Cosovereign Hopf algebras

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Abstract

A sovereign monoidal category is an autonomous monoidal category endowed with the choice of an autonomous structure and an isomorphism of monoidal functors between the associated left and right duality functors. In this paper we define and study the algebraic counterpart of sovereign monoidal categories: cosovereign Hopf algebras. We describe the universal cosovereign Hopf algebras, we study finite-dimensional cosovereign Hopf algebras via the dimension theory provided by the sovereign structure and we examine an example generalizing the quantum groups SL_q.

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1 Introduction

Monoidal category theory played a central role in the discovery of new invariants of knots and links and in the development of the theory of quantum groups.

Let us recall that a tortile tensor category (or ribbon category) is a braided monoidal category ([7]), which is autonomous (i.e. every object has a left dual, and hence also a right dual) and admits a twist ([7, 17]) compatible with duality.

The connection with knot theory is certainly best summarized in the following coherence theorem by Shum ([17]): "*the category of framed tangles (or tangles on ribbons) is the free tortile (or ribbon) category generated by an object*". This means that to any object in a tortile (or ribbon) category, one can associate an isotopy invariant of framed tangles ([17, 4, 19, 12]). We refer the reader to the book [8] for these topics.

A new structure for monoidal categories appeared in papers by Freyd and Yetter ([5, 26]). A sovereign structure on an autonomous monoidal category (i.e. with left and right duals) consists of the choice of a left and a right autonomous structure and a monoidal isomorphism between the associated left and right duality functors. A theorem of Deligne (proposition 2.11 in [26]) brought interest in sovereign structures: there is a twist on an

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autonomous braided monoidal category if and only if there is a sovereign structure on it. Maltsiniotis ([10]) studied sovereign monoidal categories in their own right. He proposed a new equivalent definition (which avoids the choice of an autonomous structure) and he showed that an axiom was redundant in [5, 26]. A sovereign structure is in fact the exact structure needed to define a trace theory.

Quantum groups and monoidal categories are linked by tannakian duality ([14, 20, 6, 15]): the reconstruction of a Hopf algebra from its finite-dimensional comodules. The use of tannakian duality is clearly illustrated in [6]: to an additional categorical structure on the category of finite-dimensional comodules, one associates an additional algebraic structure on the Hopf algebra. For example if A is a Hopf algebra, then there is a braiding ([7]) on $\mathrm{Co}_f(A)$ (finite-dimensional A-comodules) if and only if there is a cobraiding on A, i.e. a linear form on $A \otimes A$ satisfying certain conditions (see [8, 6, 16]). In the same way if A is a cobraided Hopf algebra, the category $\mathrm{Co}_f(A)$ is balanced (there is a twist on it) if and only if there is a linear form τ (called a cotwist) on A satisfying some conditions.

In this paper we investigate the algebraic structure on Hopf algebras corresponding to sovereign structures. A sovereign character on a Hopf algebra A is a *character* Φ on A such that $S^2 = \Phi * S * \Phi^{-1}$ (S is the antipode of A and $*$ is the convolution product). A cosovereign Hopf algebra is a pair (A, Φ) where A is a Hopf algebra and Φ is a sovereign character on A. We show that the category $\mathrm{Co}_f(A)$ admits a sovereign structure if and only if there is a sovereign character on A. Hence the theorem of Deligne mentioned above states in particular the bijective correspondence of cotwists and sovereign characters for a cobraided Hopf algebra. But it is easier to check the existence of a sovereign character (since it is a character). Therefore a technical simplification is brought by sovereign structures when we have applications to knot theory in mind.

We describe the universal (or free) cosovereign Hopf algebras: every finite-type cosovereign Hopf algebra is a homomorphic quotient of one of them. These algebras are parameterized by an invertible matrix. When the base field is the field of complex numbers, they already appeared in a different context: they are the algebras of representative functions on the universal compact quantum groups defined by Van Daele and Wang ([22]).

We also study finite-dimensional cosovereign Hopf algebras via the dimension theory provided by the sovereign structure (theorem 4.3). We show that if (A, Φ) is a finite-dimensional cosovereign Hopf algebra over a field of characteristic zero whose internal dimension (i.e. the dimension computed in the sovereign monoidal category $\mathrm{Co}_f(A)$) is non-zero, then the square of the antipode of A is equal to the identity (and therefore A is semisimple and cosemisimple by [9]).

We end the paper by considering examples of Hopf algebras generalizing the quantum groups SL_q, as presented in [25]. We describe the sovereign character and use a result from [2] to give a condition which ensures that they are cosemisimple.

The paper is organized as follows. In section 2 we recall the definition of a sovereign monoidal category and we introduce cosovereign (and sovereign) Hopf algebras. The universal cosovereign Hopf algebras are described in section 3. In section 4 we use the dimension theory to study finite-dimensional cosovereign Hopf algebras and in section 5 we consider the generalized quantum SL-groups.

Notations. Throughout this paper k will denote a commutative field. The category of finite-dimensional vector spaces will be denoted by $\mathrm{Vect}_f(k)$.

We assume the reader to be familiar with the theory of Hopf algebras ([8, 11]) and in particular we freely use convolution products.

