# Finite quantum groups and quantum permutation groups

Julien Bichon
Université Blaise Pascal, Clermont-Ferrand II

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# Quantum permutation algebras

We work over $k$, an algebraically closed field of characteristic zero.

## Definition

A quantum permutation algebra is a Hopf algebra generated (as an algebra) by the coefficients of a matrix $x = (x_{ij}) \in M_n(H)$ such that

1. $x$ is a permutation matrix : for all $i, j, k \in \{1, \dots, n\}$

$$\sum_{l=1}^{n} x_{li} = 1 = \sum_{l=1}^{n} x_{il}, \quad x_{ij}x_{ik} = \delta_{kj}x_{ij}, \ x_{ji}x_{ki} = \delta_{jk}x_{ji}$$

2. $x$ is a multiplicative matrix : for all $i, j \in \{1, \dots, n\}$

$$\Delta(x_{ij}) = \sum_{l=1}^{n} x_{il} \otimes x_{lj}, \ \varepsilon(x_{ij}) = \delta_{ij}, \ S(x_{ij}) = x_{ji}$$

### Definition

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### Example

$k^{\mathbb{S}_n}$ is a quantum permutation algebra with $x_{ij}(\sigma) = \delta_{i,\sigma(j)}$, for all $\sigma \in \mathbb{S}_n$.

### Definition

Let $A_s(n)$ be the universal algebra generated by the coefficients of a permutation matrix of size $n$. $A_s(n)$ is a quantum permutation algebra.

The Hopf algebra $A_s(n)$ arose first in Wang's work on compact quantum actions on finite (classical) spaces (1998).

A Hopf algebra $H$ is a quantum permutation algebra if and only if $A_s(n) \twoheadrightarrow H$ for some $n$.

### Theorem

*$A_s(n)$ is the universal cosemisimple Hopf algebra coacting on the algebra $k^n$. This means :*

1. *$A_s(n)$ is cosemisimple and $k^n$ is an $A_s(n)$-comodule algebra via*

$$k^n \longrightarrow k^n \otimes A_s(n)$$
$$e_i \longmapsto \sum_{k=1}^{n} e_k \otimes x_{ki}$$

2. *If $k^n$ is a comodule algebra over a cosemisimple Hopf algebra $H$ with coaction $\beta : k^n \longrightarrow k^n \otimes H$, then there is a unique Hopf algebra map $f : A_s(n) \longrightarrow H$ with $(1 \otimes f) \circ \alpha = \beta$*

Thus we write $A_s(n) = \mathcal{O}(S_n^+)$, where $S_n^+$ is the quantum permutation group on $n$ points, and quantum permutation algebras correspond to quantum permutation groups.

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We observe that

1. $A_s(n) \cong k^{\mathbb{S}_n}$ if $n \leq 3$,
2. $A_s(n+m) \twoheadrightarrow A_s(n) * A_s(m)$, so $\dim A_s(n) = \infty$ if $n \geq 4$.

Hence the symmetric group $\mathbb{S}_n$ has an infinite quantum analogue if $n \geq 4$ !

Banica has shown that the fusion rules of $A_s(n)$ are the same as those of $PGL_2$ (1999, when $k = \mathbb{C}$).

Early examples of quantum permutation algebras

1. $\mathcal{O}(O_{-1}(n))$ (corresponding to the quantum automorphism group of the hypercube in $\mathbb{R}^n$).
2. $(k^{\mathbb{A}_5})^\sigma$ (so that $\mathbb{A}_5$ has a quantum analogue acting faithfully on 4 points).
3. The Kac-Paljutkin algebra of dimension 8 (as well as other series of Hopf algebras studied by Masuoka).
4. Some 2-cocycle deformations of $k^{\mathbb{S}_n}$.

Several of these examples were unexpected at first sight.

