QUANTUM SUBGROUPS OF THE COMPACT QUANTUM GROUP $SU_{-1}(3)$

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Abstract. We study the (compact) quantum subgroups of the compact quantum group $SU_{-1}(3)$: we show that any such non-classical quantum subgroup is a twist of a compact subgroup of $SU(3)$ or is isomorphic to a quantum subgroup of $U_{-1}(2)$.

1. introduction

Quantum groups, named after Drinfeld's seminal work [12], are natural Hopf algebraic generalizations of usual groups, arising in several branches of mathematics. As in classical group theory, the problem of their classification is a fundamental one.

An important aspect of the classification problem for quantum groups is the determination of the quantum subgroups of the known quantum groups. Let us recall some significant contributions to this topic.

(1) Podleś [23] was the first to consider this problem, and he described the compact quantum subgroups of Woronowicz' quantum group $SU_q(2)$, for $q \in [-1, 1] \setminus \{0\}$. For other approaches, see [8] (for the finite quantum subgroups when $q = -1$) or [15] (when $q \neq -1$). See also [9] for the more general question of the classification of the quantum homogeneous spaces over $SU_q(2)$.
(2) The finite quantum subgroups of $GL_q(n)$ were classified by Müller [20], for q an odd root of unity. From this work arose in particular an infinite family of pairwise non-isomorphic Hopf algebras of the same dimension: this was one of the series of counterexamples to Kaplansky's tenth conjecture.
(3) The work of Müller was subsequently generalized by Andruskiewitsch and García in [1], where they determined the quantum subgroups of G_q, with G a connected, simply connected simple algebraic group and q a root of unity of odd order.
(4) Another generalization of Müller's work was provided by García [16], who studied the two-parameter deformations $GL_{\alpha,\beta}(n)$, and classified the quantum subgroups in the odd root of unity case.
(5) The compact quantum subgroups of $SO_{-1}(3)$ were determined by Banica and the first author in [3]: these are the compact quantum groups acting faithfully on the classical space consisting of 4 points. Here the quantum group $SO_{-1}(3)$ is different from the quantum group $SO_q(3)$ at $q = -1$ studied in [23].
(6) The compact quantum subgroups of O_n^*, the half-liberated orthogonal quantum groups from [4], were determined by Dubois-Violette and the first author in [7].

From these works emerged several new interesting classes of quantum groups, and several hints of what the classification of quantum groups should be. The approaches in (2), (3) and (4) deal with non-semisimple quantum groups and do not treat the case $q = -1$, while this is certainly the most interesting case if we have semisimple finite quantum groups in mind. The present paper is a contribution to the case $q = -1$: we determine the compact quantum subgroups of the compact quantum group $SU_{-1}(3)$, as follows.

Theorem 1.1. *Let G be a non-classical compact quantum subgroup of $SU_{-1}(3)$. Then one of the following statements holds.*

(1) *G is isomorphic to a K_{-1}, a twist at -1 of a compact subgroup $K \subset SU(3)$ containing the subgroup of diagonal matrices having ± 1 as entries.*

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(2) *G is isomorphic to a quantum subgroup of $U_{-1}(2)$.*

The quantum subgroups of $U_{-1}(2)$ can be determined by using similar techniques to those of Podleś [23] (see Remark 5.7; we shall not discuss this in detail here). Note that it follows from Theorem 1.1 and its proof that if G is a non-classical compact quantum subgroup of $SU_{-1}(3)$ acting irreducibly on $\mathbb{C}^3$, then G is isomorphic to a K_{-1}, a twist at -1 of a compact subgroup $K \subset SU(3)$ containing the subgroup of diagonal matrices having ± 1 as entries, and acting irreducibly on $\mathbb{C}^3$. Thus for any quantum subgroup of $SU_{-1}(3)$ acting irreducibly on the fundamental representation, the tensor category of representations is symmetric (in Hopf algebra terms, the Hopf algebra $\mathcal{R}(G)$ is cotriangular). This seems to be an interesting phenomenon, that does not hold in general: for instance, the quantum group $U_{-1}(2)$ has $SU_{-1}(2)$ as a subgroup, whose representation category is braided but not symmetric (see [22, 21] for related questions).

As in [3], the starting point is that $SU_{-1}(3)$ is a twist at -1 of the classical group $SU(3)$ (a 2-cocycle deformation). This furnishes a number of representation-theoretic tools, developed in Section 3, to study the C*-algebra $C(SU_{-1}(3))$ and its quotients, which are used in an essential way to prove Theorem 1.1. Note that the representation theory of twisted function algebras on finite groups is fully discussed in [14], with a precise description of the irreducible representations. However the fusion rules, which would lead to the full classification of the Hopf algebra quotients, are not discussed in [14], and we do not see any general method to compute them. What we get here in the case of $SU_{-1}(3)$ are some partial fusion rules, for some special representations of $C(SU_{-1}(3))$, which however are sufficiently generic to get the necessary information to classify the quantum subgroups.

The paper is organized as follows. Section 2 consists of preliminaries. In Section 3 we recall the twisting (2-cocycle deformation) procedure for Hopf algebras and develop the aforementioned representation-theoretic tools for representations of twisted C*-algebras of functions. In Section 4 we briefly recall how the quantum group $SU_{-1}(2m+1)$ can be obtained by twisting, and Section 5 is devoted to the proof of Theorem 1.1.

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2. Preliminaries

2.1. **Compact quantum groups.** We first recall some basic facts concerning compact quantum groups. The book [18] is a convenient reference for the topic of compact quantum groups, and all the defintions we omit can be found there. All algebras in this paper will be unital as well as all algebra morphisms, and $\otimes$ will denote the minimal tensor product of C*-algebras as well as the algebraic tensor product; this should cause no confusion.

Definition 2.1. *A **Woronowicz algebra** is a C*-algebra A endowed with a $*$-morphism $\Delta : A \to A \otimes A$ satisfying the coassociativity condition and the cancellation law*

$$\overline{\Delta(A)(A \otimes 1)} = A \otimes A = \overline{\Delta(A)(1 \otimes A)}$$

The morphism Δ is called the comultiplication of A.

The category of Woronowicz algebras is defined in the obvious way (see [26] for details). A commutative Woronowicz algebra is necessarily isomorphic with $C(G)$, the algebra of continuous functions on a compact group G, unique up to isomorphism, and the category of **compact quantum groups** is defined to be the category dual to the category of Woronowicz algebras. Hence to any Woronowicz algebra A corresponds a unique compact quantum group G according to the heuristic formula $A = C(G)$.

Woronowicz's original definition for matrix compact quantum groups [27] is still the most useful in concrete situations, and we have the following fundamental result [29].

