COREPRESENTATION THEORY OF UNIVERSAL COSOVEREIGN HOPF ALGEBRAS

Julien Bichon

Laboratoire de Mathématiques Appliquées, Université de Pau et des Pays de l'Adour, IPRA, Avenue de l'université, 64000 Pau, France. E-mail: Julien.Bichon@univ-pau.fr

Abstract

We determine the corepresentation theory of the universal cosovereign Hopf algebras, which are some natural analogues of the general linear groups in quantum group theory, for generic matrices over an algebraically closed field of characteristic zero. Our results generalize Banica's previous results in the compact case. As an application, we easily get the representation theory of the quantum automorphism group of a matrix algebra endowed with a non-necessarily tracial measure.

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1 Introduction

In this paper a quantum group is understood to be the dual object of a Hopf algebra, the latter playing the role of a function algebra. A representation of a quantum group is then a comodule (or corepresentation) over the corresponding Hopf algebra.

We study the corepresentations of the universal cosovereign Hopf algebras. The terminology "universal" is a consequence of the fact that the corresponding quantum groups satisfy, in the category of quantum groups, a universal property that is similar to the one of the general linear groups in the category of algebraic groups. The description of the representations of the general linear groups is important in classical group theory, and therefore it seems to be natural and important to describe the representations of the analogous objects in quantum group theory.

In the quantum framework the well-known correspondence between compact Lie groups and reductive complex algebraic groups fails: there exist many reductive quantum groups that do not admit a compact form. Banica [2] has described the representation theory of the universal compact quantum analogues of the unitary groups, which at the Hopf algebra level are exactly the universal cosovereign Hopf algebras that admit a compact form. We describe here the corepresentations in the generic (reductive) case, and therefore our results are generalizations of Banica's.

Let us describe the contents of the paper in more technical terms. Let k be a commutative field and let $F \in GL(n, k)$. The algebra $H(F)$ [6] is defined to be the universal algebra with generators $(u_{ij})_{1\leq i,j\leq n}$, $(v_{ij})_{1\leq i,j\leq n}$ and relations:

$$u{}^t v = {}^t v u = I_n \quad ; \quad vF{}^t uF^{-1} = F{}^t uF^{-1}v = I_n,$$

where $u = (u_{ij})$, $v = (v_{ij})$ and I_n is the identity $n \times n$ matrix. It turns out [6] that $H(F)$ is a Hopf algebra with comultiplication defined by $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ and $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$, with counit defined by $\varepsilon(u_{ij}) = \varepsilon(v_{ij}) = \delta_{ij}$ and with antipode defined by $S(u) = {}^t v$ and $S(v) = F{}^t uF^{-1}$. Furthermore, $H(F)$ is a cosovereign Hopf algebra [6]: there exists an algebra morphism $\Phi : H(F) \longrightarrow k$ such that $S^2 = \Phi * \mathrm{id} * \Phi^{-1}$. The Hopf algebras $H(F)$ have the following universal property ([6], Theorem 3.2).

Let A be a Hopf algebra and let V be a finite dimensional A-comodule isomorphic to its bidual comodule V^{**}. Then there exists a matrix $F \in GL(n, k)$ $(n = \dim V)$ such that V is an $H(F)$-comodule and such that there exists a Hopf algebra morphism $\pi : H(F) \longrightarrow A$ satisfying $(1_V \otimes \pi) \circ \beta_V = \alpha_V$, where $\alpha_V : V \longrightarrow V \otimes A$ and $\beta_V : V \longrightarrow V \otimes H(F)$ denote the coactions of A and $H(F)$ on V respectively. In particular, every cosovereign Hopf algebra of finite type is a homomorphic quotient of a Hopf algebra $H(F)$.

In view of this universal property, it is natural to say that the Hopf algebras $H(F)$ are the universal cosovereign Hopf algebras, or the free cosovereign Hopf algebras, and to see these Hopf algebras as natural analogues of the general linear groups in quantum group theory. Indeed the Hopf algebras $\mathcal{O}(GL(n))$ have exactly the same universal property when one works in the category of commutative Hopf algebras.

If $k = \mathbb{C}$ and if F is a positive matrix, the Hopf algebras $H(F)$ are nothing but the CQG algebras associated to the universal compact quantum groups introduced by Van Daele and Wang [16]. In this case the corepresentation theory has been worked out by Banica [2]: the simple corepresentations correspond to the elements of the free product $\mathbb{N} * \mathbb{N}$, and the fusion rules are described by an ingenious formula involving a new product $\odot$ on the free algebra on two generators. We generalize Banica's results to the case of an arbitrary generic matrix, over any algebraically closed field of characteristic zero. The main feature of our proof is that, thanks to Morita-like reduction techniques, we do not need any of the free probability techniques used in [2].

In order to state our main result, we need to introduce some notation and terminology.

• Let $F \in GL(n, k)$. We say that F is normalized if $\mathrm{tr}(F) = \mathrm{tr}(F^{-1})$. We say that F is normalizable if there exists $\lambda \in k^*$ such that $\mathrm{tr}(\lambda F) = \mathrm{tr}((\lambda F)^{-1})$. Over an algebraically closed field, any matrix is normalizable unless $\mathrm{tr}(F) = 0 \neq \mathrm{tr}(F^{-1})$ or $\mathrm{tr}(F) \neq 0 = \mathrm{tr}(F^{-1})$. We will only essentially consider normalized matrices F or, equivalently, normalizable matrices, since $H(\lambda F) = H(F)$.

• Let $q \in k^*$. As usual, we say that q is generic if q is not a root of unity of order $N \geq 3$. We say that a matrix $F \in GL(n, k)$ is generic if F is normalized and if the solutions of $q^2 - \mathrm{tr}(F)q + 1 = 0$ are generic.

• Let $q \in k^*$. We put $F_q = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} \in GL(2, k)$. The Hopf algebra $H(F_q)$ is denoted by $H(q)$.

• Let $F \in GL(n, k)$. The natural n-dimensional $H(F)$-comodules associated to the multiplicative matrices $u = (u_{ij})$ and $v = (v_{ij})$ are denoted by U and V, with $V = U^*$.

• We will consider the coproduct monoid $\mathbb{N} * \mathbb{N}$. Equivalently $\mathbb{N} * \mathbb{N}$ is the free monoid on two generators, which we denote, as in [2], by α and β. There is a unique antimultiplicative morphism $^{-} : \mathbb{N} * \mathbb{N} \longrightarrow \mathbb{N} * \mathbb{N}$ such that $\bar{e} = e$, $\bar{\alpha} = \beta$ and $\bar{\beta} = \alpha$ (e denotes the unit element of $\mathbb{N} * \mathbb{N}$).

We can now state our main result. Here k denotes an algebraically closed field.

Theorem 1.1 *Let $F \in GL(n, k)$ ($n \geq 2$) be a normalized matrix.*
a) Let $q \in k^*$ be such that $q^2 - \mathrm{tr}(F)q + 1 = 0$. The comodule categories over $H(F)$ and $H(q)$ are monoidally equivalent.

We assume now that k is a characteristic zero field.
b) The Hopf algebra $H(F)$ is cosemisimple if and only if F is a generic matrix.
c) Assume that F is generic. To any element $x \in \mathbb{N} * \mathbb{N}$ corresponds a simple $H(F)$-comodule U_x, with $U_e = k$, $U_\alpha = U$ and $U_\beta = V$. Any simple $H(F)$-comodule is isomorphic to one of the U_x, and $U_x \cong U_y$ if and only if $x = y$. For $x, y \in \mathbb{N} * \mathbb{N}$, we have $U_x^* \cong U_{\bar{x}}$ and

$$U_x \otimes U_y \cong \bigoplus_{\{a,b,g \in \mathbb{N} * \mathbb{N} | x = ag, y = \bar{g}b\}} U_{ab} \ .$$

It is clear from the statement that the proof of Theorem 1.1 is divided into two parts. The first part reduces the corepresentation theory of $H(F)$ to the one of $H(q)$. Then we realize $H(q)$ as a Hopf subalgebra of the free product $k[z, z^{-1}] * \mathcal{O}(SL_q(2))$, and we conclude using the classification of simple comodules of a free product of cosemisimple Hopf algebras [17], and Banica's product $\odot$ on the free algebra on two generators.

As an application of these results we get, in the generic case, the isomorphic classification of the universal cosovereign Hopf algebras and the computation of the automorphism group.

Another interesting class of universal Hopf algebras was constructed by Wang [18] in the compact quantum group framework: these are the quantum automorphism groups of finite-dimensional (measured) C^*-algebras. The corepresentation theory, similar to that of $SO(3)$, was described by Banica [3], for C^*-algebras endowed with (good) tracial measures. A special case of a general construction of [5] yields algebraic analogues of Wang's quantum automorphism groups. Using the previous results concerning $H(F)$, it is not difficult to describe the representation theory of the quantum automorphism group of a matrix algebra endowed with a non-necessarily tracial measure, reducing the computations to the case of the quantum group $SO_q(3)$.