So it becomes natural to wonder if there are lots of quantum permutation algebras. A basic obstruction to being a quantum permutation algebra is the following one :

If $H$ is a quantum permutation algebra, then $\mathrm{Hom}_{k-alg}(H,k)$ is finite and $S^2 = \mathrm{id}_H$. So if $H$ is a finite-dimensional quantum permutation algebra, then $H$ is semisimple.

So a reasonable question is :

Is any (finite dimensional) semisimple Hopf algebra a quantum permutation algebra ?
In other words, in view of the universal property of $A_s(n) = \mathcal{O}(S_n^+)$, is there a Cayley theorem for finite quantum groups ?

Naturally this leads to other more specific questions.
Is the class of finite quantum permutation algebras stable under

1. duality ?
2. extensions ?
3. 2-cocycle deformations ?

# Extensions and quantum permutation algebras

We now wish to study the stability of the class of quantum permutation algebras under extensions.

If $\Gamma$ is a finite group, the algebras $k^\Gamma$ and $k\Gamma$ are quantum permutation algebras.

**Theorem**

*Let $H$ be a Hopf algebra that fits into an exact sequence*

$$k \to k^\Gamma \to H \to kF \to k$$

*for some finite groups $\Gamma$, $F$. Assume that one of the following conditions holds :*

1. *$k^\Gamma$ is central in $H$ ;*
2. *the sequence is split ($H = k^\Gamma \# kF$) and $F$ is generated by its $\Gamma$-stable abelian subgroups ;*

*Then $H$ is a quantum permutation algebra.*

Idea of proof : we observe that $H$ is a quantum permutation algebra if and only if $H$ is generated by its commutative (right) coideal subalgebras. So we find a family of such coideal subalgebras. □

By using the theorem together with various classification results (Masuoka, Natale, Kashina, Etingof-Nikshych-Ostrik) we get

### Corollary

*Let $H$ be a semisimple Hopf algebra. Then $H$ is a quantum permutation algebra if one the following holds :*

1. $\dim H = p^3$, *with $p$ prime ;*
2. $\dim H = 2q^2$, *with $q$ prime ;*
3. $\dim H = pq^2$, *with $p > q$ prime ;*
4. $\dim H = pqr$, *with $p, q, r$ distinct primes ;*
5. $\dim H = 16$.

*In particular if* $\dim H \leq 23$, *then $H$ is a quantum permutation algebra*

## Theorem

*The Hopf algebras $k^{C_4}\#k\mathbb{S}_3$, $k^{C_5}\#k\mathbb{S}_4$, $k^{C_5}\#k\mathbb{A}_4$ (respectively associated to the group exact factorizations $\mathbb{S}_4 = \mathbb{S}_3 C_4$, $\mathbb{S}_5 = \mathbb{S}_4 C_5$, $\mathbb{A}_5 = \mathbb{A}_4 C_5$) are not quantum permutation algebras.*

Thus there exists a semisimple Hopf algebra of dimension 24 that is not a quantum permutation algebra.

## Corollary

*The class of quantum permutation algebras is not stable under extensions, duality or 2-cocycle deformations.*

Indeed, $H = k^{C_4}\#k\mathbb{S}_3$ is not a quantum permutation algebra, while $H^* = k^{\mathbb{S}_3}\#kC_4$ is a quantum permutation algebra by the first theorem. Moreover $D(H)^* \cong (D(S_4)^*)^\sigma$ for some 2-cocycle $\sigma$ (Beggs-Gould-Majid). The first theorem ensures that $D(S_4)^*$ is a quantum permutation algebra, while $D(H)^*$ is not (because $D(H)^* \twoheadrightarrow H$). $\square$

## Sketch of the proof of the theorem

We have to see that $H = k^\Gamma \# kF$ is not generated by its commutative (right) coideal subalgebras. It is not easy to have the full list of these coideal subalgebras, so instead we use the following observations :

**Lemma**

If $\pi : H \to kF$ is a surjective Hopf algebra map and if there exits a proper subgroup $F' \subsetneq F$ such that $\pi(R) \subset kF'$ for any commutative (right) coideal subalgebra $R \subset H$, then $H$ is not a quantum permutation algebra.