Theorem 2.2. *Let A be a C*-algebra endowed with a $*$-morphism $\Delta : A \to A \otimes A$. Then A is a Woronowicz algebra if and only if there exists a family of unitary matrices $(u^\lambda)_{\lambda \in \Lambda} \in M_{d_\lambda}(A)$ satisfying the following three conditions.*

(1) *The $*$-subalgebra A_0 generated by the entries u_{ij}^λ of the matrices $(u^\lambda)_{\lambda\in\Lambda}$ is dense in A.*
(2) *For $\lambda \in \Lambda$ and $i, j \in \{1, \dots, d_\lambda\}$, one has $\Delta(u_{ij}^\lambda) = \sum_{k=1}^{d_\lambda} u_{ik}^\lambda \otimes u_{kj}^\lambda$.*
(3) *For $\lambda \in \Lambda$, the transpose matrix $(u^\lambda)^t$ is invertible.*

In fact the $*$-algebra A_0 in the theorem is canonically defined, and is what we call a compact Hopf algebra (a CQG algebra in [18]): a Hopf $*$-algebra having all its finite-dimensional comodules equivalent to unitary ones, or equivalently a Hopf $*$-algebra having a positive and faithful Haar integral (see [18] for details). The counit and antipode of A_0, denoted respectively ε and S, are referred to as the counit and antipode of A. The Hopf algebra A_0 is called the **algebra of representative functions** on the compact quantum group G dual to A, with another heuristic formula $A_0 = \mathcal{R}(G)$.

Conversely, starting from a compact Hopf algebra, the universal C*-completion yields a Woronowicz algebra in the above sense: see the book [18]. In fact, in general, there are possibly several different C*-norms on A_0, in particular the reduced one (obtained from the GNS-construction associated to the Haar integral), but we will not be concerned with this problem, the compact quantum groups considered in this paper being co-amenable.

Of course, any group-theoretic statement about a compact quantum group G must be interpreted in terms of the Woronowicz algebra $C(G)$ or of the Hopf $*$-algebra $\mathcal{R}(G)$. In particular, as usual, a (compact) **quantum subgroup** $H \subset G$ corresponds to a surjective Woronowicz algebra morphism $C(G) \to C(H)$, or to a surjective Hopf $*$-algebra morphism $\mathcal{R}(G) \to \mathcal{R}(H)$.

2.2. **The quantum groups $U_{-1}(n)$ and $SU_{-1}(n)$.** In this subsection we briefly recall the definition of the compact quantum groups $U_{-1}(n)$ and $SU_{-1}(n)$ [28, 19, 24].

Definition 2.3. *The $*$-algebra $\mathcal{R}(U_{-1}(n))$ is the universal $*$-algebra generated by variables $(u_{ij})_{1\le i,j\le n}$ with relations making the matrix $u = (u_{ij})$ unitary and*

$$u_{ij}u_{kl} = (-1)^{\delta_{ik}+\delta_{jl}} u_{kl}u_{ij},\ \forall i, j, k, l$$

The C**-algebra $C(U_{-1}(n))$ is the enveloping* C**-algebra of $\mathcal{R}(U_{-1}(n))$.*

The relations $u_{ij}^* u_{kl} = (-1)^{\delta_{ik}+\delta_{jl}} u_{kl}u_{ij}^*$ automatically hold in $\mathcal{R}(U_{-1}(n))$ and $C(U_{-1}(n))$, hence the matrix u^t is also unitary. It follows that $\mathcal{R}(U_{-1}(n))$ is a compact Hopf $*$-algebra, and hence that $C(U_{-1}(n))$ is a Woronowicz algebra, with comultiplication, counit and antipode defined by

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj},\ \varepsilon(u_{ij}) = \delta_{ij},\ S(u_{ij}) = u_{ji}^*$$

The matrix $u = (u_{ij})$ forms the fundamental representation of the quantum group $U_{-1}(n)$. The quantum determinant

$$D = \sum_{\sigma\in S_n} u_{1\sigma(1)} \cdots u_{n\sigma(n)} = \sum_{\sigma\in S_n} u_{\sigma(1)1} \cdots u_{\sigma(n)n}$$

is a unitary central group-like element of $\mathcal{R}(U_{-1}(n))$.

Definition 2.4. *The $*$-algebra $\mathcal{R}(SU_{-1}(n))$ is the quotient of $\mathcal{R}(U_{-1}(n))$ by the $*$-ideal generated by $D - 1$, and the* C**-algebra $C(SU_{-1}(n))$ is the enveloping* C**-algebra of $\mathcal{R}(SU_{-1}(n))$.*

It follows, since D is group-like, that $\mathcal{R}(SU_{-1}(n))$ is a compact Hopf $*$-algebra, and that C$(SU_{-1}(n))$ is a Woronowicz algebra, with comultiplication, counit and antipode defined by the same formulas as above.

The following Lemma will be used in Section 5.

Lemma 2.5. *For any $i \in \{1, \dots, n+1\}$, there exists a surjective Hopf $*$-algebra map $\pi_i : \mathcal{R}(SU_{-1}(n+1)) \to \mathcal{R}(U_{-1}(n))$ whose kernel is the Hopf $*$-ideal generated by the elements u_{ki}, u_{ik}, $k \neq i$. In particular, if $\pi : \mathcal{R}(SU_{-1}(n+1)) \twoheadrightarrow A$ is a surjective Hopf $*$-algebra map such that for some fixed i we have $\pi(u_{ki}) = 0 = \pi(u_{ik})$ for $k \neq i$, then there exists a surjective Hopf $*$-algebra map $\mathcal{R}(U_{-1}(n)) \twoheadrightarrow A$.*

Proof. It follows from the definitions that there exists a Hopf $*$-algebra map π_i such that $\pi_i(u_{ki}) = 0 = \pi_i(u_{ik})$ for $k \neq i$, $\pi_i(u_{ii}) = D^{-1}$, $\pi_i(u_{jk}) = u_{jk}$ for $j, k < i$, $\pi_i(u_{jk}) = u_{j,k-1}$ for $j < i$ and $k > i$, $\pi_i(u_{jk}) = u_{j-1,k}$ for $j > i$ and $k < i$, $\pi_i(u_{jk}) = u_{j-1,k-1}$ for $j, k > i$. By definition π_i vanishes on I, the $*$-ideal generated by the elements in the statement of the lemma, so induces a surjective $*$-algebra map $\overline{\pi_i} : \mathcal{R}(SU_{-1}(n+1))/I \to \mathcal{R}(U_{-1}(n))$, and it is not difficult to construct an inverse isomorphism to $\overline{\pi_i}$, and hence $I = \mathrm{Ker}(\pi_i)$. The last assertion is an immediate consequence of the first one. $\square$

2.3. **Representations of C*-algebras.** In this short subsection, we collect a few useful facts on representations of $*$-algebras and C*-algebras. If A is $*$-algebra, a representation of A always means a Hilbert space representation of A, i.e. a $*$-algebra map $A \to \mathcal{B}(H)$ into the $*$-algebra of bounded operators on a Hilbert space H. As usual, the set of isomorphism classes of irreducible representations of A is denoted by $\widehat{A}$. If ρ, π are representations of A, we write $\rho \prec \pi$ if ρ is isomorphic to a sub-representation of π.

The following classical result will be a key tool. See e.g. [10] for a proof.

Theorem 2.6. *Let $A \subset B$ be an inclusion of* C**-algebras, and let ρ be an irreducible representation of A. Then there exists an irreducible representation π of B such that $\rho \prec \pi_{|A}$.*

Let A be a $*$-algebra. If $\rho : A \to \mathcal{B}(H)$ is a finite-dimensional representation, then the character of ρ is the linear map $\chi = \mathrm{tr}\rho$, where tr is the usual trace. Two finite-dimensional representations of A are isomorphic if and only if they have the same character.