This paper is organized as follows. In Section 2 we use the Hopf-Galois systems techniques of [8] to show that for a normalized matrix F, there exists $q \in k^*$ such that the comodule categories over $H(F)$ and $H(q)$ are monoidally equivalent. This section also includes results for non-normalizable matrices. In Section 3 we construct an injective algebra morphism from $H(q)$ into the free product algebra $k[z, z^{-1}] * \mathcal{O}(SL_q(2))$. In Section 4 we show that $H(q)$ is cosemisimple if and only if q is generic, and Section 5 contains the classification of the simple $H(q)$-comodules and their fusion rules in the generic case. Section 6 is devoted to some applications of Theorem 1.1: the Hopf algebras $H(F)$ are classified up to isomorphism and the automorphism group is described (in the generic case). Finally in Section 7 we use our previous results to describe the representation category of the quantum automorphism group of a matrix algebra endowed with a non-necessarily tracial measure.

2 Reduction to the two-dimensional case

This section is essentially devoted to prove part a) of Theorem 1.1. In fact we consider a more general situation and get results for non-normalizable matrices. We will use Hopf-Galois systems techniques [8]. We will not repeat here the definition a Hopf-Galois system, for which we refer to [8].

Let $E \in GL(m,k)$ and let $F \in GL(n,k)$. Recall [8] that the algebra $H(E,F)$ is the universal algebra with generators u_{ij}, v_{ij}, $1 \leq i \leq m, 1 \leq j \leq n$, and satisfying the relations

$$u\,{}^t\!v = I_m = vF\,{}^t\!uE^{-1} \quad ; \quad {}^t\!vu = I_n = F\,{}^t\!uE^{-1}v.$$

When $E = F$, we have $H(F,F) = H(F)$. In fact the Hopf algebra structure of $H(F)$ is just a particular case of the fact that if $H(E,F)$ is a non-zero algebra, then $(H(E), H(F), H(E,F), H(F,E))$ is a Hopf-Galois system (see Proposition 4.3 in [8]). Combining Proposition 4.3 and Corollary 1.4 in [8], we have the following result.

Proposition 2.1 *Assume that $H(E,F)$ is a non-zero algebra. Then the comodule categories over $H(E)$ and $H(F)$ are monoidally equivalent.* $\square$

So we have to study the algebras $H(E,F)$. It is not difficult to see that if $H(E,F) \neq \{0\}$, then $\mathrm{tr}(E) = \mathrm{tr}(F)$ and $\mathrm{tr}(E^{-1}) = \mathrm{tr}(F^{-1})$. The converse assertion will essentially follow from the next result, where some technical conditions are required.

Proposition 2.2 *Let $E \in GL(m,k)$ and let $F \in GL(n,k)$ $(m,n \geq 2)$. Assume that E is a diagonal matrix, that F is a lower-triangular matrix, that $\mathrm{tr}(E) = \mathrm{tr}(F)$ and $\mathrm{tr}(E^{-1}) = \mathrm{tr}(F^{-1})$. Then the elements (u_{ij}), $1 \leq i \leq m$, $1 \leq j \leq n$, generate a free subalgebra on mn generators. In particular $H(E,F)$ is a non zero-algebra.*

As in [7], we will use the diamond lemma [4]. Let us write down explicitly a presentation of $H(E,F)$: $H(E,F)$ is the universal algebra with generators u_{ij}, v_{ij}, $1 \leq i \leq m, 1 \leq j \leq n$, and relations:

$$u_{in}v_{jn} = \delta_{ij} - \sum_{k=1}^{n-1} u_{ik}v_{jk} \ , \quad 1 \le i,j \le m \tag{1}$$

$$v_{i1}u_{j1} = F_{11}^{-1}(E_{ij} - \sum_{k=2,l=1}^{n} F_{kl}v_{ik}u_{jl}) \ , \quad 1 \le i,j \le m \tag{2}$$

$$v_{1i}u_{1j} = \delta_{ij} - \sum_{k=2}^{m} v_{ki}u_{kj} \ , \quad 1 \le i,j \le n \tag{3}$$

$$u_{mi}v_{mj} = E_{mm}(F_{ij}^{-1} - \sum_{k=1}^{m-1} E_{kk}^{-1}u_{ki}v_{kj}) \ , \quad 1 \le i,j \le n. \tag{4}$$

We have a nice presentation to use the diamond lemma [4], of which we freely use the techniques and definitions. We only need the simplified exposition of [10]. We order the set of monomials in the following way. We order the set $\{1,\ldots,m\} \times \{1,\ldots,n\}$ lexicographically. Then we order the set $\{u_{ij}\}$ with the order induced by the preceding order, and we order the set $\{v_{ij}\}$ with the inverse order. We order the set $X = \{u_{ij}, v_{kl}\}$ in such a way that $v_{11} < u_{11}$. Finally two monomials are ordered according to their length, and two monomials of equal length are ordered lexicographically according to the order on the set X. It is clear that the order just defined is compatible with the above presentation.

Lemma 2.3 *There are exactly two inclusion ambiguities:* $(v_{11}u_{11}, v_{11}u_{11})$ *and* $(u_{mn}v_{mn}, u_{mn}v_{mn})$. *There are exactly the following overlap ambiguities.*

$$(u_{in}v_{1n}, v_{1n}u_{1j}), \quad (v_{i1}u_{m1}, u_{m1}v_{mj}), \quad 1 \le i \le m, \quad 1 \le j \le n.$$

$$(v_{1i}u_{1n}, u_{1n}v_{jn}), \quad (u_{mi}v_{m1}, v_{m1}u_{j1}), \quad 1 \le i \le n, \quad 1 \le j \le m.$$

All these ambiguities are resolvable.

Proof. It is easy to see that the ambiguities above are the only ones. Let us check that the first inclusion ambiguity is resolvable. As usual the symbol "→" means that we perform a reduction. We have:

$$F_{11}^{-1}(E_{11} - \sum_{k=2,l=1}^{n} F_{kl}v_{1k}u_{1l}) \to F_{11}^{-1}(E_{11} - \sum_{k=2,l=1}^{n} F_{kl}(\delta_{kl} - \sum_{r=2}^{m} v_{rk}u_{rl}))$$
$$= F_{11}^{-1}(E_{11} - \sum_{k=2}^{n} F_{kk} + \sum_{k=2,l=1}^{n} \sum_{r=2}^{m} F_{kl}v_{rk}u_{rl})$$

On the other hand we have:

$$1 - \sum_{k=2}^{m} v_{k1}u_{k1} \to 1 - \sum_{k=2}^{m} F_{11}^{-1}(E_{kk} - \sum_{l=2,r=1}^{n} F_{lr}v_{kl}u_{kr})$$
$$= F_{11}^{-1}(E_{11} - \sum_{k=2}^{n} F_{kk} + \sum_{k=2,l=1}^{n} \sum_{r=2}^{m} F_{kl}v_{rk}u_{rl})$$

We have used the identity $\mathrm{tr}(E) = \mathrm{tr}(F)$. Hence this inclusion ambiguity is resolvable. Also it is not difficult to check, using $\mathrm{tr}(E^{-1}) = \mathrm{tr}(F^{-1})$, that the other inclusion ambiguity is resolvable. This is left to the reader. Let us check that the first two families of overlap ambiguities are resolvable. The resolvability of the other two families will be left to the reader. First consider $(u_{in}v_{1n}, v_{1n}u_{1j}), 1 \le i \le m, 1 \le j \le n$. We have:

$$(\delta_{i1}-\sum_{k=1}^{n-1} u_{ik}v_{1k})u_{1j} \to \delta_{i1}u_{1j}-\sum_{k=1}^{n-1} u_{ik}(\delta_{kj}-\sum_{l=2}^{m} v_{lk}u_{lj}) = \delta_{i1}u_{1j}-(1-\delta_{jn})u_{ij}+\sum_{k=1}^{n-1}\sum_{l=2}^{m} u_{ik}v_{lk}u_{lj}.$$

On the other hand we have:

$$u_{in}(\delta_{nj} - \sum_{k=2}^{m} v_{kn}u_{kj}) \to \delta_{nj}u_{in} - \sum_{k=2}^{m}(\delta_{ki} - \sum_{l=1}^{n-1} u_{il}v_{kl})u_{kj} =$$

$$\delta_{nj}u_{in} - (1-\delta_{1i})u_{ij} + \sum_{k=2}^{m}\sum_{l=1}^{n-1} u_{il}v_{kl}u_{kj} = \delta_{i1}u_{1j} - (1-\delta_{jn})u_{ij} + \sum_{k=1}^{n-1}\sum_{l=2}^{m} u_{ik}v_{lk}u_{lj}.$$

Hence these ambiguities are resolvable. Let us now study the ambiguities $(v_{i1}u_{m1}, u_{m1}v_{mj})$, $1 \le i \le m, 1 \le j \le n$. We have:

$$F_{11}^{-1}(E_{im} - \sum_{k=2,l=1}^{n} F_{kl}v_{ik}u_{ml})v_{mj}$$

$$\to F_{11}^{-1}(E_{im}v_{mj} - \sum_{k=2,l=1}^{n} (F_{kl}v_{ik}(E_{mm}(F_{lj}^{-1} - \sum_{r=1}^{m-1} E_{rr}^{-1}u_{rl}v_{rj})))$$

$$= F_{11}^{-1}E_{mm}(\delta_{im}v_{mj} - (1-\delta_{j1})v_{ij} + \sum_{k=2,l=1}^{n}\sum_{r=1}^{m-1} F_{kl}E_{rr}^{-1}v_{ik}u_{rl}v_{rj}).$$