Let $A = (A, m, u, \Delta, \varepsilon, S)$ be a Hopf k-algebra. The multiplication will be denoted by m, $u : k \to A$ is the unit of A, while Δ, ε and S are respectively the comultiplication, the counit and the antipode of A. The dual Hopf algebra is denoted by A^0. The category of finite-dimensional right A-comodules will be denoted by $\mathrm{Co}_f(A)$ while the category of finite-dimensional left A-modules will be denoted by $\mathrm{Mod}_f(A)$. We only consider right comodules and left modules.

We also assume that the reader is familiar with monoidal categories ([7, 8]). We only use strict monoidal categories: by Mac Lane's coherence theorem (see [7], 1.4 for a simple proof), every monoidal category is monoidally equivalent to a strict one.

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2 Sovereign structures

In this section we recall the definition of sovereign monoidal categories and introduce their counterpart for Hopf algebras: cosovereign Hopf algebras.

We begin with duality in monoidal categories:

Definition 2.1 *Let $\mathcal{C} = (\mathcal{C}, \otimes, I)$ be a monoidal category (where I denotes the monoidal unit) and let $X \in \mathrm{ob}(\mathcal{C})$. A left dual for X is a triplet $(X^\vee, \varepsilon_X, \eta_X)$ with $X^\vee \in \mathrm{ob}(\mathcal{C})$, $\varepsilon_X : X^\vee \otimes X \longrightarrow I$ and $\eta_X : I \longrightarrow X \otimes X^\vee$ are morphisms of $\mathcal{C}$ such that:*

$$(1_X \otimes \varepsilon_X) \circ (\eta_X \otimes 1_X) = 1_X \quad \text{and} \quad (\varepsilon_X \otimes 1_{X^\vee}) \circ (1_{X^\vee} \otimes \eta_X) = 1_{X^\vee}.$$

Let X be an object of a monoidal category and suppose that X is endowed with a left dual. Then the functor $X \otimes -$ admits a left adjoint $X^\vee \otimes -$ (see [6], Section 9). We inherit the uniqueness results of adjoint functors.

Definition 2.2 *A monoidal category is said to be left autonomous if every object has a left dual. A left autonomous structure on a left autonomous monoidal category $\mathcal{C}$ is the choice for every object X of a left dual $(X^\vee, \varepsilon_X, \eta_X)$ such that $I^\vee = I$ and $\varepsilon_I = \eta_I = 1_I$.*

Let $\mathcal{C}$ be a left autonomous monoidal category and let us endow $\mathcal{C}$ with a left autonomous structure. In this way we define a functor:

$\mathbf{D}_l : (\mathcal{C}, \otimes, I) \longrightarrow (\mathcal{C}^{op}, \otimes^{op}, I)$ by $\mathbf{D}_l(X) = X^\vee$ and $\mathbf{D}_l(f) = {}^t f$

where if $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a morphism of $\mathcal{C}$, then ${}^t f : Y^\vee \longrightarrow X^\vee$ is the unique arrow such that $\varepsilon_Y \circ (1_{Y^\vee} \otimes f) = \varepsilon_X \circ ({}^t f \otimes 1_X)$.

Another choice of left autonomous structure would lead to a monoidal functor $\mathbf{D}'_l$ with $\mathbf{D}'_l \cong^{\otimes} \mathbf{D}_l$ ($\cong^{\otimes}$ means that the monoidal functors $\mathbf{D}_l$ and $\mathbf{D}'_l$ are isomorphic). Thus the choice of an autonomous structure is just a convenient way to define the duality functor.

Example 2.3 We briefly describe the left autonomous structure on $\mathrm{Vect}_f(k)$ and $\mathrm{Co}_f(A)$ where A is a Hopf algebra. Let V be a finite-dimensional vector space. Let $V^\vee = V^* = \mathrm{Hom}(V,k)$. It is well known that this procedure, with classical evaluation and coevaluation maps, defines an autonomous structure on $\mathrm{Vect}_f(k)$. Now let V be a finite-dimensional A-comodule with coaction $\alpha_V : V \longrightarrow V \otimes A$ such that $\alpha_V(v_i) = \sum_j v_j \otimes a_{ji}$ for some basis (v_i) of V. Let $V^\vee$ be the A-comodule whose underlying vector space is V^* and whose coaction $\alpha_{V^\vee} : V^\vee \longrightarrow V^\vee \otimes A$ is defined by $\alpha_{V^\vee}(v_i^*) = \sum_j v_j^* \otimes S(a_{ij})$ where (v_i^*) is the dual basis of (v_i). Then $V^\vee$, with evaluation and coevaluation maps, is a left dual for V.

Definition 2.4 *Let $\mathcal{C} = (\mathcal{C}, \otimes, I)$ be an autonomous monoidal category. A sovereign structure φ on $\mathcal{C}$ consists of a left autonomous structure on $\mathcal{C}$ and an isomorphism of monoidal functors $\varphi : 1_{\mathcal{C}} \cong^{\otimes} \mathbf{D}_l^2$. A sovereign monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, I, \varphi)$ is a left autonomous monoidal category endowed with a sovereign structure.*

Remark 2.5 The papers [5, 26, 10] use both left and right duality to define a sovereign monoidal category (an axiom was redundant as shown in [10]). The reader will easily check that the definition above is equivalent to the one in those papers.

