q(0,\tfrac{1}{y},z) + x\tfrac{1}{y}
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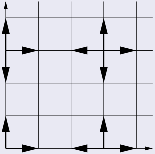
An example of resolution in the case of a finite group



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J(x,y,z)\tfrac{1}{x}\tfrac{1}{y}zQ(\tfrac{1}{x},\tfrac{1}{y},z) &= z\tfrac{1}{x}Q(\tfrac{1}{x},0,z) + z\tfrac{1}{y}Q(0,\tfrac{1}{y},z) - \tfrac{1}{x}\tfrac{1}{y} \\
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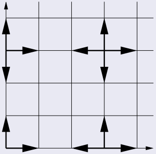
An example of resolution in the case of a finite group



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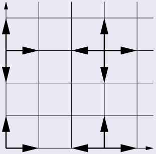
\end{aligned}$$

We obtain :

$$[x^>][y^>]\sum_{\theta\in\langle\psi,\phi\rangle}(-1)^\theta\theta\left[xyzQ(x,y,z)\right] = [x^>][y^>]\frac{-xy+\frac{1}{x}y-\frac{1}{x}\frac{1}{y}+x\frac{1}{y}}{J(x,y,z)}$$

An example of resolution in the case of a finite group



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We obtain :

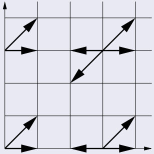
$$xyzQ(x,y,z) = [x^>][y^>]\frac{-xy + \frac{1}{x}y - \frac{1}{x}\frac{1}{y} + x\frac{1}{y}}{J(x,y,z)}$$

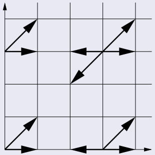
Sufficient condition for a function to be holonomic

If $F(x, y, z)$ as a formal series in z, has its coefficients in $\mathbb{C}(x)[y, \frac{1}{y}]$, then $[y^{>}]F(x, y, z)$ is algebraic on $\mathbb{C}(x, y, z)$.
If in addition $[y^{>}]F(x, y, z)$ as a formal series in z, has its coefficients in $\mathbb{C}[x, \frac{1}{x}, y]$, then $[x^{>}][y^{>}]F(x, y, z)$ is holonomic on $\mathbb{C}(x, y, z)$.

79 problems : finite and infinite groups

- 79 problems
  - 23 admit a finite group
    - 19 are solved *via* “orbit sums”
    - 3 are solved *via* “half-orbit sums”
    - 1 is not solved
  - 56 have an infinite group
    - 2 are solved
    - 54 are not solved

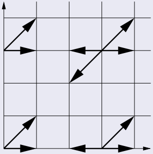
Gessel's walk 

Gessel's walk 

2001 : conjecture of I. Gessel

$$q(0,0,2k) = 16^k \frac{(5/6)_k (1/2)_k}{(2)_k (5/3)_k}, \qquad (a)_k = a(a+1)\cdots(a+k-1).$$

Gessel's walk



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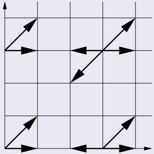
$$q(0,0,2k) = 16^k \frac{(5/6)_k(1/2)_k}{(2)_k(5/3)_k}, \qquad (a)_k = a(a+1)\cdots(a+k-1).$$

2009 : computer-aided proof of the conjecture

M. Kauers, C. Koutschan, D. Zeilberger.

Gessel's walk



2001 : conjecture of I. Gessel

$$q(0,0,2k) = 16^k \frac{(5/6)_k (1/2)_k}{(2)_k (5/3)_k}, \qquad (a)_k = a(a+1)\cdots(a+k-1).$$