Now assume that A is a Hopf $*$-algebra. The trivial representation is ε, the counit of A. Let $\rho : A \to \mathcal{B}(H)$ be a finite-dimensional representation of A. Recall that the dual representation $\rho^\vee : A \to \mathcal{B}(\overline{H})$ (where $\overline{H}$ is the conjugate Hilbert space of H) is defined by $\rho^\vee(a)(\overline{x}) = \overline{\rho(S(a^*))(x)}$, for any $a \in A$ and $x \in H$. We have $\varepsilon \prec \rho \otimes \rho^\vee$, and when ρ is irreducible, this property characterizes the irreducible representation $\rho^\vee$ up to isomorphism.

3. 2-cocycle deformations

We now recall the usual twisting (2-cocycle deformation) construction for Hopf algebras, which is dual to the theory initiated by Drinfeld, and developed by Doi [11]. We also develop the representation theoretic machinery needed to study the quotients of a twisting of a Hopf algebra of representative functions on a compact group.

Let Q be a Hopf $*$-algebra. We use Sweedler's notation $\Delta(x) = x_1 \otimes x_2$. Recall (see e.g. [11]) that a **unitary 2-cocycle** on Q is a convolution invertible linear map $\sigma : Q \otimes Q \longrightarrow \mathbb{C}$ satisfying

$$\sigma(x_1, y_1)\sigma(x_2y_2, z) = \sigma(y_1, z_1)\sigma(x, y_2z_2)$$
$$\sigma^{-1}(x, y) = \overline{\sigma(S(x)^*, S(y)^*)}$$

and $\sigma(x, 1) = \sigma(1, x) = \varepsilon(x)$, for $x, y, z \in Q$. Here σ^{-1} denotes the convolution inverse of σ.

Following [11] and [25], we associate various $*$-algebras to a unitary 2-cocycle.

• First consider the $*$-algebra ${}_\sigma Q$. As a vector space we have ${}_\sigma Q = Q$ and the product and involution of ${}_\sigma Q$ are defined to be

$$\{x\}\{y\} = \sigma(x_1, y_1)\{x_2y_2\}, \quad \{x\}^* = \sigma^{-1}(x_2^*, S(x_1)^*)\{x_3^*\}, \quad x, y \in Q,$$

where an element $x \in Q$ is denoted $\{x\}$, when viewed as an element of ${}_\sigma Q$.

• We also have the $*$-algebra $Q_{\sigma^{-1}}$. As a vector space we have $Q_{\sigma^{-1}} = Q$ and the product and involution of $Q_{\sigma^{-1}}$ are defined to be

$$\langle x\rangle\langle y\rangle = \sigma^{-1}(x_2, y_2)\langle x_1y_1\rangle, \quad \langle x\rangle^* = \sigma(S(x_3)^*, x_2^*)\langle x_1^*\rangle, \quad x, y \in Q.$$

where an element $x \in Q$ is denoted $\langle x\rangle$, when viewed as an element of $Q_{\sigma^{-1}}$. The unitary cocycle condition ensures that ${}_\sigma Q$ and $Q_{\sigma^{-1}}$ are associative $*$-algebras with 1 as a unit. The algebras ${}_\sigma Q$ and $Q_{\sigma^{-1}}$ are in fact anti-isomorphic, see e.g. [5].

If Q is a compact Hopf algebra, then the Haar integral on Q, viewed as a linear map on ${}_\sigma Q$ and $Q_{\sigma^{-1}}$, is still a faithful state (this can been seen by using the orthogonality relations

[27, 18]). We denote by $C^*_r({}_\sigma Q)$ and $C^*_r(Q_{\sigma^{-1}})$ the respective C*-completions obtained from the GNS-constructions associated to the Haar integral.

• Finally we have the Hopf $*$-algebra $Q^\sigma = {}_\sigma Q_{\sigma^{-1}}$. As a coalgebra $Q^\sigma = Q$. The product and involution of Q^σ are defined to be

$$[x][y] = \sigma(x_1, y_1)\sigma^{-1}(x_3, y_3)[x_2y_2], \quad [x]^* = \sigma(S(x_5)^*, x_4^*)\sigma^{-1}(x_2^*, S(x_1)^*)[x_3^*] \quad x, y \in Q,$$

where an element $x \in Q$ is denoted $[x]$, when viewed as an element of Q^σ, and we have the following formula for the antipode of Q^σ:

$$S^\sigma([x]) = \sigma(x_1, S(x_2))\sigma^{-1}(S(x_4), x_5)[S(x_3)].$$

The Hopf algebras Q and Q^σ have equivalent tensor categories of comodules [25]. If Q is a compact Hopf algebra, then Q^σ is also a compact Hopf algebra, the Haar integral on Q^σ being the one of Q, and the C*-tensor categories of unitary comodules over Q and Q^σ are equivalent [6]. If $Q = \mathcal{R}(G)$, the algebra of representative functions on a compact group G, we denote by $C(G)^\sigma$ the enveloping C*-algebra of $\mathcal{R}(G)^\sigma$.

Very often unitary 2-cocycles are induced by simpler quotient Hopf $*$-algebras (quantum subgroups). More precisely let $\pi : Q \to L$ be a Hopf $*$-algebra surjection and let $\sigma : L \otimes L \to \mathbb{C}$ be a unitary 2-cocycle on L. Then $\sigma_\pi = \sigma \circ (\pi \otimes \pi) : Q \otimes Q \to \mathbb{C}$ is a unitary 2-cocycle. In what follows the cocycle σ_π will simply be denoted by σ, this should cause not cause any confusion.

We first record the following elementary result from [3].

Proposition 3.1. *Let $\pi : Q \to L$ be a Hopf $*$-algebra surjection and let $\sigma : L \otimes L \to \mathbb{C}$ be a unitary 2-cocycle. Denote by $[\pi] : Q^\sigma \to L^\sigma$ the map $[x] \mapsto [\pi(x)]$. Then there is a bijection between the following data.*

1. *Pairs (f, R) where R is a Hopf $*$-algebra and $f : Q \to R$ is a surjective Hopf $*$-algebra map such that there exists a Hopf $*$-algebra map $g : R \to L$ satisfying $g \circ f = \pi$.*
2. *Pairs (f', R') where R' is a Hopf $*$-algebra and $f' : Q^\sigma \to R'$ is a surjective Hopf $*$-algebra map such that there exists a Hopf $*$-algebra map $g' : R' \to L$ satisfying $g' \circ f' = [\pi]$.*

Similarly, the following result is essentially contained in [3].

Proposition 3.2. *Let $\pi : Q \to L$ be a Hopf $*$-algebra surjection and let $\sigma : L \otimes L \to \mathbb{C}$ be a unitary 2-cocycle on L. We have an injective $*$-algebra map*

$$\begin{aligned} \theta : Q^\sigma &\longrightarrow Q \otimes {}_\sigma L \otimes L_{\sigma^{-1}} \\ [x] &\longmapsto x_2 \otimes \{\pi(x_1)\} \otimes \langle \pi(x_3) \rangle \end{aligned}$$

that induces an isomorphism to the subalgebra of coinvariant elements

$$Q^\sigma \simeq (Q \otimes {}_\sigma L \otimes L_{\sigma^{-1}})^{\mathrm{co}(L^{\mathrm{cop}} \otimes L)}$$

where the respective right coactions of $L^{\mathrm{cop}} \otimes L$ on Q and ${}_\sigma L \otimes L_{\sigma^{-1}}$ are defined by

$$Q \to Q \otimes L^{\mathrm{cop}} \otimes L \qquad {}_\sigma L \otimes L_{\sigma^{-1}} \to {}_\sigma L \otimes L_{\sigma^{-1}} \otimes L^{\mathrm{cop}} \otimes L$$

$$x \mapsto x_2 \otimes \pi(x_1) \otimes \pi(x_3) \qquad \{\pi(x)\} \otimes \langle \pi(y) \rangle \mapsto \{\pi(x_1)\} \otimes \langle \pi(y_2) \rangle \otimes S^{-1}\pi(x_2) \otimes S\pi(y_1)$$

If moreover Q and L are cosemisimple and h_Q and h_L denote their respective Haar integrals, we have $(h_Q \otimes h_L \otimes h_L)\theta = h_Q = h_{Q^\sigma}$.