Example 2.6 i) Let us describe the sovereign structure ψ on $\mathrm{Vect}_f(k)$. We endow $\mathrm{Vect}_f(k)$ with the autonomous structure of example 2.3. Let V be a finite-dimensional vector space. Then $\psi_V : V \longrightarrow V^{**}$ is the classical identification with the bidual vector space.
ii) The reader will easily imagine the definition of a sovereign functor. The category of diagrams ([26]) is the free sovereign monoidal category generated by an object ([5]). An autonomous braided monoidal category admits a sovereign structure if and only there is a twist on it ([26], see also [10]).

We now come into the heart of the subject:

Definition 2.7 *Let A be a Hopf algebra. i) A sovereign character on A is a character Φ on A such that $S^2 = \Phi * id * \Phi^{-1}$. A cosovereign Hopf algebra is a pair (A, Φ) where A is a Hopf algebra and Φ is a sovereign character on A.*
ii) A sovereign element of A is a group-like element Φ such that $S^2(a) = \Phi a \Phi^{-1}$ for all $a \in A$. A sovereign Hopf algebra is a pair (A, Φ) where A is a Hopf algebra and Φ is a sovereign element of A.

It is clear that if (A, Φ) is a cosovereign (resp. sovereign) Hopf algebra then (A^0, Φ) is a sovereign (resp. cosovereign) Hopf algebra. The antipode of the underlying Hopf algebra of a sovereign or cosovereign Hopf algebra is bijective.

Example 2.8 It is easy to check that Sweedler's famous 4-dimensional Hopf algebra (see [8], p. 67) admits a sovereign element (the non-trivial group-like element) and a sovereign character (the non-trivial character). Sweedler's algebra clearly shows a way to construct sovereign Hopf algebras. Let H_n be the quotient of the free algebra $k\{X_1, \dots, X_n, \Phi, \Phi^{-1}\}$ by the two-sided ideal generated by the relations $\Phi\Phi^{-1} = 1 = \Phi^{-1}\Phi$. Then H_n is a Hopf algebra with comultiplication $\Delta(X_i) = 1 \otimes X_i + X_i \otimes \Phi$, $\Delta(\Phi) = \Phi \otimes \Phi$, with counit $\varepsilon(X_i) = 0$, $\varepsilon(\Phi) = 1$ and with antipode $S(X_i) = -X_i\Phi^{-1}$ and $S(\Phi) = \Phi^{-1}$. It is clear that Φ is a sovereign element in H_n. For more examples of this kind, see [18].

Proposition 2.9 *i) Let (A, Φ) be a cosovereign Hopf algebra. Then the sovereign character Φ defines a sovereign structure on* $\mathrm{Co}_f(A)$*.*
ii) Let (A, Φ) be a sovereign Hopf algebra. Then there is a sovereign structure on $\mathrm{Mod}_f(A)$*.*

Proof. i) We use the autonomous structure on $\mathrm{Co}_f(A)$ described in example 2.3. Let V be a finite-dimensional A-comodule with coaction α_V. We define a linear map $\varphi_V : V \longrightarrow V^{\vee\vee}$ as follows: $\varphi_V = (\psi_V \otimes \Phi) \circ \alpha_V$ (ψ_V was defined in 2.6). Then φ_V is a map of comodules, because $S^2 * \Phi = \Phi * id$. It is easily seen that if $f : V \longrightarrow W$ is a map of comodules, then ${}^t f \circ \varphi_W = \varphi_V \circ {}^t f$. In this way we get a bijective natural transformation $\varphi : 1_{\mathcal{C}} \longrightarrow \mathbf{D}_l^2$. Finally φ is a morphism of monoidal functors since Φ is a character of A and therefore we have defined a sovereign structure on $\mathrm{Co}_f(A)$.
ii) Let A^0 be the dual Hopf algebra of A. The categories $\mathrm{Mod}_f(A)$ and $\mathrm{Co}_f(A^0)$ are monoidally equivalent and (A^0, Φ) is a cosovereign Hopf algebra. □

Clearly we want a converse of proposition 2.9. In fact a more precise statement is available in the tannakian reconstruction setting, which we now briefly describe.

Let $\mathcal{C}$ be a left autonomous category and let $F : \mathcal{C} \longrightarrow \mathrm{Vect}_f(k)$ be a monoidal functor. Let us recall ([20, 6, 15, 14])that there is a universal Hopf algebra $\mathrm{End}^\vee(F) = \mathrm{coend}(F)$ such that F factorizes through a monoidal functor $\overline{F} : \mathcal{C} \longrightarrow \mathrm{Co}_f(\mathrm{End}^\vee(F))$ followed by the forgetful functor. The Hopf algebra $\mathrm{End}^\vee(F)$ is a representative for the functor $V \mapsto Nat(F, F \otimes V)$ from vector spaces to sets. Indeed we have for every vector space V an isomorphism (natural in V, we write $A = \mathrm{End}^\vee(F)$)

$$\theta_V : \mathrm{Hom}_k(A, V) \longrightarrow Nat(F, F \otimes V)$$

defined by $\theta_V(f) = (1_F \otimes f) \circ \alpha$ where $\alpha = \theta_A(1_A)$. In particular we have $\mathrm{End}^\vee(F)^* \cong \mathrm{End}(F)$ and $\mathrm{Hom}_{k-alg}(\mathrm{End}^\vee(F), k) \cong \mathrm{Aut}^\otimes(F)$. ($\mathrm{Aut}^\otimes(F)$ denotes the group of automorphism of the monoidal functor F). When $\mathcal{C} = \mathrm{Co}_f(A)$ is the category of finite-dimensional comodules of a Hopf algebra and F is the forgetful functor, then $A \cong \mathrm{End}^\vee(F)$. Conversely, one can characterize the categories $\mathrm{Co}_f(A)$ for some Hopf algebra A (see [14, 15, 6]).