On the other hand we have:

$$v_{i1}(E_{mm}(F_{1j}^{-1} - \sum_{k=1}^{m-1} E_{kk}^{-1}u_{k1}v_{kj})$$

$$\to E_{mm}(\delta_{1j}F_{11}^{-1}v_{i1} - \sum_{k=1}^{m-1} E_{kk}^{-1}(F_{11}^{-1}(\delta_{ik}E_{ii} - \sum_{l=2,r=1}^{n} F_{lr}v_{il}u_{kr}))v_{kj})$$

$$= E_{mm}F_{11}^{-1}(\delta_{1j}v_{i1} - (1-\delta_{im})v_{ij} + \sum_{k=1}^{m-1}\sum_{l=2,r=1}^{n-1} E_{kk}^{-1}F_{lr}v_{il}u_{kr}v_{kj})$$

$$= F_{11}^{-1}E_{mm}(\delta_{im}v_{mj} - (1-\delta_{j1})v_{ij} + \sum_{k=2,l=1}^{n}\sum_{r=1}^{m-1} F_{kl}E_{rr}^{-1}v_{ik}u_{rl}v_{rj}).$$

Hence these ambiguities are resolvable. □

Proof of Proposition 2.2. Since our order is compatible with the presentation, and since all the ambiguities are resolvable, we can use the diamond lemma [4]: the reduced monomials form a basis of $H(E,F)$, and in particular the monomials in elements of the set $\{u_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\}$ are linearly independent, and hence the elements of this set generate a free subalgebra on mn generators. In particular $H(E,F)$ is a non-zero algebra. $\square$

We can now easily prove the following slightly more general result.

Proposition 2.4 *Let $E \in GL(m,k)$ and let $F \in GL(n,k)$ $(m,n \geq 2)$. Assume that $\mathrm{tr}(E) = \mathrm{tr}(F)$ and $\mathrm{tr}(E^{-1}) = \mathrm{tr}(F^{-1})$. Then $H(E,F)$ is a non-zero algebra.*

Proof. Since we want to prove that $H(E,F)$ is a non-zero algebra, we can assume that k is algebraically closed. For matrices $P \in GL(m,k)$ and let $Q \in GL(n,k)$, the algebras $H(E,F)$ and $H(PEP^{-1}, QFQ^{-1})$ are isomorphic ([8], Proposition 4.2), thus we can assume that the matrices E and F are lower-triangular. Consider $G \in GL(m,k)$ a diagonal matrix such that $\mathrm{tr}(G) = \mathrm{tr}(E) = \mathrm{tr}(F)$ and $\mathrm{tr}(G^{-1}) = \mathrm{tr}(E^{-1}) = \mathrm{tr}(F^{-1})$. By the proof of Proposition 4.3 in [8], there exists an algebra morphism $\delta : H(E,F) \longrightarrow H(E,G) \otimes H(G,F)$ such that $\delta(u_{ij}) = \sum_{k=1}^{m} u_{ik} \otimes u_{kj}$. Also there exists an algebra morphism $\phi : H(E,G) \longrightarrow H(G,E)^{\mathrm{op}}$ such that $\phi(u) = {}^t v$. Thus we have an algebra morphism $\delta' : H(E,F) \longrightarrow H(G,E)^{\mathrm{op}} \otimes H(G,F)$ such that $\delta'(u_{ij}) = \sum_{k=1}^{m} v_{ki} \otimes u_{kj}$. By the proof of Proposition 2.2, the elements (v_{ij}), (u_{ij}) are linearly independent elements of $H(G,E)$ and $H(G,F)$ respectively. Hence it is clear that $H(E,F)$ is a non-zero algebra. $\square$

Combining Propositions 2.1 and 2.4, we can state the main result of the section, which contains part a) of Theorem 1.1. Recall that for $q \in k^*$, we put $H(q) = H(F_q)$ where $F_q = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} \in GL(2,k)$. Note also that if k is algebraically closed, any $F \in GL(2,k)$ is normalizable.

Corollary 2.5 *Let $F \in GL(n,k)$ $(n \geq 2)$ and assume that k is algebraically closed.*
a) Assume that F is normalizable. Then there exists $q \in k^*$ such that we have an equivalence of monoidal categories:

$$\mathrm{Comod}(H(F)) \cong^{\otimes} \mathrm{Comod}(H(q)).$$

If F is normalized, we take q as a solution of the equation $q^2 - \mathrm{tr}(F)q + 1 = 0$.
b) Assume that F is not normalizable. Let $E \in GL(3,k)$ be any matrix such that $\mathrm{tr}(E) = 0$ and $\mathrm{tr}(E^{-1}) \neq 0$. Then we have an equivalence of monoidal categories:

$$\mathrm{Comod}(H(F)) \cong^{\otimes} \mathrm{Comod}(H(E)).$$

Proof. a) Let $\lambda \in k^*$ be such that $\mathrm{tr}(\lambda F) = \mathrm{tr}((\lambda F)^{-1})$, and let $q \in k^*$ be a solution of $q^2 - \mathrm{tr}(\lambda F)q + 1 = 0$. This equation is equivalent to $\mathrm{tr}(F_q^{-1}) = \mathrm{tr}(F_q) = q + q^{-1} = \mathrm{tr}(\lambda F) = \mathrm{tr}((\lambda F)^{-1}))$. By Proposition 2.4, $H(F_q, F)$ is a non-zero algebra, and we conclude using Proposition 2.1.

b) Since F is not normalizable and since the base field is algebraically closed, we have $\mathrm{tr}(F) = 0 \neq \mathrm{tr}(F^{-1})$ or $\mathrm{tr}(F) \neq 0 = \mathrm{tr}(F^{-1})$. Since the Hopf algebras $H(F)$ and $H({}^tF^{-1})$ are isomorphic ([6], Proposition 3.3), we can assume that $\mathrm{tr}(F) = 0 \neq \mathrm{tr}(F^{-1})$. Since k is algebraically closed, there always exists $E \in GL(3,k)$ satisfying $\mathrm{tr}(E) = 0$ and $\mathrm{tr}(E^{-1}) \neq 0$, and we conclude as in part a). □

Recall that the fundamental n-dimensional comodule of $H(F)$ associated to the multiplicative matrix (u_{ij}) is denoted by U. The following result reflects the “freeness” of $H(F)$.

Corollary 2.6 *Let $F \in GL(n,k)$. The comodules $U^{\otimes k}$, $k \in \mathbb{N}$, are simple non-equivalent $H(F)$-comodules.*

Proof. We can assume that k is algebraically closed. If $n = 1$ then $H(F)$ is just the algebra of Laurent polynomials $k[z, z^{-1}]$, so the result is immediate. Assume now that $n \geq 2$. First assume that F is a diagonal matrix. By Proposition 2.2 the monomials in the elements u_{ij} form a linearly independent subset of $H(F)$, and hence the comodules $U^{\otimes k}$, $k \in \mathbb{N}$, are simple non-equivalent $H(F)$-comodules. Now assume that F is a lower-triangular matrix. Take $E \in GL(n,k)$ a diagonal matrix such that $\mathrm{tr}(E) = \mathrm{tr}(F)$ and $\mathrm{tr}(E^{-1}) = \mathrm{tr}(F^{-1})$. The monoidal category equivalence of Proposition 2.1 transforms the $H(F)$-comodule U into the $H(E)$-comodule U (see [15, 14, 8] for the construction). Hence we conclude by the diagonal case. This finishes the proof since the Hopf algebras $H(PFP^{-1})$ and $H(F)$ are isomorphic for $P \in GL(m,k)$ ([6]). □

Corollary 2.7 *Let $F \in GL(n,k)$ be a non-normalizable matrix. Then the Hopf algebra $H(F)$ is not cosemisimple.*

Proof. By the preceding corollary U is a simple $H(F)$-comodule. We can assume that k is algebraically closed. If $H(F)$ was cosemisimple, and since ${}^tF^{-1}$ is an intertwiner between U and U^{**} (see the Proof of Theorem 3.2 in [6]), then we would have by ([10], Proposition 15, chapter 11, or the original reference [12]) $\mathrm{tr}(F) \neq 0$ and $\mathrm{tr}(F^{-1}) \neq 0$, which would contradict our assumption. □

3 The algebra $H(q)$

This section is devoted to the construction of an algebra embedding of $H(q) = H(F_q)$ into $k[z, z^{-1}] * \mathcal{O}(SL_q(2))$. This embedding will be used later to study the corepresentation theory of $H(q)$.