**Lemma**

*Let $H = k^\Gamma \# kF$ and $\pi = \epsilon \otimes \mathrm{id} : H \to kF$. Let $R \subseteq H$ be a commutative right coideal subalgebra. Then $\pi(R) = kT$, where $T$ is an abelian subgroup of $F$, and we have :*

(i) *If $k^\Gamma \subseteq R$, then $T$ acts trivially on $\Gamma$ via $\triangleleft$.*

(ii) *If $k^\Gamma \cap R = k1$, then $T$ is stable under the action $\triangleright$ of $\Gamma$.*

Now assume that $H = k^{C_5} \# k\mathbb{S}_4$ (exact factorization $\mathbb{S}_5 = \mathbb{S}_4 C_5$ and actions : $C_5 \stackrel{\triangleleft}{\leftarrow} C_5 \times \mathbb{S}_4 \stackrel{\triangleright}{\rightarrow} \mathbb{S}_4$).
If $R$ is a commutative right coideal subalgebra of $H$, then $R \cap k^{C_5}$ is a right coideal subalgebra of $k^{C_5}$, hence a Hopf subalgebra of $k^{C_5}$ and thus $\dim(R \cap k^{C_5})$ divides 5. We are in the situation of the previous lemma : we have $\pi(R) = kT$ where $T$ is an abelian subgroup of $\mathbb{S}_4$ and either $T$ acts trivially on $C_5$ via $\triangleleft$ or $T$ is stable under the action $\triangleright$ of $C_5$.
The only subgroup of $\mathbb{S}_4$ that acts trivially on $C_5$ is $\{1\}$, and the only abelian subgroups of $\mathbb{S}_4$ that are stable under the action $\triangleright$ of $C_5$ are contained in $\langle(1324)\rangle = F'$. Thus $\pi(R) \subset kF'$, and we conclude by the first lemma. $\square$

**Question**

What is the smallest dimension that a self dual non quantum permutation algebra can have ?

# Some quantum permutation algebras obtained by 2-cocycle deformations

We have seen that the class of quantum permutation algebras is not stable under 2-cocycle deformations. We wish to show however that large classes of quantum permutation algebras can be constructed in this way.

Let $\Gamma$ be an abelian group and let $\sigma \in Z^2(\Gamma, k^*)$. The character group $\widehat{\Gamma}$ acts faithfully on the twisted group algebra $k_\sigma\Gamma$ by $\chi.g = \chi(g)g$ ($\chi \in \widehat{\Gamma}$, $g \in \Gamma$), hence $\widehat{\Gamma} \subset \mathrm{Aut}(k_\sigma\Gamma)$.

**Theorem**

*Let $\Gamma$ be a finite abelian group and let $\sigma \in Z^2(\Gamma, k^*)$. Let $G$ be a linear algebraic group with $\widehat{\Gamma} \subset G \subset \mathrm{Aut}(k_\sigma\Gamma)$. Then $\sigma$ induces a 2-cocycle $\sigma'$ on $\mathcal{O}(G)$ such that $\mathcal{O}(G)^{\sigma'}$ is a quantum permutation algebra (non commutative if the only subgroup of $\widehat{\Gamma}$ that is normal in $G$ is $\{1\}$ and if $k_\sigma\Gamma$ is non commutative).*

Examples : $\widehat{\Gamma} = C_2^n \subset G \subset O_n(k) \subset \mathrm{Aut}(Cl_n(k))$
$\widehat{\Gamma} = C_n \times C_n \subset G \subset PGL_n(k) = \mathrm{Aut}(M_n(k))$

**Question**

If $G$ is a finite group and $\sigma$ is a 2-cocycle on $k^G$, is $(k^G)^\sigma$ a quantum permutation algebra ?