As shown is [16], the square of the antipode of a cobraided Hopf algebra is a coinner automorphism ($S^2 = \lambda * id * \lambda^{-1}$ for some linear form λ), and the result follows from the canonical isomorphism of an object with its bidual in a braided category. Here we have:

Proposition 2.10 *Let $\mathcal{C} = (\mathcal{C}, \otimes, I, \varphi)$ be a sovereign monoidal category and let $F : \mathcal{C} \longrightarrow \mathrm{Vect}_f(k)$ be a monoidal functor. Then there is a sovereign character on the Hopf algebra $\mathrm{End}^\vee(F)$. In particular If A is a Hopf algebra, there is a sovereign character on A if and only if there is a sovereign structure on $\mathrm{Co}_f(A)$.*

Proof. We first define an automorphism of the monoidal functor F. We have an isomorphism of monoidal functors $F \circ \mathbf{D}_l^2 \cong^{\otimes} \mathbf{D}_l^2 \circ F$ since a monoidal functor commutes with duality. Let u be defined by the composition

$$F \xrightarrow{F(\varphi)} F \circ \mathbf{D}_l^2 \cong^{\otimes} \mathbf{D}_l^2 \circ F \xrightarrow{\psi_F^{-1}} F$$

where ψ^{-1} is defined in example 2.6. It is clear that u is an automorphism of the monoidal functor F, and thus we get a character Φ of A such that $\theta_k(\Phi) = u = (1_F \otimes \Phi) \circ \alpha$. Let X be any object of $\mathcal{C}$: the isomorphisms $F(\varphi_X)$ and $F(X^{\vee\vee}) \cong F(X)^{\vee\vee}$ are $\mathrm{End}^\vee(F)$-comodule morphisms and hence so is the map $\varphi'_{F(X)} = \psi_{F(X)} \circ u = (\psi_{F(X)} \otimes \Phi) \circ \alpha_{F(X)}$. This means that $\alpha_{F(X)^{\vee\vee}} \circ \varphi'_{F(X)} = (\varphi'_{F(X)} \otimes id) \circ \alpha_{F(X)}$. Now let us remark that $\alpha_{F(X)^{\vee\vee}} \circ \psi_{F(X)} = (\psi_{F(X)} \otimes S^2) \circ \alpha_{F(X)}$. It is easy to see that this implies

$$\alpha_{F(X)^{\vee\vee}} \circ \varphi'_{F(X)} = (\psi_{F(X)} \otimes (S^2 * \Phi)) \circ \alpha_{F(X)} \ .$$

On the other hand

$$(\varphi'_{F(X)} \otimes id) \circ \alpha_{F(X)} = (\psi_{F(X)} \otimes (\Phi * id)) \circ \alpha_{F(X)} \ .$$

Using the isomorphism θ_A, we get $S^2 * \Phi = \Phi * id$, and hence Φ is a sovereign character. The second assertion follows immediately from the first one and the reconstruction of a Hopf algebra from its finite-dimensional comodules. □

Remark 2.11 If the functor of the above proposition is sovereign, then the square of the antipode of $\mathrm{End}^\vee(F)$ is equal to the identity.

Let (A, σ) be a cobraided Hopf algebra (σ is a bilinear form on A satisfying some conditions, see [8]). As a particular case of Deligne's theorem ([26], prop. 2.11) we have a bijective correspondence between sovereign characters and cotwists ([8]) on A. One can check this result directly: if τ is a cotwist on A, then $\Phi := \tau^{-1} * \lambda^{-1}$ is a sovereign character (λ is the convolution invertible linear form defined by $\lambda(x) = \sum \sigma(x_1, S(x_2))$). Conversely if Φ is sovereign character, then $\tau := \lambda^{-1} * \Phi^{-1}$ is a cotwist.

3 Universal cosovereign Hopf algebras

In this section we introduce the universal cosovereign Hopf algebras and study some of their properties. By universal we mean that every finite type cosovereign Hopf algebra is a homomorphic quotient of one of them (a quantum subgroup in the language of quantum groups).

Notations. We will use matrix notations. If $u = (u_{ij})$ is a $n \times n$ matrix with values in any algebra, the transpose matrix of u will be denoted by ${}^t u$.

Definition 3.1 *Let $F \in GL_n(k)$. The algebra $H(F)$ is the universal algebra with generators $(u_{ij})_{1\leq i,j\leq n}$, $(v_{ij})_{1\leq i,j\leq n}$ and relations:*

$$u{}^t v = {}^t v u = 1 \quad ; \quad vF{}^t uF^{-1} = F{}^t uF^{-1}v = 1$$

The algebras $H(F)$ are closely related to the universal compact quantum groups of Van Daele and Wang [22]. We go back to this subject later in the section. The following result justify the expression universal cosovereign Hopf algebras for the algebras $H(F)$.