Let $q \in k^*$. The algebra $H(q)$ has 8 generators. We put $\alpha = u_{11}$, $\beta = u_{12}$, $\gamma = u_{21}$, $\delta = u_{22}$, $\alpha^* = v_{11}$, $\beta^* = v_{12}$, $\gamma^* = v_{21}$, $\delta^* = v_{22}$. Let us rewrite the presentation of $H(q)$: it is the universal algebra with generators α, β, γ, δ, α^*, β^*, γ^*, δ^* and satisfying the relations:

$$\begin{cases} \beta\beta^* = 1 - \alpha\alpha^* \\ \beta\delta^* = -\alpha\gamma^* \\ \delta\beta^* = -\gamma\alpha^* \\ \delta\delta^* = 1 - \gamma\gamma^* \end{cases} \quad \begin{cases} \alpha^*\alpha = 1 - q^2\beta^*\beta \\ \alpha^*\gamma = -q^2\beta^*\delta \\ \gamma^*\alpha = -q^2\delta^*\beta \\ \gamma^*\gamma = q^2(1 - \delta^*\delta) \end{cases} \quad \begin{cases} \alpha^*\alpha = 1 - \gamma^*\gamma \\ \alpha^*\beta = -\gamma^*\delta \\ \beta^*\alpha = -\delta^*\gamma \\ \beta^*\beta = 1 - \delta^*\delta \end{cases} \quad \begin{cases} \gamma\gamma^* = q^2(1 - \alpha\alpha^*) \\ \gamma\delta^* = -q^2\alpha\beta^* \\ \delta\gamma^* = -q^2\beta\alpha^* \\ \delta\delta^* = 1 - q^2\beta\beta^* \end{cases}$$

Note that the fourth relation of the first family and that the first relation of the second family are redundant. We have left these redundant relations in order to use the results of Section 2, where some redundant relations were also present.

We define now an algebra extension of $H(q)$, which will be denoted by $H^+(q)$. This algebra will be shown to be isomorphic with $k[z, z^{-1}] * \mathcal{O}(SL_q(2))$.

Definition 3.1 *The algebra $H^+(q)$ is the universal algebra with generators α, β, γ, δ, α^*, β^*, γ^*, δ^*, t, t^{-1}, and satisfying the relations of $H(q)$ and:*

$$tt^{-1} = 1 = t^{-1}t \ ; \ t^{-1}\alpha = \delta^* t \ ; \ t^{-1}\beta = -q^{-1}\gamma^* t \ ; \ t^{-1}\gamma = -q\beta^* t \ ; \ t^{-1}\delta = \alpha^* t.$$

There is an obvious algebra morphism $H(q) \longrightarrow H^+(q)$.

Lemma 3.2 *The natural algebra morphism $H(q) \longrightarrow H^+(q)$ is injective.*

Proof. We will use again the diamond lemma, since we have not been able to find a more direct way to prove our lemma. First we order the set $\{\alpha, \beta, \gamma, \delta, \alpha^*, \beta^*, \gamma^*, \delta^*, t, t^{-1}\}$ in the following way:

$$\delta^* < \gamma^* < \beta^* < \alpha^* < \alpha < \beta < \gamma < \delta < t^{-1} < t.$$

Two monomials of different length are ordered according to their length and two monomials of equal length are ordered lexicographically according to the above order. In order to resolve some ambiguities, let us rewrite the presentation of $H^+(q)$: $H^+(q)$ is the universal algebra with generators α, β, γ, δ, α^*, β^*, γ^*, δ^*, t, t^{-1}, and satisfying the relations of $H(q)$ and

$$tt^{-1} = t^{-1}t \ ; \ t^{-1}t = 1 \ ; \ t^{-1}\alpha = \delta^* t \ ; \ t\delta^* = \alpha t^{-1} \ ; \ t^{-1}\beta = -q^{-1}\gamma^* t \ ; \ t\gamma^* = -q\beta t^{-1} \ ;$$
$$t^{-1}\gamma = -q\beta^* t \ ; \ t\beta^* = -q^{-1}\gamma t^{-1} \ ; \ t^{-1}\delta = \alpha^* t \ ; \ t\alpha^* = \delta t^{-1}.$$

It is clear the order just defined is compatible with this presentation. There are the ambiguities of Lemma 2.3, which were shown to be resolvable there, there are no other inclusion ambiguities and the following overlap ambiguities:

$$\begin{array}{l}
(tt^{-1}, t^{-1}\alpha) \ ; \ (tt^{-1}, t^{-1}\beta) \ ; \ (tt^{-1}, t^{-1}\gamma) \ ; \ (tt^{-1}, t^{-1}\delta) \ ; \\
(t^{-1}t, t\delta^*) \ ; \ (t^{-1}t, t\gamma^*) \ ; \ (t^{-1}t, t\beta^*) \ ; \ (t^{-1}t, t\alpha^*) \ ; \\
(t^{-1}\beta, \beta\beta^*) \ ; \ (t^{-1}\beta, \beta\delta^*) \ ; \ (t\gamma^*, \gamma^*\alpha) \ ; \ (t\gamma^*, \gamma^*\gamma) \ ; \\
(t^{-1}\gamma, \gamma\gamma^*) \ ; \ (t^{-1}\gamma, \gamma\delta^*) \ ; \ (t\beta^*, \beta^*\alpha) \ ; \ (t\beta^*, \beta^*\beta) \ ; \\
(t^{-1}\delta, \delta\beta^*) \ ; \ (t^{-1}\delta, \delta\delta^*) \ ; \ (t^{-1}\delta, \delta\gamma^*) \ ; \ (t^{-1}\delta, \delta\delta^*) \ ; \\
(t\alpha^*, \alpha^*\alpha) \ ; \ (t\alpha^*, \alpha^*\gamma) \ ; \ (t\alpha^*, \alpha^*\alpha) \ ; \ (t\alpha^*, \alpha^*\beta).
\end{array}$$

These ambiguities are easily seen to be resolvable: this is left to the reader. Hence by the diamond lemma the reduced monomials form a basis of $H^+(q)$. It is clear that the reduced monomials of $H(q)$ (for the reductions of Section 2) are still reduced monomials in $H^+(q)$, and hence the images under $H(q) \to H^+(q)$ of the elements of a basis of $H(q)$ are still linealy independant elements, which proves that our algebra map is injective. $\square$

Recall that $\mathcal{O}(SL_q(2))$ is the universal algebra with generators a, b, c, d and relations

$$ba = qab \ ; \ ca = qac \ ; \ db = qbd \ ; \ dc = qcd \ ; \ cb = bc = q(ad-1) \ ; da = qbc + 1.$$

The algebra just defined is $\mathcal{O}(SL_{q^{-1}}(2))$ in [10]. Our convention does not change the resulting Hopf algebra, up to isomorphism. Now consider the free product $k[z, z^{-1}] * \mathcal{O}(SL_q(2))$, that is the coproduct of $k[z, z^{-1}]$ and of $\mathcal{O}(SL_q(2))$ in the category of unital algebras. We have the following result:

Lemma 3.3 *There exists a unique algebra isomorphism* $\tilde{\pi} : H^+(q) \to k[z, z^{-1}] * \mathcal{O}(SL_q(2))$ *such that*

$$\tilde{\pi}(\alpha) = za, \ \tilde{\pi}(\beta) = zb, \ \tilde{\pi}(\gamma) = zc, \ \tilde{\pi}(\delta) = zd, \ \tilde{\pi}(\alpha^*) = dz^{-1}, \ \tilde{\pi}(\beta^*) = -q^{-1}cz^{-1},$$
$$\tilde{\pi}(\gamma^*) = -qbz^{-1}, \ \tilde{\pi}(\delta^*) = az^{-1}, \ \tilde{\pi}(t) = z, \ \tilde{\pi}(t^{-1}) = z^{-1}.$$

Proof. It is a direct verification to check the existence of the algebra morphism $\tilde{\pi}$. Let us construct an inverse isomorphism. First there is an algebra morphism $\rho_1 : k[z, z^{-1}] \longrightarrow H^+(q)$ defined by $\rho_1(z) = t$. It is also a direct verification to check the existence of an algebra morphism $\rho_2 : \mathcal{O}(SL_q(2)) \longrightarrow H^+(q)$ such that

$$\rho_2(a) = t^{-1}\alpha = \delta^* t, \ \rho_2(b) = t^{-1}\beta = -q^{-1}\gamma^* t, \ \rho_2(c) = t^{-1}\gamma = -q\beta^* t, \ \rho_2(d) = t^{-1}\delta = \alpha^* t.$$

Using the universal property of the free product, we have a unique algebra morphism $\rho : k[z, z^{-1}] * \mathcal{O}(SL_q(2)) \longrightarrow H^+(q)$ extending ρ_1 and ρ_2. It is straightforward to check that $\tilde{\pi}$ and ρ are mutually inverse isomorphisms. $\square$

We arrive at the main result of the section.

Proposition 3.4 *There exists an injective algebra morphism* $\pi : H(q) \longrightarrow k[z, z^{-1}] * \mathcal{O}(SL_q(2))$ *such that*

$$\pi(\alpha) = za \ , \ \pi(\beta) = zb \ , \ \pi(\gamma) = zc \ , \ \pi(\delta) = zd \ ,$$
$$\pi(\alpha^*) = dz^{-1}, \ \pi(\beta^*) = -q^{-1}cz^{-1}, \pi(\gamma^*) = -qbz^{-1}, \ \pi(\delta^*) = az^{-1}.$$

Proof. The algebra morphism announced is just the composition of the injective algebra morphisms of Lemmas 3.2 and 3.3, so is itself injective $\square$

4 Cosemisimplicity of $H(q)$

In this section, where k is assumed to be an algebraically closed field of characteristic zero, we show that $H(q)$ is cosemisimple if and only if q is generic.