2009 : computer-aided proof of the conjecture

M. Kauers, C. Koutschan, D. Zeilberger.

2009 : computer-aided proof of the algebraicity of $Q(x,y,z)$

A. Bostan, M. Kauers.

Results (1/2)

$$K(x,0,z)Q(x,0,z) - K(0,0,z)Q(0,0,z) =$$
$$\frac{1}{2\pi\imath}\int_{\mathcal{M}_z} \frac{tY_0(t,z)}{zA_{+1}(t)}\left[\frac{\partial_t w(t,z)}{w(t,z)-w(x,z)} - \frac{\partial_t w(t,z)}{w(t,z)-w(0,z)}\right]\mathrm{d}t.$$

$$K(0,y,z)Q(0,y,z) - K(0,0,z)Q(0,0,z) =$$
$$\frac{1}{2\pi\imath}\int_{\widetilde{\mathcal{M}}_z} \frac{X_0(t,z)t}{zB_{+1}(t)}\left[\frac{\partial_t \widetilde{w}(t,z)}{\widetilde{w}(t,z)-\widetilde{w}(y,z)} - \frac{\partial_t \widetilde{w}(t,z)}{\widetilde{w}(t,z)-\widetilde{w}(0,z)}\right]\mathrm{d}t.$$

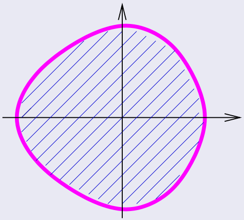
$Q(0,0,z)$.

X_0, Y_0 are the roots of the kernel $K(x,y,z) = xyz\left[\sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z\right]$ and

$$\sum_{(i,j)\in\mathcal{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y)x^i = \sum_{j=-1}^{1} A_j(x)y^j.$$

Results (2/2)

The curves $\mathcal{M}_z$ and $\widetilde{\mathcal{M}}_z$ are symmetrical w.r.t. the horizontal axis.



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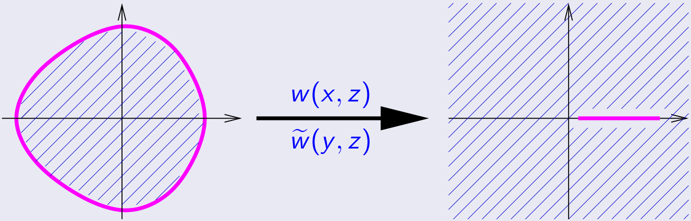

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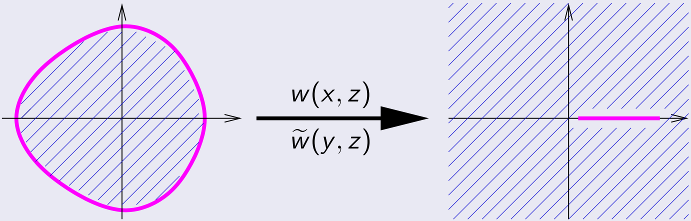

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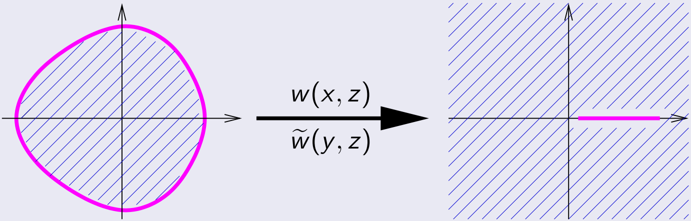

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If the group $\langle \psi, \phi \rangle$ is finite, then w and $\widetilde{w}$ are algebraic.

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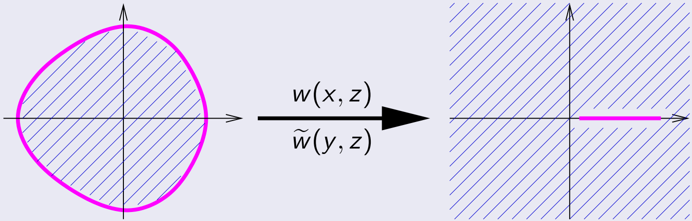

w and $\widetilde{w}$ are explicit.

If the group $\langle\psi,\phi\rangle$ is finite, then w and $\widetilde{w}$ are algebraic.

If the group $\langle\psi,\phi\rangle$ is infinite, then w and $\widetilde{w}$ are non-holonomic.