Proof. It follows from the definitions that θ is a $*$-algebra map and that $(\mathrm{id}_Q \otimes \varepsilon \otimes \varepsilon)\theta = \mathrm{id}_{Q^\sigma}$, hence θ is injective. It is a direct verification to check that $\theta(Q^\sigma) \subset (Q \otimes {}_\sigma L \otimes L_{\sigma^{-1}})^{\mathrm{co}(L^{\mathrm{cop}} \otimes L)}$, and that θ induces the announced isomorphism, with inverse $(\mathrm{id}_Q \otimes \varepsilon \otimes \varepsilon)$. The last assertion is immediate. □

We now specialize to the case $Q = \mathcal{R}(G)$, the algebra of representative on a classical compact group G.

Proposition 3.3. *Let G be a compact group, let $\Gamma \subset G$ be a closed subgroup and let σ be a unitary 2-cocycle on $\mathcal{R}(\Gamma)$. Put $B = \mathrm{C}^*_r({}_\sigma\mathcal{R}(\Gamma)) \otimes \mathrm{C}^*_r(\mathcal{R}(\Gamma)_{\sigma^{-1}})$. Then there exists a* C*-*algebra embedding*

$$\theta : C(G)^\sigma \longrightarrow C(G) \otimes B$$

inducing a C*-*algebra isomorphism*

$$C(G)^\sigma \simeq (C(G) \otimes B)^{\Gamma^{\mathrm{op}} \times \Gamma}$$

for some natural actions of $\Gamma^{\mathrm{op}} \times \Gamma$ on G and B.

Proof. The restriction map $\mathcal{R}(G) \to \mathcal{R}(\Gamma)$ enables us to use the previous proposition. The previous injective $*$-algebra map $\theta : \mathcal{R}(G)^\sigma \to \mathcal{R}(G) \otimes {}_\sigma\mathcal{R}(\Gamma) \otimes \mathcal{R}(\Gamma)_{\sigma^{-1}}$ induces a $*$-algebra map $C(G)^\sigma \longrightarrow C(G) \otimes B$, still denoted θ (recall that $C(G)^\sigma$ is the enveloping C*-algebra of $\mathcal{R}(G)^\sigma$). The co-amenability of $\mathcal{R}(G)^\sigma$ [2] and the last observation in the previous proposition show that θ is injective at the C*-algebra level. The coactions of the previous proposition induce actions of $\Gamma^{\mathrm{op}} \times \Gamma$ on $\mathcal{R}(G)$ and on ${}_\sigma\mathcal{R}(\Gamma) \otimes \mathcal{R}(\Gamma)_{\sigma^{-1}}$, and hence on $C(G)$ and on B. We have, by the previous proposition, an isomorphism $\mathcal{R}(G)^\sigma \simeq (\mathcal{R}(G) \otimes {}_\sigma\mathcal{R}(\Gamma) \otimes \mathcal{R}(\Gamma)_{\sigma^{-1}})^{\Gamma^{\mathrm{op}} \times \Gamma}$, and hence, since $(\mathcal{R}(G) \otimes {}_\sigma\mathcal{R}(\Gamma) \otimes \mathcal{R}(\Gamma)_{\sigma^{-1}})^{\Gamma^{\mathrm{op}} \times \Gamma}$ is dense in $(C(G) \otimes B)^{\Gamma^{\mathrm{op}} \times \Gamma}$, an isomorphism

$$C(G)^\sigma \simeq (C(G) \otimes B)^{\Gamma^{\mathrm{op}} \times \Gamma}$$

This gives the announced result. □

Remark 3.4. The right action of $\Gamma^{\mathrm{op}} \times \Gamma$ on G in the previous result is given by

$$G \times (\Gamma^{\mathrm{op}} \times \Gamma) \longrightarrow G$$
$$(g, (r, s)) \longmapsto rgs$$

The C^*-algebra $(C(G) \otimes B)^{\Gamma^{\mathrm{op}} \times \Gamma}$ is naturally identified with $C(G \times_{\Gamma^{\mathrm{op}} \times \Gamma} B)$, the algebra of continuous functions $f : G \to B$ such that $f(g.(r,s)) = (r,s)^{-1}.f(g)$, $\forall g \in G$, $\forall (r,s) \in \Gamma^{\mathrm{op}} \times \Gamma$. Thus it follows that $C(G)^\sigma$ is (the algebra of sections on) a continuous bundle of C*-algebras over the orbit space $G/(\Gamma^{\mathrm{op}} \times \Gamma) \simeq \Gamma \setminus G/\Gamma$, with fiber at an orbit $\Gamma g \Gamma$ the fixed point algebra $B^{(\Gamma^{\mathrm{op}} \times \Gamma)_g}$, where $(\Gamma^{\mathrm{op}} \times \Gamma)_g = \{(r,s) \in \Gamma \times \Gamma,\ rgs = g\}$: see e.g. Lemma 2.2 in [13]. Hence the representation theory of $C(G)^\sigma$ is determined by the representation theory of the fibres $B^{(\Gamma^{\mathrm{op}} \times \Gamma)_g}$.

The following result will be our main tool to study the representations and quotients of a Woronowicz algebra of type $C(G)^\sigma$.

Proposition 3.5. *Let G be a compact group, let $\Gamma \subset G$ be a closed subgroup and let σ be a unitary 2-cocycle on $\mathcal{R}(\Gamma)$. Then for each $g \in G$ we have a $*$-algebra map*

$$\theta_g : C(G)^\sigma \longrightarrow \mathrm{C}^*_r({}_\sigma\mathcal{R}(\Gamma)) \otimes \mathrm{C}^*_r(\mathcal{R}(\Gamma)_{\sigma^{-1}})$$
$$\mathcal{R}(G)^\sigma \ni [f] \longmapsto f_2(g)\{f_{1|\Gamma}\} \otimes \langle f_{3|\Gamma}\rangle \in {}_\sigma\mathcal{R}(\Gamma) \otimes \mathcal{R}(\Gamma)_{\sigma^{-1}}$$

If Γ is finite, then $\dim(\mathrm{Im}(\theta_g)) = |\Gamma g \Gamma|$.

Assume moreover that ${}_\sigma\mathcal{R}(\Gamma)$ and $\mathcal{R}(\Gamma)_{\sigma^{-1}}$ are full matrix algebras, so that θ_g defines a representation of dimension $|\Gamma|$ of $\mathcal{R}(G)^\sigma$.