Theorem 3.2 *Let $F \in GL_n(k)$. Then $H(F)$ is a Hopf algebra:*

$$\begin{aligned} \text{with comultiplication } \Delta(u_{ij}) &= \sum_k u_{ik} \otimes u_{kj} \quad ; \quad \Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}, \\ \text{with counit } \varepsilon(u_{ij}) &= \delta_{ij} = \varepsilon(v_{ij}), \\ \text{with antipode } S(u) &= {}^t v \quad ; \quad S(v) = F{}^t uF^{-1}. \end{aligned}$$

There is a sovereign character Φ_F on $H(F)$ defined by $\Phi_F(u) = {}^t F^{-1}$ and $\Phi_F(v) = F$ and hence $(H(F), \Phi_F)$ is a cosovereign Hopf algebra.

If A is a Hopf algebra and V is a finite-dimensional A-comodule with coaction $\alpha_V : V \longrightarrow V \otimes A$ such that $V \cong V^{\vee\vee}$, then there is a matrix $F \in GL_n(k)$ ($n = \dim(V)$), a coaction $\beta_V : V \longrightarrow V \otimes H(F)$ and a Hopf algebra morphism $\pi : H(F) \longrightarrow A$ such that $(1_V \otimes \pi) \circ \beta_V = \alpha_V$. In particular for every finite type cosovereign Hopf algebra (A, Φ), there is a surjective Hopf algebra morphism $\pi : H(F) \longrightarrow A$ for some $F \in GL_n(k)$.

Proof. It is easily seen that the maps Δ and ε are well defined algebra morphisms and thus $H(F)$ is a bialgebra. In the same way the formulas of the theorem give rise to a well defined anti-homomorphism S which is clearly an antipode for $H(F)$. The character Φ_F is easily seen to be well defined on $H(F)$. We have $S^2(u) = {}^t F^{-1} {}^t u {}^t F = \Phi_F(u) u \Phi_F^{-1}(u)$ and $S^2(v) = FvF^{-1} = \Phi_F(v) v \Phi_F^{-1}(v)$. Therefore $S^2 = \Phi_F * id * \Phi_F^{-1}$ and $(H(F), \Phi_F)$ is a cosovereign Hopf algebra.

Let A be a Hopf algebra and let V be a finite dimensional A-comodule with basis $v_1, \dots, v_n$ such that $\alpha_V(v_i) = \sum_j v_j \otimes a_{ji}$ and $V \cong V^{\vee\vee}$. Hence there is a matrix $K \in GL_n(k)$ such that $S^2(a)K = Ka$ (a is the matrix (a_{ij})). Let $F = {}^t K^{-1}$: there is a

Hopf algebra morphism $\pi : H(F) \longrightarrow A$ defined by $\pi(u) = a$ and $\pi(v) = {}^tS(a)$ (since $({}^tS(a))^{-1} = {}^tS^2(a)$). A coaction $\beta_V : V \longrightarrow V \otimes H(F)$ is defined by $\beta_V(v_i) = \sum_j v_j \otimes u_{ji}$. Clearly π satisfies the requirement in the theorem. The last assertion is straightforward. $\square$

The next result reduces the list of the algebras $H(F)$:

Proposition 3.3 *Let F and $K \in GL_n(k)$ and let $\lambda \in k^*$. Then $H(\lambda F) = H(F)$, $H(F) \cong H(KFK^{-1})$ and $H(F) \cong H({}^tF^{-1})$ (as Hopf algebras).*

Proof. The first statement is obvious. A Hopf algebra isomorphism $\phi : H(F) \to H(KFK^{-1})$ is defined by $\phi(u) = {}^tKu{}^tK^{-1}$ and $\phi(v) = K^{-1}vK$. A Hopf algebra isomorphism $\psi : H(F) \longrightarrow H({}^tF^{-1})$ is defined by $\psi(u) = v$ and $\psi(v) = FuF^{-1}$. $\square$

Proposition 3.4 *Let $F \in GL_n(k)$ such that the elements u_{ij} of $H(F)$ are linearly independent (for example the identity element of $GL_n(k)$). If $\mathrm{Tr}(F) = 0$ or $\mathrm{Tr}(F^{-1}) = 0$ then $H(F)$ is not cosemisimple.*

Proof. One can assume that the base field is algebraically closed. Let us assume that $H(F)$ is cosemisimple. The n-dimensional comodule U associated to the elements u_{ij} is irreducible by the assumption. The matrix ${}^tF^{-1}$ intertwines U and $U^{\vee\vee}$ and then $\mathrm{Tr}(F)$ and $\mathrm{Tr}(F^{-1})$ are non-zero by [3], proposition 3.5. $\square$

We now make contact with the theory of compact quantum groups. We assume that the reader is familiar with Hopf $*$-algebras [21] and with the algebraic theory of compact quantum groups as in [3]. We now assume the base field to be the field of complex numbers.

Notations. Let $a = (a_{ij})$ where A is a $*$-algebra. Then the matrix (a_{ij}^*) is denoted by $\overline{a}$ and the matrix ${}^t\overline{a}$ is denoted by a^*.