First let us recall that if A and B are Hopf algebras, their free product may be endowed with a natural Hopf algebra structure, induced by the Hopf algebras structures of A and B. For example $k[z, z^{-1}] * \mathcal{O}(SL_q(2))$ is a Hopf algebra, and by a straightforward verification, we have the following result.

Proposition 4.1 *The injective algebra morphism $\pi : H(q) \longrightarrow k[z, z^{-1}] * \mathcal{O}(SL_q(2))$ is a Hopf algebra morphism.* □

Wang [17] has studied free products of Hopf algebras at the compact quantum group level. His results may be adapted to arbitrary cosemisimple Hopf algebras without difficulties. Let us recall the main results. In the following A and B denote cosemisimple Hopf algebras.

• The Hopf algebra $A * B$ is still cosemisimple. This may be shown as follows. Consider the Haar functionals (see e.g. [10]) h_A and h_B on A and B respectively, and form their free product $h_A * h_B$ as in [1], Proposition 1.1. Then $h_A * h_B$ is a Haar functional on $A * B$ (see [17], Theorem 3.8) and thus $A * B$ is a cosemisimple Hopf algebra

• An $A * B$-comodule is said to be a simple alternated $A * B$-comodule if it has the form $V_1 \otimes \ldots \otimes V_n$, where each V_i is a simple non-trivial A-comodule or B-comodule, and if V_i is an A-comodule, then V_{i+1} is a B-comodule, and conversely. A simple alternated $A * B$-comodule is a simple $A * B$-comodule, and every non-trivial simple $A * B$-comodule is isomorphic with a simple alternated $A * B$-comodule (see [17], Theorem 3.10).

• Let V and W be simple alternated $A * B$-comodules. Assume that V ends by an A-comodule and that W begins by a B-comodule. Then $V \otimes W$ is decomposed into a direct sum of simple alternated comodules according to the decomposition of tensor products of A-comodules. The same thing holds for B.

We will use these results to prove the following fact.

Proposition 4.2 *Let $q \in k^*$. Then $H(q)$ is cosemisimple if and only if q is generic.*

Proof. We will use the following well-known fact. Let $A \subset B$ be a Hopf algebra inclusion. Then an A-comodule is semisimple if and only if it is semisimple as a B-comodule. In particular if B is cosemisimple, so is A. First assume that q is generic. Then it is well-known that $\mathcal{O}(SL_q(2))$ is cosemisimple (see e.g. [10]), and since $k[z, z^{-1}]$ is also cosemisimple, we have that $k[z, z^{-1}] * \mathcal{O}(SL_q(2))$ is cosemisimple, and so is $H(q)$ by Proposition 4.1.

Let us now assume that q is a root of unity of order $N \geq 3$. We will construct a non semisimple $H(q)$-comodule. Put $N_0 = N/2$ if N is even and $N_0 = N$ if N is odd. Let V_1 be the fundamental two-dimensional $\mathcal{O}(SL_q(2))$-comodule. One can deduce from the results of [11] that $V_1^{\otimes N_0}$ is not a semisimple $\mathcal{O}(SL_q(2))$-comodule. For $i \in \mathbb{Z}$ we denote by Z^i the one-dimensional comodule associated to the group-like element z^i of $k[z, z^{-1}]$. Using π, we view $H(q)$ as a Hopf subalgebra of $k[z, z^{-1}] * \mathcal{O}(SL_q(2))$ and by the construction of π, $Z \otimes V_1$ and $V_1 \otimes Z^{-1}$ are $H(q)$-comodules. Then $V_1^{\otimes 2} = V_1 \otimes Z^{-1} \otimes Z \otimes V_1$ is an $H(q)$-comodule. Assume that N_0 is even: $N_0 = 2k$. Then $V_1^{\otimes N_0} = V_1^{\otimes 2k}$ is an $H(q)$-comodule. Since $V_1^{\otimes N_0}$ is not a semisimple $\mathcal{O}(SL_q(2))$-comodule, it is not a semisimple $k[z, z^{-1}] * \mathcal{O}(SL_q(2))$-comodule, and so is not a semisimple $H(q)$-comodule. Assume now that N_0 is odd: $N_0 = 2k + 1$. We have seen that $V_1^{\otimes 2k}$ is an $H(q)$-comodule, and hence $Z \otimes V_1^{\otimes N_0} = Z \otimes V_1 \otimes V_1^{\otimes 2k}$ is also an $H(q)$-comodule. If $Z \otimes V_1^{\otimes N_0}$ was a semisimple $H(q)$-comodule, it would be a semisimple $k[z, z^{-1}] * \mathcal{O}(SL_q(2))$-comodule, and $V_1^{\otimes N_0} = Z^{-1} \otimes Z \otimes V_1^{\otimes N_0}$ would be a semisimple $k[z, z^{-1}] * \mathcal{O}(SL_q(2))$-comodule, and hence a semisimple $\mathcal{O}(SL_q(2))$-comodule. Thus $Z \otimes V_1^{\otimes N_0}$ is not a semisimple $H(q)$-comodule: this concludes our proof. $\square$

Proposition 4.2, combined with part a) of Theorem 1.1, proves part b) of Theorem 1.1.

5 Corepresentations of $H(q)$, q generic

In this section k is still an algebraically closed field of characteristic zero, and $q \in k^*$ is generic. We describe the simple $H(q)$-comodules and their fusion rules, thereby completing the proof of Theorem 1.1.

Let us begin with some preliminaries. We consider the monoid $\mathbb{N} * \mathbb{N}$, the free product (=coproduct) of two copies of the monoid $\mathbb{N}$. Equivalently $\mathbb{N} * \mathbb{N}$ is the free monoid on two generators α and β (this should not cause any confusion with the elements α and β of $H(q)$). There is a unique antimultiplicative morphism $^- : \mathbb{N} * \mathbb{N} \longrightarrow \mathbb{N} * \mathbb{N}$ such that $\bar{e} = e$, $\bar{\alpha} = \beta$ and $\bar{\beta} = \alpha$ (e denotes the unit element of $\mathbb{N} * \mathbb{N}$). Let $k[\mathbb{N} * \mathbb{N}]$ be the monoid algebra of $\mathbb{N} * \mathbb{N}$: $k[\mathbb{N} * \mathbb{N}]$ is also the free algebra on two generators. Banica [2] has introduced a new product $\odot$ on $k[\mathbb{N} * \mathbb{N}]$. The following lemma is Lemma 3 in [2], where the proof can be found.

Lemma 5.1 *Consider the map* $\odot : \mathbb{N} * \mathbb{N} \times \mathbb{N} * \mathbb{N} \longrightarrow k[\mathbb{N} * \mathbb{N}]$ *defined by*

$$x \odot y = \sum_{x = ag, y = \bar{g}b} ab \ , \ x, y \in \mathbb{N} * \mathbb{N} \ ,$$

and extend $\odot$ *to* $k[\mathbb{N} * \mathbb{N}]$ *by bilinearity. Then* $(k[\mathbb{N} * \mathbb{N}], +, \odot)$ *is an associative* k*-algebra, with* e *as unit element. Furthermore* $(k[\mathbb{N} * \mathbb{N}], +, \odot)$ *is still the free algebra on two generators: if* B *is any algebra and* $u, v \in B$*, there exists a unique algebra morphism* $\psi : (k[\mathbb{N} * \mathbb{N}], +, \odot) \longrightarrow B$ *such that* $\psi(\alpha) = u$ *and* $\psi(\beta) = v$. $\square$

We will need some character theory. Let A be a Hopf algebra and let V be a finite-dimensional A-comodule with corresponding coalgebra map $\Phi_V : V^* \otimes V \longrightarrow A$. Recall (see e.g. [10]) that the character of V is defined to be $\chi_V := \Phi_V(\mathrm{id}_V)$. If V and W are finite-dimensional A-comodules, then $\chi(V \oplus W) = \chi(V) + \chi(W)$, $\chi(V \otimes W) = \chi(V)\chi(W)$ and $V \cong W \iff \chi(V) = \chi(W)$.

Recall [11, 10] that $\mathcal{O}(SL_q(2))$ is cosemisimple and has a complete family of simple comodules $(V_i)_{i \in \mathbb{N}}$, with $V_0 = k$ and $\dim(V_i) = i + 1$, for $i \in \mathbb{N}$, and

$$V_i \otimes V_1 \cong V_1 \otimes V_i \cong V_{i-1} \oplus V_{i+1}, \text{ for } i \in \mathbb{N}^*.$$

As in the preceding section, for $i \in \mathbb{Z}$, we denote by Z^i the one-dimensional comodule corresponding to the element z^i of $k[z, z^{-1}]$. We identify $H(q)$ with a Hopf subalgebra of $k[z, z^{-1}] * \mathcal{O}(SL_q(2))$, via the morphism π of Propositions 3.4 and 4.1. Under this identification, the canonical two-dimensional comodules U and V of $H(q)$ (see the notation in Section 1) correspond to the simple alternated comodules $Z \otimes V_1$ and $V_1 \otimes Z^{-1}$.

Proposition 5.2 *There exists a unique algebra morphism* $\psi : (k[\mathbb{N} * \mathbb{N}], +, \odot) \longrightarrow H(q)$ *such that* $\psi(\alpha) = \chi(Z \otimes V_1)$ *and* $\psi(\beta) = \chi(V_1 \otimes Z^{-1})$. *Moreover for all* $x \in \mathbb{N} * \mathbb{N}$, $\psi(x)$ *is the character of a simple* $H(q)$*-comodule.*

The first assertion is a direct consequence of Lemma 5.1. To prove the second one, we need a couple of lemmas.