1. *Every irreducible representation of $C(G)^\sigma$ is isomorphic to a subrepresentation of θ_g for some $g \in G$. In particular every irreducible representation of $C(G)^\sigma$ is finite-dimensional and has dimension at most $|\Gamma|$.*
2. *The representation θ_g is irreducible if and only if $|\Gamma g \Gamma| = |\Gamma|^2$, if and only if $\#\{(s,t) \in \Gamma \times \Gamma \mid sgt = g\} = 1$. Any irreducible representation of dimension $|\Gamma|$ of $C(G)^\sigma$ is isomorphic to an irreducible representation θ_g as above.*
3. *For $g, h \in G$, we have $\theta_g \simeq \theta_h \iff \Gamma g \Gamma = \Gamma h \Gamma$.*
4. *For $g, h \in G$, we have $\theta_g \otimes \theta_h \simeq \oplus_{s \in \Gamma} \theta_{gsh}$.*
5. *Assume furthermore that Γ is abelian. Then each $s \in \Gamma$ defines a 1-dimensional representation ε_s of $C(G)^\sigma$, and for $s \in \Gamma$, we have $\theta_s \simeq \oplus_{t \in \Gamma} \varepsilon_t$.*

Proof. The representations θ_g are defined using the previous embedding θ, by $\theta_g = (\mathrm{ev}_g \otimes \mathrm{id} \otimes \mathrm{id})\theta$, where ev_g is the evaluation at g. We assume now that Γ is finite. As a linear space, we view $\mathrm{C}^*_r({}_\sigma\mathcal{R}(\Gamma)) \otimes \mathrm{C}^*_r(\mathcal{R}(\Gamma)_{\sigma^{-1}})$ as $C(\Gamma \times \Gamma)$. Consider the continuous linear map

$$\begin{aligned}\theta'_g : C(G) &\longrightarrow C(\Gamma \times \Gamma)\\ f &\longmapsto ((s,t) \mapsto f(sgt))\end{aligned}$$

For $f \in \mathcal{R}(G)$, we have $\theta'_g(f) = \theta_g([f])$, hence $\theta'_g(\mathcal{R}(G)) = \theta_g(\mathcal{R}(G)^\sigma)$ and $\theta'_g(C(G)) = \theta_g(C(G)^\sigma)$ by the density of $\mathcal{R}(G)$ and the finite-dimensionality of the target space. We have $\mathrm{Ker}(\theta'_g) = \{f \in C(G) \mid f_{|\Gamma g\Gamma} = 0\} = I$ and since $\theta'_g(C(G)) \simeq C(G)/I \simeq C(\Gamma g \Gamma)$, we have $\dim(\theta'_g(C(G)) = |\Gamma g\Gamma| = \dim(\theta_g(C(G)^\sigma))$.

Assume now that ${}_\sigma\mathcal{R}(\Gamma)$ and $\mathcal{R}(\Gamma)_{\sigma^{-1}}$ are full matrix algebras. By counting dimensions, ${}_\sigma\mathcal{R}(\Gamma) \otimes \mathcal{R}(\Gamma)_{\sigma^{-1}} \cong M_{|\Gamma|}(\mathbb{C})$. The irreducible representations of $C(G) \otimes {}_\sigma\mathcal{R}(\Gamma) \otimes \mathcal{R}(\Gamma)_{\sigma^{-1}}$ all are of the form $\mathrm{ev}_g \otimes \mathrm{id} \otimes \mathrm{id}$, and since θ defines an embedding $C(G)^\sigma \hookrightarrow C(G) \otimes {}_\sigma\mathcal{R}(\Gamma) \otimes \mathcal{R}(\Gamma)_{\sigma^{-1}}$, it follows from Theorem 2.6 that any irreducible representation of $C(G)^\sigma$ is isomorphic to a subrepresentation of some θ_g, and hence is finite-dimensional of dimension $\leq |\Gamma|$. This proves (1). The matrix representation θ_g is irreducible if and only if θ_g is surjective, if and only if $|\Gamma g \Gamma| = \dim(\mathrm{Im}(\theta_g)) = |\Gamma|^2$, and this proves (2).

Consider now the linear map

$$\begin{aligned}\chi'_g : C(G) &\longrightarrow \mathbb{C}\\ f &\mapsto \frac{1}{|\Gamma|}\sum_{s,t\in\Gamma} f(sgt)\end{aligned} \tag{3.1}$$

Let χ_g be the character of θ_g. Let us check that $\chi_g([f]) = \chi'_g(f)$ for any $f \in \mathcal{R}(G)$. By the density of $\mathcal{R}(G)$ and $\mathcal{R}(G)^\sigma$, this will show that for $g, h \in G$, we have $\chi_g = \chi_h \iff \chi'_g = \chi'_h$. Consider the normalized Haar integral $h : C(\Gamma) \to \mathbb{C}$, $f \mapsto \frac{1}{|\Gamma|}\sum_{s\in\Gamma} f(s)$. Then h, viewed as a linear map on ${}_\sigma\mathcal{R}(\Gamma)$, is still a trace since it is invariant under the natural ergodic action of the finite group Γ on the matrix algebra ${}_\sigma\mathcal{R}(\Gamma)$ (arising from the canonical coaction of $\mathcal{R}(\Gamma)$ on ${}_\sigma\mathcal{R}(\Gamma)$), and hence we have $h = \frac{1}{\sqrt{|\Gamma|}}\mathrm{tr}$, where tr is the usual trace. Thus we have, for $f \in \mathcal{R}(G)$,

$$\begin{aligned}\chi_g([f]) = (\mathrm{tr} \otimes \mathrm{tr})\theta_g([f]) &= |\Gamma|(h \otimes h)\theta_g([f]) = |\Gamma|(h \otimes h)(f_2(g)\{f_{1_{|\Gamma}}\} \otimes \langle f_{3_{|\Gamma}}\rangle)\\ &= \frac{1}{|\Gamma|}\sum_{s,t\in\Gamma} f_1(s)f_2(g)f_3(t) = \frac{1}{|\Gamma|}\sum_{s,t\in\Gamma} f(sgt) = \chi'_g(f)\end{aligned}$$

Let $g, h \in G$. If $\Gamma g\Gamma = \Gamma h\Gamma$, then $\chi'_g = \chi'_h$, and hence $\chi_g = \chi_h$, and it follows that $\theta_g \simeq \theta_h$. Conversely, assume that $\Gamma g \Gamma \neq \Gamma h \Gamma$, and let $f \in C(G)$ be such that $f_{|\Gamma g\Gamma} = 0$ and $f_{|\Gamma h\Gamma} = 1$. We have $\chi'_g(f) = 0$ and $\chi'_h(f) = 1$: this shows that $\chi_g \neq \chi_h$ and hence that θ_g and θ_h are not isomorphic. This proves (3).

For $g, h \in G$, let us show that $(\chi_g \otimes \chi_h)\Delta = \sum_{s\in\Gamma}\chi_{gsh}$. This will prove (4). For f in $\mathcal{R}(G)$, we have

$$\begin{aligned}(\chi_g \otimes \chi_h)\Delta([f]) = \chi_g([f_1])\chi_h([f_2]) &= \frac{1}{|\Gamma|^2}\sum_{r,s,t,u\in\Gamma} f_1(rgs)f_2(tsu)\\ &= \frac{1}{|\Gamma|^2}\sum_{r,s,t,u\in\Gamma} f(rgsthu) = \frac{1}{|\Gamma|}\sum_{r,s,u\in\Gamma} f(rgshu) = \sum_{s\in\Gamma}\chi_{gsh}([f])\end{aligned}$$

and we have the result by density of $\mathcal{R}(G)^\sigma$ in $C(G)^\sigma$.