Definition 3.5 *A Hopf $*$-algebra is a complex algebra A, which is a $*$-algebra and whose coproduct $\Delta : A \longrightarrow A \otimes A$ is a $*$-homomorphism. A CQG algebra is Hopf $*$-algebra A such that every finite-dimensional A-comodule is unitarizable. In other words for every matrix $a = (a_{ij}) \in M_n(A)$ such that $\Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}$ and $\varepsilon(a_{ij}) = \delta_{ij}$, there is a matrix $F \in GL_n(\mathbb{C})$ such that the matrix FaF^{-1} is unitary.*

A CQG algebra may be thought as the algebra of representative functions on a compact quantum group. A Hopf $*$-algebra is CQG if and only if there is a faithful Haar measure on it ([3], 3.10). In particular a CQG algebra is cosemisimple. In the next result we find a necessary and sufficient condition for the Hopf algebra $H(F)$ to admit a CQG algebra structure. We say that a matrix $F \in GL_n(\mathbb{C})$ is relatively positive if there is a scalar $\lambda \in \mathbb{C}^*$ such that λF is a positive matrix.

Proposition 3.6 *Let $F \in GL_n(\mathbb{C})$. Then $H(F)$ admits a CQG algebra structure if and only if F is conjugate to a relatively positive matrix.*

Proof. Let us decompose the comodule U associated to the u_{ij}'s into a direct sum of irreducible comodules $U_1 \oplus ... \oplus U_p$. Let u_i be the matrix associated to the comodule U_i. Then $S^2(u_i) = K_i u_i K_i^{-1}$ for some invertible matrices K_i. By [3], proposition 3.6, the matrices K_i are conjugate to relatively positive matrices, and so is F, which is conjugate to $K_1 \oplus ... \oplus K_p$.

Conversely we can assume that F is a positive matrix by proposition 4.3. It is easy to see that a Hopf $*$-algebra structure is defined on $H(F)$ by letting $\overline{u} = v$ (at this point we only use $F^* = F$). The matrix u is unitary. Let $K = \sqrt{F}$. The matrix KvK^{-1} is unitary. It follows that $H(F)$ is a CQG algebra since it is generated by the entries of the matrices u and v ([3], proposition 2.4, S. Wang pointed out that the proof of this proposition in [3] is false. The conclusion is true however). □

When F is a positive matrix, it is easy to see that $H(F)$ is the CQG algebra of representative functions on the compact quantum group $A_u(F)$ defined by Van Daele and Wang in theorem 1.3 of [22] (in fact it is sufficient to consider positive matrices to get the family of universal compact quantum groups). In that case the representation semi-ring of $H(F)$ is described by Banica in [1]: the irreducible comodules are labeled by the free product $\mathbb{N} * \mathbb{N}$.

Woronowicz has shown ([24], theorem 5.6) that a CQG algebra always has a sovereign character. For a general cosemisimple Hopf algebra A, there is a convolution invertible λ such that $S^2 = \lambda * id * \lambda^{-1}$ ([9]) and the map $\sigma = \lambda * id * \lambda$ is an algebra morphism (the modular homomorphism of theorem 5.6 in [24]). However it does not seem to be clear that a cosemisimple Hopf algebra always has a sovereign character.

4 Dimension theory

Maltsiniotis has shown in [10] that a sovereign structure is exactly the one needed to define a trace theory. In particular one gets a dimension theory. In this section we study this dimension theory for cosovereign Hopf algebras.

Definition 4.1 *Let $\mathcal{C} = (\mathcal{C}, \otimes, I, \varphi)$ be a sovereign monoidal category and let $X \in \mathrm{ob}(\mathcal{C})$. The left (resp. right) dimension of X is the element of* $\mathrm{End}(X)$ *defined by*

$$\dim_\varphi^l(X) = \varepsilon_X \circ (1_{X^\vee} \otimes \varphi_X^{-1}) \circ \eta_{X^\vee}$$

$$(\text{Resp.} \quad \dim_\varphi^r(X) = \varepsilon_{X^\vee} \circ (\varphi_X \otimes 1_{X^\vee}) \circ \eta_X)$$

If (A, Φ) is a cosovereign Hopf algebra (resp. sovereign Hopf algebra) the left and right dimension of an object V of $\mathrm{Co}_f(A)$ (resp. $\mathrm{Mod}_f(A)$) are denoted by $\dim_\Phi^l(V)$ and $\dim_\Phi^r(V)$.

Here are the basic properties of the dimensions:

Proposition 4.2 *(see [10], corollary 3.5.25). Let $\mathcal{C} = (\mathcal{C}, \otimes, I, \varphi)$ be a sovereign monoidal category and let X and Y be objects of $\mathcal{C}$.*

i) $\dim^l_\varphi(I) = 1_I = \dim^r_\varphi(I)$
ii) If $X \cong Y$ *then* $\dim^l_\varphi(X) = \dim^l_\varphi(Y)$ *and* $\dim^r_\varphi(X) = \dim^r_\varphi(Y)$
iii) $\dim^r_\varphi(X) = \dim^l_\varphi(X^\vee)$; $\dim^l_\varphi(X) = \dim^r_\varphi(X^\vee)$.
iv) If $\mathrm{End}(I)$ *is central* ($u \otimes f = f \otimes u$ *for all morphisms* f *of* $\mathcal{C}$ *and all* $u \in \mathrm{End}(I)$) *then* $\dim^l_\varphi(X \otimes Y)) = \dim^l_\varphi(X)\dim^l_\varphi(Y)$ *and* $\dim^r_\varphi(X \otimes Y) = \dim^r_\varphi(X)\dim^r_\varphi(Y)$.

