Sums of two S-units via Frey-Hellegouarch curves
(joint work with Michael A. Bennett)

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- B. M. M. de Weger (around 1990) : algorithm ; complete list of solutions for $S = \{2, 3, 5, 7\}$.

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How many primitive solutions are there ?

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Since $2 \notin S$, then z is necessarily *even* and E has multiplicative reduction at 2 with $\mathrm{ord}_2(\Delta(E)) \equiv 0 \pmod{n}$.

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The set of primitive solutions is finite if and only if $2 \notin S$.

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Proposition

For $S = \{2, 3, 5, 7\}$ and $S = \{2, 3, p\}$ $p < 100$ prime, the conductor $N(E)$ of E satisfies $N(E) < 350,000$ unless the solution corresponds to :

$$3^4 + 1 = 2 \cdot 41, \quad 2 \cdot 3^3 - 1 = 53, \quad 3^{10} - 1 = 2 \cdot 61 \cdot 22^2, \quad 3^4 + 2 = 83,$$

$$3^4 - 2 = 79, \quad 3^5 + 1 = 61 \cdot 2^2, \quad 2^3 \cdot 3^2 + 1 = 73, \quad 2^3 \cdot 3^2 - 1 = 71,$$

$$3^4 + 2^3 = 89, \quad 3^4 - 2^3 = 73.$$

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Pretending this is a $(n, n, 3)$ generalized Fermat equation bounds n :

$$n \leq (1 + \sqrt{2})^{(N_0+1)/6}$$

where $N_0 = 3^5 \cdot 5^2 \cdot 7^2$.

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We have to lower this upper bound !

Preliminary results

Lemma

Let x, y be coprime nonzero integers and $n \geq 5$ be a prime number. Then the greatest prime divisor $P(x^n + y^n)$ of $x^n + y^n$ satisfies

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Proposition

Let x and w be coprime positive $\{3, 5, 7\}$-units, $y = \pm 1$ and let $n \geq 5$ be a prime number such that $x + y = wz^n$ for some positive integer z. Then, $y = -1$, $n = 5$, $z = 2$ and

$$(x, w) = (7^4, 3 \cdot 5^2) \quad \text{or} \quad (3^2 \cdot 5^2, 7).$$

Remaining equations

$$
\begin{aligned}
3^\alpha 5^\beta + (-1)^\delta 7^\gamma = z^n, &\quad (\alpha,\beta) \not\equiv (0,0) \mod n, \quad \gamma \not\equiv 0 \mod n \\
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5^\beta 7^\gamma + (-1)^\delta 3^\alpha = z^n, &\quad (\beta,\gamma) \not\equiv (0,0) \mod n, \quad \alpha \not\equiv 0 \mod n \\
3^\alpha + (-1)^\delta 5^\beta = 7^\gamma z^n, &\quad \alpha,\beta > 0, \quad 0 < \gamma \le n-1 \\
3^\alpha + (-1)^\delta 7^\gamma = 5^\beta z^n, &\quad \alpha,\gamma > 0 \quad 0 < \beta \le n-1 \\
5^\beta + (-1)^\delta 7^\gamma = 3^\alpha z^n, &\quad \beta,\gamma > 0, \quad 0 < \alpha \le n-1
\end{aligned}
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where α, β and γ are nonnegative integers, $n \ge 5$ is prime and $\delta \in \{0,1\}$.

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   - 1.b) Assume α and β odd.
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     - 1.b.ii) If $\gamma = 1$ and $n = 7$, then $\delta = 1$ and

$$\text{sieving modulo } 2^3 \cdot 7^2 \cdot 29 \cdot 43 \cdot 113 \Rightarrow (\alpha, \beta) \equiv (3, 1) \pmod 7.$$

$\rightsquigarrow$ Thue equation $z^7 - 3^3 5 y^7 = 7$
$\rightsquigarrow$ solution $3^3 \cdot 5 - 7 = 2^7$.

A sample equation : $3^\alpha 5^\beta + (-1)^\delta 7^\gamma = z^n$

Modulo 8 : $\alpha \equiv \beta \pmod 2 + (\alpha \text{ is odd}) \Leftrightarrow (\delta \equiv \gamma \pmod 2)$.