Lemma 5.3 *For all* $n \in \mathbb{N}$, *we have:*

$$\psi((\alpha\beta)^n) = \chi(Z \otimes V_{2n} \otimes Z^{-1}) \quad ; \quad \psi((\beta\alpha)^n) = \chi(V_{2n}) \quad ;$$

$$\psi((\alpha\beta)^n\alpha) = \chi(Z \otimes V_{2n+1}) \quad ; \quad \psi((\beta\alpha)^n\beta) = \chi(V_{2n+1} \otimes Z^{-1}).$$

Proof. We prove the lemma by induction on n. For $n = 0$, the result is clear. Now assume that the lemma has been proved for $n \geq 0$. We have $(\alpha\beta)^n\alpha \odot \beta = (\alpha\beta)^{n+1} + (\alpha\beta)^n$, and so

$$\begin{aligned}
\psi((\alpha\beta)^{n+1}) &= \psi((\alpha\beta)^n\alpha)\psi(\beta) - \psi((\alpha\beta)^n) \\
&= \chi(Z \otimes V_{2n+1})\chi(V_1 \otimes Z^{-1}) - \chi(Z \otimes V_{2n} \otimes Z^{-1}) \text{ (by induction)} \\
&= \chi(Z \otimes V_{2n} \otimes Z^{-1}) + \chi(Z \otimes V_{2n+2} \otimes Z^{-1}) - \chi(Z \otimes V_{2n} \otimes Z^{-1}) \\
&= \chi(Z \otimes V_{2(n+1)} \otimes Z^{-1}).
\end{aligned}$$

Using $(\beta\alpha)^n\beta \odot \alpha = (\beta\alpha)^{n+1} + (\beta\alpha)^n$, one shows in the same way that $\psi((\beta\alpha)^{n+1}) = \chi(V_{2(n+1)})$. We have $(\alpha\beta)^{n+1} \odot \alpha = (\alpha\beta)^{n+1}\alpha + (\alpha\beta)^n\alpha$, and hence

$$\psi((\alpha\beta)^{n+1}\alpha) = \psi((\alpha\beta)^{n+1})\psi(\alpha) - \psi((\alpha\beta)^n\alpha).$$

We have already shown that $\psi((\alpha\beta)^{n+1}) = \chi(Z \otimes V_{2(n+1)} \otimes Z^{-1})$, and by induction $\psi((\alpha\beta)^n\alpha) = \chi(Z \otimes V_{2n+1})$, so we have:

$$\begin{aligned}
\psi((\alpha\beta)^{n+1}\alpha) &= \chi(Z \otimes V_{2n+2} \otimes Z^{-1} \otimes Z \otimes V_1) - \chi(Z \otimes V_{2n+1}) \\
&= \chi(Z \otimes V_{2n+1}) + \chi(Z \otimes V_{2n+3}) - \chi(Z \otimes V_{2n+1}) = \chi(Z \otimes V_{2(n+1)+1}).
\end{aligned}$$

One shows in a similar manner that $\psi((\beta\alpha)^{n+1}\beta) = \chi(V_{2(n+1)+1} \otimes Z^{-1})$: this concludes the proof. $\square$

Lemma 5.4 *Let* $x \in \mathbb{N} * \mathbb{N}$. *Then:*

- $\psi(x\alpha) = \chi(X \otimes V_i)$, *for some* $i \in \mathbb{N}^*$, *where* $X = k$ *or* X *is a simple alternated comodule ending by* Z *or* Z^{-1}.
- $\psi(\alpha x) = \chi(Z \otimes X)$, *where* X *is a simple alternated comodule beginning by some* V_i, $i \in \mathbb{N}^*$.
- $\psi(x\beta) = \chi(X \otimes Z^{-1})$, *where* X *is a simple alternated comodule ending by some* V_i, $i \in \mathbb{N}^*$.
- $\psi(\beta x) = \chi(V_i \otimes X)$, *for some* $i \in \mathbb{N}^*$, *where* $X = k$ *or* X *is a simple alternated comodule beginning by* Z *or* Z^{-1}.

Proof. We first prove the lemma for elements x as in Lemma 5.3. Let $x = (\alpha\beta)^n$. Then using Lemma 5.3, we have

$$\psi(x\alpha) = \psi((\alpha\beta)^n\alpha) = \chi(Z \otimes V_{2n+1}) \ ,$$
$$\psi(\alpha x) = \psi(\alpha(\alpha\beta)^n) = \psi(\alpha \odot (\alpha\beta)^n)) = \psi(\alpha)\psi((\alpha\beta)^n) =$$
$$\chi(Z \otimes V_1)\chi(Z \otimes V_{2n} \otimes Z^{-1}) = \chi(Z \otimes V_1 \otimes Z \otimes V_{2n} \otimes Z^{-1}) \ ,$$
$$\psi(x\beta) = \psi((\alpha\beta)^n\beta) = \chi(Z \otimes V_{2n} \otimes Z^{-1})\chi(V_1 \otimes Z^{-1}) = \chi(Z \otimes V_{2n} \otimes Z^{-1} \otimes V_1 \otimes Z^{-1}) \ ,$$
$$\psi(\beta x) = \psi(\beta(\alpha\beta)^n) = \psi((\beta\alpha)^n\beta) = \chi(V_{2n+1} \otimes Z^{-1}) \ .$$

Similar computations show that the lemma is true for $x = (\beta\alpha)^n$, $x = (\alpha\beta)^n\alpha$ or $x = (\beta\alpha)^n\beta$.

We now prove the lemma for an arbitrary element $x \in \mathbb{N} * \mathbb{N}$ using an induction on the length n of x. If $n = 0$, the result is obviously true. Let us assume that the lemma has been proved for elements of length $\leq n$ ($n \geq 0$), and let x be an element of length $n+1$. If x is one of the elements of Lemma 5.3, the result has already been proved so we can assume that $x = y\alpha^2 z$ or that $x = y\beta^2 z$. For example assume that $x = y\alpha^2 z$. We have

$$\psi(x\alpha) = \psi(y\alpha^2 z\alpha) = \psi(y\alpha \odot \alpha z\alpha) = \psi(y\alpha)\psi(\alpha z\alpha).$$

By induction, we have $\psi(y\alpha) = X \otimes V_i$ for $i \in \mathbb{N}^*$ and $X = k$ or X is a simple alternated $k[z, z^{-1}] * \mathcal{O}(SL_q(2))$-comodule ending by Z or Z^{-1}. Also by induction $\psi(\alpha z\alpha) = \chi(Z \otimes Y \otimes V_j)$ for $j \in \mathbb{N}^*$, and $Y = k$ or Y is a simple alternated comodule ending by Z or Z^{-1} and beginning by some V_k, $k \in \mathbb{N}^*$. So finally $\psi(x\alpha) = \chi(X \otimes V_i \otimes Z \otimes Y \otimes V_j)$ and $X \otimes V_i \otimes Z \otimes Y$ is a simple alternated comodule ending by Z or Z^{-1}. We also have

$$\psi(\alpha x) = \psi(\alpha y\alpha^2 z) = \psi(\alpha y\alpha \odot z\alpha) = \psi(\alpha y\alpha)\psi(\alpha z).$$

By induction we have $\psi(\alpha y\alpha) = \chi(Z \otimes X \otimes V_i)$, $i \in \mathbb{N}^*$, and $X = k$ or X is a simple alternated comodule beginning by some V_j, $j \in \mathbb{N}^*$ and ending by Z or Z^{-1}. Also $\psi(\alpha z) = \chi(Z \otimes Y)$ where Y is a simple alternated comodule beginning by some V_k, $k \in \mathbb{N}^*$. Hence $\psi(\alpha x) = \chi(Z \otimes X \otimes V_i \otimes Z \otimes Y)$, where $X \otimes V_i \otimes Z \otimes Y$ is a simple alternated comodule beginning by some V_j, $j \in \mathbb{N}^*$. Let us now compute $\psi(x\beta)$:

$$\psi(x\beta) = \psi(y\alpha^2 z\beta) = \psi(y\alpha \odot \alpha z\beta) = \psi(y\alpha)\psi(\alpha z\beta).$$

By induction $\psi(y\alpha) = \chi(X \otimes V_i)$ where $X = k$ or X is a simple alternated comodule ending by Z or Z^{-1}. Also $\psi(\alpha z\beta) = \chi(Z \otimes Y \otimes Z^{-1})$ where Y is a simple alternated comodule beginning by some V_j and ending by some V_k, $j, k \in \mathbb{N}^*$. So $\psi(x\beta) = \chi(X \otimes V_i \otimes Z \otimes Y \otimes Z^{-1})$, where $X \otimes V_i \otimes Z \otimes Y$ is a simple alternated comodule ending by some V_k, $k \in \mathbb{N}^*$. Let us finally compute $\psi(\beta x)$:

$$\psi(\beta x) = \psi(\beta y\alpha^2 z) = \psi(\beta y\alpha \odot \alpha z) = \psi(\beta y\alpha)\psi(\alpha z).$$