Let (A, Φ) be a cosovereign Hopf algebra and let V be a finite-dimensional A-comodule with basis (v_i) such that $\alpha_V(v_i) = \sum_j v_j \otimes a_{ji}$. Then $\dim^l_\Phi(V) = \sum_i \Phi^{-1}(a_{ii})$ and $\dim^r_\Phi(V) = \sum_i \Phi(a_{ii})$. Let (A, Φ) be a sovereign Hopf algebra and let V be a finite-dimensional A-module. We have $\dim^l_\Phi(V) = \mathrm{Tr}(\Phi^{-1})$ and $\dim^r_\Phi(V) = \mathrm{Tr}(\Phi)$ where Φ and Φ^{-1} are considered as operators on V.

The Hopf algebras $H(F)$ clearly show that anything can happen with the dimension theory of sovereign monoidal categories. Let U be the obvious n-dimensional comodule associated with the matrix u. Then $\dim^l_{\Phi_F}(U) = \mathrm{Tr}(F)$ and $\dim^r_{\Phi_F}(U) = \mathrm{Tr}(F^{-1})$. These two scalars may not coincide. It is also clear in this example that the dimension of an object may take any value with respect to different sovereign structure.

On the other hand, for a finite-dimensional (co)sovereign Hopf algebra in characteristic zero, the dimension theory completely determines the sovereign structure.

Theorem 4.3 *Let* (A, Φ) *be a finite-dimensional cosovereign (or sovereign) Hopf algebra over a field of characteristic zero. If* $\dim^l_\Phi(A) \neq 0$ *or* $\dim^r_\Phi(A) \neq 0$ *then* $S \circ S = 1_A$ *and* A *is semisimple and cosemisimple.*

Proof. We can assume that k is algebraically closed and that (A, Φ) is a cosovereign Hopf algebra (the sovereign case is dual). We consider A as an A-comodule via the comultiplication. For every finite-dimensional A-comodule there is an A-comodule isomorphism $A \otimes V \cong A^{\dim(V)}$ (this is a well-known trick in Hopf algebra theory, see [11] proposition 3.1.4 in the particular case $V = A$, the only one we need). By proposition 4.2 we have $\dim^l_\Phi(A)\dim^l_\Phi(V) = \dim(V)\dim^l_\Phi(A)$ (the dimension is clearly additive on direct sums). In particular $\dim^l_\Phi(A) = \dim(A)$ if $\dim^l_\Phi(A) \neq 0$. Let $e_1, \ldots, e_n$ be a basis of A such that $\Delta(e_i) = \sum_j e_j \otimes a_{ji}$. Let F be the matrix $F = (\Phi^{-1}(a_{ij}))$. The matrix F can be assumed to be triangular. We have $(\Phi^{-1})^{*k} = \varepsilon$ for some integer k since Φ is a character and hence $F^k = I$. This means that the elements $\Phi^{-1}(a_{ii})$ are k-th roots of unity. But $\dim^l_\Phi(A) = \dim(A) = n = \sum_i \Phi^{-1}(a_{ii})$ and therefore $\Phi^{-1}(a_{ii}) = 1$ for all i since the base field is of characteristic 0. This also implies that F is a diagonal matrix and $F = I$. Then $\Phi = \varepsilon$ (the elements a_{ij} generate A) and $S \circ S = 1_A$. Now A is semisimple and cosemisimple by [9] theorem 4.3. The proof is the same if $\dim^r_\Phi(A) \neq 0$. $\square$

5 Quantum SL-groups

We now examine a class of examples closely related to the quantum groups $SU(n)$ of [25].

Let $V = k^n$ and let $e_1, \dots, e_n$ be the canonical basis with dual basis $e_1^*, \dots, e_n^*$. Let $N \geq 2$ be an integer and let $E : V^{\otimes N} \longrightarrow k$ be a linear map. Let $E(i_1, \dots, i_N) = E(e_{i_1} \otimes \dots \otimes e_{i_N})$. We say that E is left non-degenerate if the linear map

$$V^{\otimes N-1} \longrightarrow V^*, \quad e_{i_1} \otimes \dots \otimes e_{i_{N-1}} \longmapsto \sum_k E(i_1, \dots, i_{N-1}, k) e_k^*$$

is surjective. In this case there are scalars $\lambda(i_1, \dots, i_N)$ such that

$$(\star) \qquad \sum_{j_1, \dots, j_{N-1}} \lambda(i, j_1, \dots, j_{N-1}) E(j_1, \dots, j_{N-1}, k) = \delta_{ik}, \quad 1 \leq i, k \leq n.$$

We say that E is right non-degenerate if the linear map

$$V^{\otimes N-1} \longrightarrow V^*, \quad e_{i_1} \otimes \dots \otimes e_{i_{N-1}} \longmapsto \sum_k E(k, i_1, \dots, i_{N-1}) e_k^*$$

is surjective. In that case there are scalars $\mu(i_1, \dots, i_N)$ such that

$$(\star\star) \qquad \sum_{j_1, \dots, j_{N-1}} E(k, j_1, \dots, j_{N-1}) \mu(j_1, \dots, j_{N-1}, i) = \delta_{ik}, \quad 1 \leq i, k \leq n.$$