Assume finally that Γ is abelian. Then $\mathcal{R}(\Gamma)$ is cocommutative and $\mathcal{R}(\Gamma)^\sigma = \mathcal{R}(\Gamma)$. For $s \in \Gamma$, the $*$-algebra map $\varepsilon_s : \mathcal{R}(G)^\sigma \to \mathbb{C}$ is obtained by composing the restriction $\mathcal{R}(G)^\sigma \to \mathcal{R}(\Gamma)^\sigma = \mathcal{R}(\Gamma)$ with the evaluation at s. For $s \in \Gamma$ and f in $\mathcal{R}(G)$, we have

$$\chi_s([f]) = \frac{1}{|\Gamma|}\sum_{r,t\in\Gamma} f(rst) = \frac{1}{|\Gamma|}\sum_{r,t\in\Gamma}\varepsilon_{rst}([f]) = \sum_{r\in\Gamma}\varepsilon_r([f])$$

and again we get the result by density of $\mathcal{R}(G)^\sigma$ in $C(G)^\sigma$. □

We arrive at a useful criterion to show that a quotient of a twisted function algebra on compact group is still a twisted function algebra on a compact subgroup.

Theorem 3.6. *Let G be a compact group and let σ be a unitary 2-cocycle on $\mathcal{R}(G)$ induced by a finite abelian subgroup $\Gamma \subset G$ such that ${}_\sigma\mathcal{R}(\Gamma)$ is a full matrix algebra. Let A be a Woronowicz algebra quotient of $C(G)^\sigma$. Then all the irreducible representations of the* C*-*algebra A have dimension $\leq |\Gamma|$, and if A has an irreducible representation of dimension $|\Gamma|$, then there exists a compact subgroup $\Gamma \subset K \subset G$ such that $A \simeq C(K)^\sigma$.*

Proof. We are in the situation of Proposition 3.5, since the algebras ${}_\sigma\mathcal{R}(\Gamma)$ and $\mathcal{R}(\Gamma)_{\sigma^{-1}}$ are anti-isomorphic. Thus if ρ is an irreducible representation of A of dimension $|\Gamma|$, then $\rho\pi$ is also an irreducible representation of $C(G)^\sigma$ (with $\pi : C(G)^\sigma \to A$ being the given quotient map), and so there exists $g \in G$ such that $\rho\pi \simeq \theta_g$. That is, θ_g factors through a representation of A. The isomorphisms from 3.5

$$\theta_g \otimes \theta_{g^{-1}} \simeq \oplus_{s\in\Gamma}\theta_{gsg^{-1}} \simeq \theta_1 \oplus (\oplus_{s\in\Gamma, s\neq 1}\theta_{gsh}) \simeq (\oplus_{s\in\Gamma}\varepsilon_s) \oplus (\oplus_{s\in\Gamma, s\neq 1}\theta_{gsh})$$

show that $\theta_{g^{-1}}$ is the dual of the representation θ_g of $C(G)^\sigma$. Thus, $\theta_{g^{-1}}$ factors through a representation of A, as do all the simple constituents of $\theta_g \otimes \theta_{g^{-1}}$. In particular, each ε_s, $s \in \Gamma$, defines a representation A, and we get a surjective $*$-algebra map $A \to \mathcal{R}(\Gamma)$, which is automatically a coalgebra map. We conclude by Proposition 3.1. □

4. Application to $SU_{-1}(2m+1)$ and $U_{-1}(2m+1)$

From now on we assume that $n = 2m + 1$ is odd. We recall how the quantum groups $SU_{-1}(2m+1)$ and $U_{-1}(2m+1)$ can be obtained by 2-cocycle deformation, using a 2-cocycle induced from the group $\mathbb{Z}_2^{2m}$, and then use the results of the previous section to get information on their quantum subgroups.

We denote by $\mathbb{Z}_2$ the cyclic group on two elements, and we use the identification

$$\mathbb{Z}_2^{2m} = \langle t_1, \ldots, t_{2m+1} \mid t_i t_j = t_i t_j,\ t_1^2 = \cdots = t_{2m+1}^2 = 1 = t_1 \cdots t_{2m+1}\rangle$$

Let $\sigma : \mathbb{Z}_2^{2m} \times \mathbb{Z}_2^{2m} \to \{\pm 1\}$ be the unique bicharacter such that

$$\sigma(t_i, t_j) = -1 = -\sigma(t_j, t_i) \text{ for } 1 \leq i < j \leq 2m$$

$$\sigma(t_i, t_i) = (-1)^m \text{ for } 1 \leq i \leq 2m+1$$

$$\sigma(t_i, t_{2m+1}) = (-1)^{m-i} = -\sigma(t_{2m+1}, t_i) \ \text{ for } 1 \leq i \leq 2m$$

It is well-known that the twisted group algebra $\mathbb{C}_\sigma\mathbb{Z}_2^{2m}$ is isomorphic to the matrix algebra $M_{2^m}(\mathbb{C})$.

There exists a surjective Hopf $*$-algebra morphism

$$\pi : \mathcal{R}(SU(2m+1)) \to \mathbb{C}\mathbb{Z}_2^{2m}$$
$$u_{ij} \longmapsto \delta_{ij} t_i$$

induced by the restriction of functions to Γ, the subgroup of $SU(2m+1)$ formed by diagonal matrices having ± 1 as entries, composed with the Fourier transform $\mathcal{R}(\Gamma) \simeq \mathbb{C}\widehat{\Gamma} \simeq \mathbb{C}\mathbb{Z}_2^{2m}$. Thus we may form the twisted Hopf algebra $\mathcal{R}(SU(2m+1))^\sigma$, and it is not difficult to check that there exists a surjective Hopf $*$-algebra map $\mathcal{R}(SU_{-1}(2m+1)) \to \mathcal{R}(SU(2m+1))^\sigma$, $u_{ij} \mapsto [u_{ij}]$, which is known to be an isomorphism (there are several ways to show this, a simple one being to invoke the presentation Theorem 3.5 in [17]). Hence we have $C(SU_{-1}(2m+1)) \simeq C(SU(2m+1))^\sigma$, with σ induced from the subgroup $\Gamma \simeq \mathbb{Z}_2^{2m}$, and we are in the framework of Theorem 3.6. Similarly $C(U_{-1}(2m+1)) \simeq C(U(2m+1))^\sigma$.

If K is a compact subgroup of $SU(2m+1)$ with $\Gamma \subset K$, we denote by K_{-1} the compact quantum group corresponding to the Woronowicz algebra $C(K)^\sigma$. With this language, the following result is an immediate consequence of Theorem 3.6.

Theorem 4.1. *Let G be a compact quantum subgroup of $SU_{-1}(2m+1)$. Then all the irreducible representations of the* C*-*algebra $C(G)$ *have dimension* $\leq 4^m$, *and if* $C(G)$ *has an irreducible dimension of dimension* 4^m, *then there exists a compact subgroup* $\Gamma \subset K \subset SU(2m+1)$ *such that* $G \simeq K_{-1}$.

A similar statement holds as well with $SU_{-1}(2m+1)$ replaced by $U_{-1}(2m+1)$.

5. Quantum subgroups of $SU_{-1}(3)$

This section is devoted to the proof of Theorem 1.1. The case studied here corresponds to $m = 1$ in Section 4. We first need some preliminary results, and we begin by fixing some notation.