By induction $\psi(\beta y\alpha) = \chi(V_i \otimes X \otimes V_j)$ for $i, j \in \mathbb{N}^*$, and X is a simple alternated comodule beginning by Z or Z^{-1} and ending by Z or Z^{-1}. Also $\psi(\alpha z) = \chi(Z \otimes Y)$, where Y is an alternated simple comodule beginning by some V_k, $k \in \mathbb{N}^*$. So $\psi(\beta x) = \chi(V_i \otimes X \otimes V_j \otimes Z \otimes Y)$ where $X \otimes V_j \otimes Z \otimes Y$ is a simple alternated comodule beginning by Z or Z^{-1}. Very similar computations prove the result for $x = y\beta^2 z$, and conclude the proof of Lemma 5.4. $\square$

Proposition 5.2 is a direct consequence of Lemma 5.4. $\square$

We can now easily list the simple $H(q)$-comodules, and describe their fusion rules. For $x \in \mathbb{N} * \mathbb{N}$, let U_x be a simple $H(q)$-comodule such that $\chi(U_x) = \psi(x)$. We have $U_e = k$, $U_\alpha = U$ and $U_\beta = V$, for the notations of the introduction. We have

$$\chi(U_x \otimes U_y) = \chi(U_x)\chi(U_y) = \psi(x)\psi(y) = \psi(x \odot y) = \psi(\sum_{x=ag, y=\bar{g}b} ab) = \chi(\bigoplus_{x=ag, y=\bar{g}b} U_{ab}),$$

and hence

$$U_x \otimes U_y \cong \bigoplus_{x=ag, y=\bar{g}b} U_{ab} \ .$$

By Lemma 5.4 we have $U_x \cong k$ if and only if $x = e$, and using the last formula, we see that $\mathrm{Hom}(k, U_x \otimes U_y) \neq (0)$ if and only if $y = \bar{x}$. This implies that $U_x^* \cong U_{\bar{x}}$ and that $U_x \cong U_y$ if and only if $x = y$. Thus we have a family of simple $H(q)$-comodules $(U_x)_{x \in \mathbb{N} * \mathbb{N}}$ whose coefficients generate A as an algebra, containing the trivial comodule and stable under tensor products: using e.g. the orthogonality relations [10] we conclude that any simple $H(q)$-comodule is isomorphic with a comodule U_x.

The preceding discussion concludes the proof of Theorem 1.1: there just remain to be said that the monoidal category equivalence $\mathrm{Comod}(H(F)) \cong^{\otimes} \mathrm{Comod}(H(q))$ transforms the fundamental n-dimensional comodules U and V of $H(F)$ into the fundamental 2-dimensional comodules U and V of $H(q)$.

Lemma 5.1, Proposition 5.2 and Theorem 1.1 combined together also yield the description of the Grothendieck K_0-ring of the category $\mathrm{Comod}_{\mathrm{f}}(H(F))$.

Corollary 5.5 *Let $F \in GL(n,k)$ $(n \geq 2)$ be a generic matrix. Then we have a ring isomorphism*

$$K_0(\mathrm{Comod}_{\mathrm{f}}(H(F)) \cong \mathbb{Z}\{X, Y\} \ .$$

6 Some applications

We use Theorem 1.1 to prove a few structural results concerning the Hopf algebras $H(F)$, for generic matrices. Again k is an algebraically closed field of characteristic zero.

Let us begin with the isomorphic classification. For universal compact quantum groups, this was done by Wang [19]. Since we use the same type of arguments, we will be a little concise.

Proposition 6.1 *Let $E \in GL(m,k)$, $F \in GL(n,k)$ $(m,n \geq 2)$ be generic matrices. The Hopf algebras $H(E)$ and $H(F)$ are isomorphic if and only if one of the two conditions hold.*
i) $m = n$ and there exists $P \in GL(n,k)$ such that $F = \pm PEP^{-1}$.
ii) $m = n$ and there exists $P \in GL(n,k)$ such that ${}^tF^{-1} = \pm PEP^{-1}$.

Proof. Let $f : H(E) \longrightarrow H(F)$ be a Hopf algebra isomorphism, and denote by $f_* : \text{Comod}(H(E)) \longrightarrow \text{Comod}(H(F))$ the functor induced by f. By [19] U and V are the simple $H(E)$-comodules (resp. $H(F)$-comodules) with the strictly smallest dimension, and hence we have $f_*(U) \cong U$ or $f_*(U) \cong V$. If $f_*(U) \cong U$, then $m = n$ and there exists $P \in GL(n,k)$ such that $f(u) = {}^tPu{}^tP^{-1}$ and necessarily $f(v) = P^{-1}vP$. Since f is well-defined and since U is simple, it is easy to check that $F = \pm PEP^{-1}$. If $f_*(U) \cong V$, then $m = n$ and there exists $P \in GL(n,k)$ such that $f(u) = {}^tPv{}^tP^{-1}$ and necessarily $f(v) = P^{-1}{}^tF^{-1}v{}^tFP$. Since f is well-defined and since U and V are simple, it is easy to check that ${}^tF^{-1} = \pm PEP^{-1}$.

Conversely, if $F = \pm PEP^{-1}$, it is easy to check that there exists a Hopf algebra isomorphism $f : H(E) \longrightarrow H(F)$ such that $f(u) = {}^tPu{}^tP^{-1}$ and $f(v) = P^{-1}vP$. if ${}^tF^{-1} = \pm PEP^{-1}$, it is easy to check that there exists a Hopf algebra isomorphism $f : H(E) \longrightarrow H(F)$ such that $f(u) = {}^tPv{}^tP^{-1}$ and $f(v) = P^{-1}{}^tF^{-1}u{}^tFP$. $\square$

Let us now compute the automorphism group of the Hopf algebra $H(F)$. Let $F \in GL(n,k)$. Put

$$X_0(F) = \{K \in GL(n,k) \mid KFK^{-1} = F\}\ ,\ Y(F) = \{K \in GL(n,k) \mid KFK^{-1} = {}^tF^{-1}\},$$

and $X(F) = X_0(F)/k^*$. Then $X(F)$ is a group. For $N \in \mathbb{N} \cup \{\infty\}$, the cyclic group of order N is denoted by C_N.

Proposition 6.2 *Let $F \in GL(n,k)$ $(n \geq 2)$ be a generic matrix.*
a) Assume that $Y(F) = \emptyset$. Then $X(F) \cong \text{Aut}_{\text{Hopf}}(H(F))$.
b) Assume that $Y(F) \neq \emptyset$. Let $K \in Y(F)$, and put $N = \min\{p \in \mathbb{N} \cup \{\infty\} \mid (F^{-1}{}^tK^{-1}K)^p \in k^*\}$. Then we have an exact sequence of groups

$$1 \longrightarrow C_N \longrightarrow X(F) \rtimes C_{2N} \longrightarrow \text{Aut}_{\text{Hopf}}(H(F)) \longrightarrow 1$$

In particular, if there exists $K \in GL(n,k)$ such that $F = {}^tK^{-1}K$, then $K \in Y(F)$, we can take $N = 1$ and we have an isomorphism $X(F) \rtimes C_2 \cong \text{Aut}_{\text{Hopf}}(H(F))$.

Proof. Let $K \in X_0(F)$. Then there exists a Hopf algebra automorphism ϕ_K of $H(F)$ such that $\phi_K(u) = {}^tKu{}^tK^{-1}$ and $\phi_K(v) = K^{-1}vK$. This gives a group morphism $\phi : X(F) \longrightarrow \text{Aut}_{\text{Hopf}}(H(F))$, injective since the comodule U is simple. Now consider $f \in \text{Aut}_{\text{Hopf}}(H(F))$. Then by the proof of Proposition 6.1, either there exists $K \in X_0(F)$ (recall that $\text{tr}(F) \neq 0$ since F is generic) such that $f(u) = {}^tKu{}^tK^{-1}$, either there exists $K \in Y(F)$ such that $f(u) = {}^tKv{}^tK^{-1}$. If $Y(F) = \emptyset$, then $f = \phi_K$ and the morphism ϕ is an isomorphism. Assume now that $Y(F) \neq \emptyset$ and let $K \in Y(F)$. Then there exists

$\psi_K \in \mathrm{Aut}_{\mathrm{Hopf}}(H(F))$ such that $\psi_K(u) = {}^tKv^tK^{-1}$ and $\psi_K(v) = K^{-1t}F^{-1}u^tFK$. Let $f \in \mathrm{Aut}_{\mathrm{Hopf}}(H(F))$. Then by the proof of Proposition 6.1 there exists $K \in X_0(F)$ such that $f = \phi_K$ or there exists $M \in Y(F)$ such that $f = \psi_M$. We have $\psi_M = \phi_{{}^tM^{-1t}K} \circ \psi_K$, and thus $G = \phi(X(F))\langle\psi_K\rangle$. For $M, L \in Y(F)$, we have $\psi_M \circ \psi_L = \phi_{F^{-1t}M^{-1}L}$, hence $|\langle\psi_K\rangle| = 2N$. Also $\langle\psi_K\rangle \cap \phi(X(F)) = \langle\phi_{F^{-1t}K^{-1}K}\rangle$ and $|\langle\psi_K\rangle \cap \phi(X(F))| = N$. We have $\psi_K \circ \phi_L \circ \psi_K^{-1} = \phi_{F^{-1t}K^{-1t}L^{-1t}KF}$ and hence $\phi(X(F))$ is a normal subgroup of $\mathrm{Aut}_{\mathrm{Hopf}}(H(F))$. We can now use a well-know result in group theory: if G is a group with two subgroups H and K such that $G = HK$, such that H is normal in G and such that $H \cap K$ is abelian, then we have a group exact sequence

$$1 \longrightarrow H \cap K \longrightarrow H \rtimes K \longrightarrow G \longrightarrow 1.$$

The last assertion is immediate. $\square$

7 Quantum automorphism groups of matrix algebras

In his paper [18], Wang described the quantum automorphism group of a finite-dimensional C^*-algebra endowed with a trace, the term quantum automorphism group (or quantum symmetry group) being understood in the sense of Manin [13]. We refer the reader to [13] or [18] for these ideas. The representation theory of such quantum automorphism groups was described by Banica [3] in the case of good traces, and is similar to the one of $SO(3)$.