Theorem 5.1 *Let $E : V^{\otimes N} \longrightarrow k$ be a left and right non-degenerate linear map. Let $SL(E)$ be the universal algebra with generators $(a_{ij})_{1 \leq i,j \leq n}$ and relations:*

$$(5.1.1) \qquad \sum_{j_1, \dots, j_N} E(j_1, \dots, j_N) a_{j_1 i_1} \dots a_{j_N i_N} = E(i_1, \dots, i_N) 1, \quad 1 \leq i_1, \dots, i_N \leq n$$

$$(5.1.2) \qquad \sum_{j_1, \dots, j_N} E(j_1, \dots, j_N) a_{i_1 j_1} \dots a_{i_N j_N} = E(i_1, \dots, i_N) 1, \quad 1 \leq i_1, \dots, i_N \leq n$$

i) Then $SL(E)$ is a Hopf algebra with bijective antipode.
ii) Assume that there are invertible scalars $(\beta_i)_{1 \leq i \leq n}$ such that $\beta_i E(j_1, \dots, j_{N-1}, i) = E(i, j_1, \dots, j_{N-1})$ for all $i, j_1, \dots, j_{N-1}$. Then there is a sovereign character Φ_β on $SL(E)$ such that $\Phi_\beta(a_{ij}) = \delta_{ij} \beta_i$.
iii) If k is a field of characteristic zero and if the field $\mathbb{Q}(E(j_1, \dots, j_N)_{1 \leq i_1, \dots, i_N \leq n})$ can be ordered, then $SL(E)$ is cosemisimple.
iv) If $k = \mathbb{C}$ and $E(j_1, \dots, j_N) \in \mathbb{R}$ for all $j_1, \dots, j_N$, then $SL(E)$ admits a CQG algebra structure.

Proof. i) It is easily seen that $SL(E)$ is a bialgebra with coproduct $\Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}$ and counit $\varepsilon(a_{ij}) = \delta_{ij}$. Let us show that the matrix $a = (a_{ij})$ is invertible. Let us consider equation 5.1.1. Multiplying by $\lambda(k, i_1, \dots, i_{N-1})$ and summing over $i_1, \dots, i_{N-1}$, we get that a is left invertible (we use $(\star)$). In the same way a is right invertible (use 5.1.2 and $(\star\star)$) and therefore a is invertible and $SL(E)$ is a Hopf algebra by [23], theorem 1. Let us

show that ${}^t a$ is invertible. By 5.1.2 and $(\star)$ ${}^t a$ is right invertible and by 5.1.1 and $(\star\star)$ ${}^t a$ is left invertible. Hence the antipode of $SL(E)$ is invertible.
ii) The character Φ_β is easily seen to be well defined. Let us consider equation 5.1.2: multiplying on the left by $S(a_{ki_1})$ and summing over i_1, we get

$$(\star\star\star) \qquad \sum_{j_2,\ldots,j_N} E(k, j_2, \ldots j_N) a_{i_2 j_2} \ldots a_{i_N j_N} = \sum_{i_1} S(a_{ki_1}) E(i_1, \ldots, i_N).$$

We have

$$\begin{aligned} & \sum_i \sum_k E(k, i_2, \ldots, i_N) S(a_{ik}) \beta_i^{-1} a_{ji} \\ = & \sum_i \sum_{j_2,\ldots,j_N} E(i, j_2, \ldots, j_N) \beta_i^{-1} a_{i_2 j_2} \ldots a_{i_N j_N} a_{ji} \quad \text{by } (\star\star\star) \\ = & \sum_i \sum_{j_2,\ldots,j_N} E(j_2, \ldots j_N, i) a_{i_2 j_2} \ldots a_{i_N j_N} a_{ji} \\ = & E(i_2, \ldots, i_N, j) \quad \text{(by 5.1.2)} = \beta_j^{-1} E(j, i_2, \ldots, i_N). \end{aligned}$$

Using $(\star\star)$ we get

$$\sum_i S(a_{il}) \beta_i^{-1} a_{ji} = \delta_{jl} \beta_j^{-1} \text{ for all } j, l.$$

The inverse of the matrix ${}^t a$ is ${}^t S^{-1}(a)$ and hence we have $S^{-1}(a_{il}) = \beta_i^{-1} S(a_{il}) \beta_j = \Phi_\beta^{-1} * S * \Phi_\beta(a_{il})$. This means that Φ_β is a sovereign character on $SL(E)$.
iii) There is an algebra automorphism τ of $SL(E)$ defined by $\tau(a_{ij}) = a_{ji}$. Hence by [2], 4.7 $SL(E)$ is cosemisimple. Statement iv) follows from [2], 4.6. □

In a special case the $SL(E)$ construction gives the quantum groups SL_q. Let $N = n$ and let $q \in k^*$. Let $E_q : V^{\otimes n} \longrightarrow k$ defined by $E_q(i_1, \ldots i_N) = 0$ if two indices are equal and otherwise $E_q(i_1, \ldots i_N) = (-q)^{l(\sigma)}$ where $l(\sigma)$ is the length of the permutation $\sigma(k) = i_k$. The Hopf algebras $SL(E_q)$ and $SL_q(n)$ are isomorphic. This fact can be proved using the same proof as in Rosso's comparison of the quantum SL groups of Woronowicz ([25]) and Drinfeld ([13], theorem 6). See also [21] for useful computations. The elements β_i of theorem 5.1 are given by $\beta_i = (-q)^{2i-1-n}$ and therefore $SL_q(n)$ admits a sovereign character Φ defined by $\Phi(a_{ij}) = \delta_{ij}(-q)^{2i-1-n}$.

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