For a permutation $\nu \in S_3$, we put

$$SU(3)^\nu = \{g = (g_{ij}) \in SU(3) \mid g_{ij} = 0 \text{ if } \nu(j) \neq i\}$$

and also

$$SU(3)^\Sigma = \cup_{\nu \in S_3} SU(3)^\nu.$$

For $g \in SU(3)^\Sigma$, we denote by ν_g the unique element of S_3 such that $g \in SU(3)^{\nu_g}$.

The following result is easily verified (and has an obvious generalization for any n).

Lemma 5.1. *Any element* $g = (g_{ij}) \in SU(3)^\Sigma$ *defines a* $*$-*algebra map* $\varepsilon_g : C(SU_{-1}(3)) \to \mathbb{C}$ *such that* $\varepsilon_g(u_{ij}) = \epsilon(\nu_g) g_{ij}$ *(where* $\epsilon(\nu_g)$ *is the signature of* ν_g*). Conversely any* 1*-dimensional representation of* $C(SU_{-1}(3))$ *arises in this way.*

As is the previous section, the subgroup of $SU(3)$ formed by diagonal matrices having ± 1 as entries is denoted Γ. In the case $g \in \Gamma$, then ε_g is of course the representation of the same name from Proposition 3.5.

We denote by $SU(3)_{\text{reg}}$ the subset of matrices in $SU(3)$ for which there exists a row or a column having no zero coefficient.

Recall from Section 4 and Proposition 3.5 that each $g \in SU(3)$ defines a representation

$$\theta_g : C(SU_{-1}(3)) \longrightarrow \mathbb{C}_\sigma\Gamma \otimes \mathbb{C}_\sigma\Gamma \simeq M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \simeq M_4(\mathbb{C})$$

The twisted group algebra $\mathbb{C}_\sigma\Gamma$ is presented by generators T_1, T_2, T_3 and relations $T_1^2 = -1 = T_2^2 = T_3^2$, $1 = T_1T_2T_3$, $T_iT_j = -T_jT_i$ if $i \neq j$ (where in the notation of the previous sections, $T_i = \{t_i\} = \langle t_i \rangle$). With this notation, the representation θ_g ($g \in SU(3)$) has the following form

$$\begin{aligned} \theta_g : C(SU_{-1}(3)) &\longrightarrow \mathbb{C}_\sigma\Gamma \otimes \mathbb{C}_\sigma\Gamma \\ u_{ij} &\longmapsto g_{ij} T_i \otimes T_j \end{aligned}$$

Lemma 5.2. *The representation* θ_g *is irreducible if and only if* $g \in SU(3)_{\text{reg}}$. *If* $g \in SU(3)^\Sigma$, *then* θ_g *is isomorphic to a direct sum of one-dimensional representations.*

Proof. The first assertion follows directly from (2) in Proposition 3.5. The second assertion follows from the fact that if $g \in SU(3)^\Sigma$, the algebra $\theta_g(C(SU_{-1}(3))$ is commutative (this is clear from the above description of θ_g). □

Our next aim is to describe the tensor products $\varepsilon_g \otimes \theta_h$.

Lemma 5.3. *Let* $g \in SU(3)^\Sigma$ *and let* $h \in SU(3)$. *Then the representations* $\varepsilon_g \otimes \theta_h$ *and* θ_{gh} *are isomorphic.*

Proof. Put $g = (\delta_{i,\nu(j)} a_i)$ with $\nu \in S_3$. We have, for any i, j,

$$(\varepsilon_g \otimes \theta_h)\Delta(u_{ij}) = \sum_k \varepsilon_g(u_{ik}) h_{kj} T_k \otimes T_j = \epsilon(\nu) a_i h_{\nu^{-1}(i)j} T_{\nu^{-1}(i)} \otimes T_j$$

It is straightforward to check that there exists an automorphism α_ν of $\mathbb{C}_\sigma\Gamma$ such that $\alpha_\nu(T_i) = \varepsilon(\nu) T_{\nu(i)}$ for any i. We have

$$\alpha_\nu \otimes \text{id}(\varepsilon_g \otimes \theta_h)\Delta(u_{ij}) = a_i h_{\nu^{-1}(i)j} T_i \otimes T_j = \theta_{gh}(u_{ij})$$

and hence, since α_ν is (necessarily) an inner automorphism of the matrix algebra $\mathbb{C}_\sigma\Gamma$, we conclude that the representations $\varepsilon_g \otimes \theta_h$ and θ_{gh} are isomorphic. $\square$

Before going into the proof of Theorem 1.1, we need a final piece of notation. For $1 \leq i, j \leq 3$, we put

$$SU(3)^{[i,j]} = \{g = (g_{ij}) \in SU(3) \mid g_{ik} = 0 \text{ if } k \neq j,\ g_{kj} = 0 \text{ if } i \neq k,\ g \notin SU(3)^\Sigma\}$$

Proof of Theorem 1.1. Let $G \subset SU_{-1}(3)$ be a non-classical compact quantum subgroup, with corresponding surjective Woronowicz algebra map $\pi : C(SU_{-1}(3)) \to C(G)$. Recall that we have to prove that one of the following assertion holds.

(1) There exists a compact subgroup $\Gamma \subset K \subset SU(3)$ such that G is isomorphic to K_{-1}.
(2) G is isomorphic to a quantum subgroup of $U_{-1}(2)$.

We already know from Theorem 4.1 that if $C(G)$ has an irreducible representation of dimension 4, then (1) holds. So we assume that $C(G)$ has all its irreducible representation of dimension < 4.

We denote by X the set of (isomorphism classes) of irreducible representations of $C(G)$ having dimension d satisfying $1 < d < 4$. We remark that X is non-empty since $C(G)$ is non-commutative.

Let $\rho \in X$. Then ρ defines an irreducible representation $\rho\pi$ of $C(SU_{-1}(3))$, and hence by Proposition 3.5 there exists $g \in SU(3)$ such that $\rho\pi \prec \theta_g$. If $g \in SU(3)_{\text{reg}}$, then by Lemma 5.2 θ_g is irreducible and $\rho\pi \simeq \theta_g$ has dimension 4, which contradicts our assumptions. Hence $g \notin SU(3)_{\text{reg}}$. If $g \in SU(3)^\Sigma$, then by Lemma 5.2 θ_g is a direct sum of representations of dimension 1, hence ρ has dimension 1, which again contradicts our assumption, and hence $g \notin SU(3)^\Sigma$. Thus there exist i, j such that $g \in SU(3)^{[i,j]}$. Suppose that $i \neq j$. Then $\rho\pi \otimes \rho\pi \prec \theta_g \otimes \theta_g \simeq \oplus_{s\in\Gamma}\theta_{gsg}$ (by Proposition 3.5). For any $s \in \Gamma$, $sg \in SU(3)^{[i,j]}$ and it is a direct matrix computation to check that $gsg \in SU(3)_{\text{reg}}$, so the constituents of this decomposition are irreducible representations. By a dimension argument there exists $s \in \Gamma$ such that $\rho\pi \otimes \rho\pi \simeq \theta_{gsg}$, and hence $\rho \otimes \rho$ is irreducible of dimension 4; this is a contradiction.