In [5] we proposed a natural categorical generalization of Wang's construction, yielding in particular an algebraic analogue of the quantum automorphism group of a finite-dimensional measured algebra. We will see that in the case of a measured matrix algebra with a non-necessarily tracial measure, the results of the present paper enable us to describe the representation theory of such a quantum group, reducing the computations to the case of the quantum $SO(3)$-group.

Recall [5] that a measured algebra is a pair (Z, ϕ) where Z is an algebra and $\phi : Z \longrightarrow k$ is a linear map such that the bilinear form $Z \times Z \longrightarrow k$, $(a, b) \mapsto \phi(ab)$, is non-degenerate. We will only be concerned here by the example $(M_n(k), \mathrm{tr_F})$ where $F \in GL(n, k)$ and $\mathrm{tr}_F = \mathrm{tr}({}^tF^{-1}-)$. The quantum automorphism groups of $(M_n(k), \mathrm{tr_F})$, denoted $A_{aut}(M_n(k), \mathrm{tr_F})$, may be described as follows (see [18] for details). As an algebra $A_{aut}(M_n(k), \mathrm{tr_F})$ is the universal algebra with generators X_{ij}^{kl}, $1 \le i, j, k, l \le n$, and satisfying the relations ($1 \le i, j, k, l, r, s \le n$):

$$\sum_t X_{rt}^{ij} X_{ts}^{kl} = \delta_{jk} X_{rs}^{il} \ ; \ \sum_{t,p} F_{tp} X_{kl}^{it} X_{rs}^{pj} = F_{lr} X_{ks}^{ij} \ ; \ \sum_t X_{ij}^{tt} = \delta_{ij} \ ; \ \sum_{t,p} F_{tp}^{-1} X_{tp}^{ij} = F_{ij}^{-1}.$$

It has a natural Hopf algebra structure given by

$$\Delta(X_{ij}^{kl}) = \sum_{r,s} X_{ij}^{rs} \otimes X_{rs}^{kl} \ ; \ \varepsilon(X_{ij}^{kl}) = \delta_{ik}\delta_{jl} \ ; \ S(X_{ij}^{kl}) = \sum_{r,s} F_{jr} F_{sl}^{-1} X_{sk}^{ri} \ , 1 \le i, j, k, l \le n.$$

Let $E \in GL(m, k)$ and $F \in GL(n, k)$. Let us define the algebra $A_{is}(M_m(k), \mathrm{tr_E}; M_n(k), \mathrm{tr}_F)$ to be the universal algebra with generators X_{ij}^{kl}, $1 \le i, j \le m$, $1 \le k, l \le n$, and satisfying the relations:

$$\sum_{t=1}^{m} X_{rt}^{ij} X_{ts}^{kl} = \delta_{jk} X_{rs}^{il} \quad , \quad 1 \le i,j,k,l \le n \quad , \quad 1 \le r,s \le m \; ;$$

$$\sum_{t,p=1}^{n} F_{tp} X_{kl}^{it} X_{rs}^{pj} = E_{lr} X_{ks}^{ij} \quad , \quad 1 \le i,j \le n \quad , \quad 1 \le k,l,r,s \le m \; ;$$

$$\sum_{t=1}^{n} X_{ij}^{tt} = \delta_{ij} \; , \; 1 \le i,j \le m \quad ; \quad \sum_{t,p=1}^{m} E_{tp}^{-1} X_{tp}^{ij} = F_{ij}^{-1} \quad , \quad 1 \le i,j \le n.$$

Lemma 7.1 *Let $E \in GL(m,k)$ and let $F \in GL(n,k)$ $(m,n \ge 2)$ with $\mathrm{tr}(E) = \mathrm{tr}(F)$ and $\mathrm{tr}(E^{-1}) = \mathrm{tr}(F^{-1})$. Then $A_{is}(M_m(k), \mathrm{tr}_E; M_n(k), \mathrm{tr}_F)$ is a non-zero algebra.*

Proof. It is straightforward to check that there exists a unique algebra morphism $\varphi : A_{is}(M_m(k), \mathrm{tr}_E; M_n(k), \mathrm{tr}_F) \longrightarrow H(E,F)$ such that $\varphi(X_{ij}^{kl}) = u_{ik}v_{jl}$ for $1 \le i,j \le n$, $1 \le k,l \le m$. The elements $u_{ik}v_{jl}$ are non-zero elements of $H(E,F)$ by Section 2, and hence $A_{is}(M_m(k), \mathrm{tr}_E; M_n(k), \mathrm{tr}_F)$ is a non-zero algebra. $\square$

We arrive at the main result of the section:

Theorem 7.2 *Let $E \in GL(m,k)$ and let $F \in GL(n,k)$ $(m,n \ge 2)$ with $\mathrm{tr}(E) = \mathrm{tr}(F)$ and $\mathrm{tr}(E^{-1}) = \mathrm{tr}(F^{-1})$. Then the comodule categories over $A_{aut}(M_m(k), \mathrm{tr}_E)$ and $A_{aut}(M_n(k), \mathrm{tr}_F)$ are monoidally equivalent. In particular, if $\mathrm{tr}(F) = \mathrm{tr}(F^{-1})$ and if there exists $q \in k^*$ such that $q^2 - \mathrm{tr}(F)q + 1 = 0$, then the comodule categories over $A_{aut}(M_n(k), \mathrm{tr}_F)$ and $\mathcal{O}(SO_{q^{1/2}}(3))$ are monoidally equivalent.*

Proof. Let us show that

$$(A_{aut}(M_m(k), \mathrm{tr}_E), A_{aut}(M_n(k), \mathrm{tr}_F), A_{is}(M_m(k), \mathrm{tr}_E; M_n(k), \mathrm{tr}_F), A_{is}(M_n(k), \mathrm{tr}_F; M_m(k), \mathrm{tr}_E))$$

is a Hopf-Galois system [8]. First by Lemma 7.1 all these algebras are non-zero. Let $G \in GL(p,k)$. It is a direct computation to check that there exists a unique algebra morphism

$$\delta_{E,F}^{G} : A_{is}(M_m(k), \mathrm{tr}_E; M_n(k), \mathrm{tr}_F) \longrightarrow A_{is}(M_m(k), \mathrm{tr}_E; M_p(k), \mathrm{tr}_G) \otimes A_{is}(M_p(k), \mathrm{tr}_G; M_n(k), \mathrm{tr}_F)$$

such that $\delta_{E,F}^{G}(X_{ij}^{kl}) = \sum_{r,s} X_{ij}^{rs} \otimes X_{rs}^{kl}$. Also there exists a unique algebra morphism $\phi : A_{is}(M_n(k), \mathrm{tr}_F; M_m(k), \mathrm{tr}_E) \longrightarrow A_{is}(M_m(k), \mathrm{tr}_E; M_n(k), \mathrm{tr}_F)^{\mathrm{op}}$ such that $\phi(X_{ij}^{kl}) = \sum_{r,s} F_{jl} E_{sl}^{-1} X_{sk}^{ri}$. With these structural morphisms, it is immediate to check that we indeed have a Hopf-Galois system. Hence using Corollary 1.4 of [8], we have our monoidal category equivalence. Now assume that $\mathrm{tr}(F) = \mathrm{tr}(F^{-1})$ and that there exists $q \in k^*$ such that $q^2 - \mathrm{tr}(F)q + 1 = 0$. Put $\mathrm{tr}_q = \mathrm{tr}_{F_q}$. Then we have an equivalence of monoidal categories:

$$A_{aut}(M_n(k), \mathrm{tr}_F) \cong^{\otimes} A_{aut}(M_2(k), \mathrm{tr}_q).$$

Finally it may be shown that $A_{aut}(M_2(k), \mathrm{tr}_q)$ and $\mathcal{O}(SO_{q^{1/2}}(3))$ are isomorphic. One considers first the Hopf algebra morphism $A_{aut}(M_2(k), \mathrm{tr}_q) \longrightarrow \mathcal{O}(SL_q(2))$ obtained using the adjoint corepresentation of the canonical two-dimensional $\mathcal{O}(SL_q(2))$-comodule. This Hopf algebra morphism is injective, and using [9], we arrive at the desired conclusion. $\square$

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