We have thus proved that for any $\rho \in X$, there exists $i \in \{1, 2, 3\}$ and $g \in SU(3)^{[i,i]}$ such that $\rho\pi \prec \theta_g$. Assume that there exist $\rho, \rho' \in X$ with $\rho\pi \prec \theta_g$, $\rho'\pi \prec \theta_{g'}$ for $g \in SU(3)^{[i,i]}$, $g' \in SU(3)^{[j,j]}$ and $i \neq j$. Then $\rho\pi \otimes \rho'\pi \prec \theta_g \otimes \theta_{g'} \simeq \oplus_{s\in\Gamma}\theta_{gsg'}$. Once again, for any $s \in \Gamma$, $gsg' \in SU(3)_{\text{reg}}$, and we conclude as before that $\rho \otimes \rho'$ is an irreducible representation of dimension 4, a contradiction.

Thus we have proved that there exists $i \in \{1, 2, 3\}$ such that for any $\rho \in X$, we have $\rho\pi \prec \theta_g$ for some $g \in SU(3)^{[i,i]}$, and hence $\rho\pi(u_{ik}) = 0 = \rho\pi(u_{ki})$ for any $k \neq i$ and $\rho \in X$.

Let ϕ be a 1-dimensional representation of $C(G)$. By Lemma 5.1, there exists $\nu \in S_3$ and $g \in SU(3)^\nu$ such that $\phi\pi = \varepsilon_g$. Let $\rho \in X$ with $\rho\pi \prec \theta_h$ for $h \in SU(3)^{[i,i]}$. Then $\phi\pi \otimes \rho\pi \prec \varepsilon_g \otimes \theta_h \simeq \theta_{gh}$ by Lemma 5.3. It is straightforward to check that $gh \in SU(3)^{[\nu(i),i]}$. By a previous case we must have $\nu(i) = i$. Hence $\phi\pi(u_{ik}) = 0 = \phi\pi(u_{ki})$ for any $k \neq i$.

Summarizing, we have shown that for any $\rho \in \widehat{C(G)}$, we have $\rho\pi(u_{ik}) = 0 = \rho\pi(u_{ki})$ for any $k \neq i$. The irreducible representations of a C*-algebra separate its elements, so we conclude that $\pi(u_{ik}) = 0 = \pi(u_{ki})$ for any $k \neq i$, and by Lemma 2.5, we are in situation (2). This concludes the proof. $\square$

Corollary 5.4. *Let G be a non-classical compact quantum subgroup of $SU_{-1}(3)$ acting irreducibly on $\mathbb{C}^3$. Then G is isomorphic to a K_{-1}, a twist at -1 of a compact subgroup $K \subset SU(3)$ containing the subgroup of diagonal matrices having ± 1 as entries, and acting irreducibly on $\mathbb{C}^3$.*

Proof. We have shown in the previous proof that if $C(G)$ does not have an irreducible representation of dimension 4, then the fundamental 3-dimensional representation of G is not irreducible. Thus if G acts irreducibly on $\mathbb{C}^3$, there exist an irreducible representation of dimension 4 of $C(G)$ and a compact subgroup $\Gamma \subset K \subset SU(3)$ such that G is isomorphic to K_{-1}, and K acts irreducibly on $\mathbb{C}^3$ since G does. $\square$

Remark 5.5. The proof of Theorem 1.1 works as well by replacing $SU(3)$ by $SO(3)$. In particular one recovers, under a less precise form, the results of [3]: if $G \subset SO_{-1}(3)$ is a non-classical compact quantum subgroup, then either there exists a compact subgroup $\Gamma \subset K \subset SO(3)$ such that G is isomorphic to K_{-1} or G is isomorphic to a quantum subgroup of $O_{-1}(2)$.

Remark 5.6. Corollary 5.4 also holds with $SU_{-1}(3)$ replaced by $U_{-1}(3)$ (and $SU(3)$ replaced by $U(3)$), with a similar proof.

Remark 5.7. Although $U_{-1}(2)$ is a twist of $U(2)$ at -1 (by the subgroup of diagonal matrices having ± 1 as entries), it is not true that all of its quantum subgroups are twists of subgroups of $U(2)$ at -1: for instance, the quantum group $SU_{-1}(2)$ is not. In fact, the techniques of Section 3 above are not appropriate for this example: they furnish representations of dimension 4, which cannot be irreducible since the irreducible representations of $C(U_{-1}(2))$ all have dimension 1 or 2 (as an algebra, one has $C(U_{-1}(2)) \simeq C(SU_{-1}(2)) \otimes C(\mathbb{T})$).

One can use the techniques from [8] (see Theorem 3.5) to describe the finite quantum subgroups. In general, it seems to us that the best way to analyse the quantum subgroups of $U_{-1}(2)$ is to follow the same method used by Podleś in [23], as follows.

(1) The first step is to give a parameterization of the irreducible representations of $C(U_{-1}(2))$ by elements of $U(2)$, with the dimension 1 representations corresponding to diagonal or anti-diagonal matrices, and the dimension 2 representations corresponding to matrices having no zero coefficient (let us denote by $U(2)_{\text{reg}}$ the set of such matrices).
(2) The second step consists of describing the fusion rules of the irreducible representations of $C(U_{-1}(2))$.
(3) With this information, one can deduce that the non commutative Woronowicz algebra quotients $C(U_{-1}(2)) \to A$ correspond, via their spectrum, to the closed subspaces $X \subseteq U(2)$ satisfying the following conditions:
    (a) $X \cap U(2)_{\text{reg}} \neq \emptyset$,
    (b) If $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in X \cap U(2)_{\text{reg}}$, then for any ϵ_1, $\epsilon_2 \in \{\pm 1\}$,
$$\begin{pmatrix} \epsilon_1 g_{11} & \epsilon_2 g_{12} \\ \epsilon_2 g_{21} & \epsilon_1 g_{22} \end{pmatrix} \in X \cap U(2)_{\text{reg}},$$
    (c) $\pm I_2 \in X$,
    (d) X is stable under the Podleś law
$$\begin{aligned} U(2) \times U(2) &\longrightarrow U(2) \\ (g,h) &\longmapsto \begin{cases} -gh & \text{if } g,\, h \text{ are anti-diagonal} \\ gh & \text{otherwise.} \end{cases} \end{aligned}$$

Note that if X satisfies (c), then it satisfies (d) if and only X is a subgroup of $U(2)$. If, in addition, it contains the group of all diagonal matrices having ± 1 as entries, then it automatically satisfies (b): this is the case which corresponds to twists of classical subgroups of $U(2)$.

We do not know a simple way to express the quantum subgroups of $U_{-1}(2)$ in terms of those of $SU_{-1}(2)$ by using a crossed coproduct, as is done in [15, Corollary 4.8] for the case $q \neq \pm 1$. In this respect, the case $q = -1$ is similar to the classical case $q = 1$, where only short exact sequences can be written in general.

As a last remark concerning the $U_{-1}(2)$ case, we would like to point out that an alternative approach might be to use the results in [7]: the Hopf $*$-algebra $\mathcal{R}(U_{-1}(2))$ is a quotient of the Hopf algebra $A_u^{**}(2)$ defined in Example 4.10 of [7], and hence it can be seen as a half-commutative orthogonal Hopf algebra. The results in [7] then give a parameterization the quantum subgroups of $U_{-1}(2)$ in terms of certain subgroups of $U(4)$. This parameterization is certainly not the most convenient.

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