1. Assume $n \geq 7$
   - 1.a) Assume α and β even.
     - 1.a.i) If $\beta > 0$, then $(n, n, 2)$ (with square $= 3^\alpha 5^\beta$) + level lowering (to level $N_0 = 14$) $\rightsquigarrow$ contradiction.
     - 1.a.ii) If $\beta = 0$ and $n = 7$, then local sieve modulo 49 (and 43 if $\gamma = 1$) $\rightsquigarrow$ contradiction.
     - 1.a.iii) If $\beta = 0$ and $n \geq 11$, then $(n, n, 3)$ (with cube $= 3^\alpha 5^\beta$) + level lowering (to $N_0 \mid 3 \cdot 7^2$ or $3^3 \cdot 7^2$) $\rightsquigarrow$ contradiction.
   - 1.b) Assume α and β odd.
     - 1.b.i) If $\gamma \geq 2$ and $n = 7$, then local sieve modulo $2^3 \cdot 7^2 \cdot 29 \cdot 43 \cdot 71$ $\rightsquigarrow$ contradiction.
     - 1.b.ii) If $\gamma = 1$ and $n = 7$, then $\delta = 1$ and

       sieving modulo $2^3 \cdot 7^2 \cdot 29 \cdot 43 \cdot 113 \Rightarrow (\alpha, \beta) \equiv (3, 1) \pmod 7$.

       $\rightsquigarrow$ Thue equation $z^7 - 3^3 5 y^7 = 7$
       $\rightsquigarrow$ solution $3^3 \cdot 5 - 7 = 2^7$.

A sample equation : $3^\alpha 5^\beta + (-1)^\delta 7^\gamma = z^n$ (continued)

1.b.iii) If $n \geq 11$, then $(n, n, 3)$ (with cube $= (-1)^\delta 7^\gamma$) + level lowering (to $N_0 \mid 3 \cdot 5 \cdot 7^2$ or $N_0 = 3^4 \cdot N_1$ with $N_1 \mid 5 \cdot 7^2$, if $\alpha \geq 3$ or $\alpha = 1$ respectively) $\Rightarrow \alpha = 1$ and $n = 11$.

A sample equation : $3^\alpha 5^\beta + (-1)^\delta 7^\gamma = z^n$ (continued)

1.b.iii) If $n \geq 11$, then $(n, n, 3)$ (with cube $= (-1)^\delta 7^\gamma$) + level lowering (to $N_0 \mid 3 \cdot 5 \cdot 7^2$ or $N_0 = 3^4 \cdot N_1$ with $N_1 \mid 5 \cdot 7^2$, if $\alpha \geq 3$ or $\alpha = 1$ respectively) $\Rightarrow \alpha = 1$ and $n = 11$. $(n, n, n) \rightsquigarrow 23 \nmid z$ and $67 \nmid z$.

1.b.iii) If $n \geq 11$, then $(n, n, 3)$ (with cube $= (-1)^\delta 7^\gamma$) + level lowering (to $N_0 \mid 3 \cdot 5 \cdot 7^2$ or $N_0 = 3^4 \cdot N_1$ with $N_1 \mid 5 \cdot 7^2$, if $\alpha \geq 3$ or $\alpha = 1$ respectively) $\Rightarrow \alpha = 1$ and $n = 11$.
$(n, n, n) \rightsquigarrow 23 \nmid z$ and $67 \nmid z$.
Sieve modulo 23 and 67 $\rightsquigarrow$ contradiction.

1.b.iii) If $n \geq 11$, then $(n, n, 3)$ (with cube $= (-1)^\delta 7^\gamma$) + level lowering (to $N_0 \mid 3 \cdot 5 \cdot 7^2$ or $N_0 = 3^4 \cdot N_1$ with $N_1 \mid 5 \cdot 7^2$, if $\alpha \geq 3$ or $\alpha = 1$ respectively) $\Rightarrow \alpha = 1$ and $n = 11$.
$(n, n, n) \rightsquigarrow 23 \nmid z$ and $67 \nmid z$.
Sieve modulo 23 and 67 $\rightsquigarrow$ contradiction.

2. Assume $n = 5$

1.b.iii) If $n \geq 11$, then $(n, n, 3)$ (with cube $= (-1)^\delta 7^\gamma$) + level lowering (to $N_0 \mid 3 \cdot 5 \cdot 7^2$ or $N_0 = 3^4 \cdot N_1$ with $N_1 \mid 5 \cdot 7^2$, if $\alpha \geq 3$ or $\alpha = 1$ respectively) $\Rightarrow \alpha = 1$ and $n = 11$.
$(n, n, n) \rightsquigarrow 23 \nmid z$ and $67 \nmid z$.
Sieve modulo 23 and 67 $\rightsquigarrow$ contradiction.

2. Assume $n = 5 \rightsquigarrow$ Thue-Mahler equation $z^5 - 3^a 5^b y^5 = 7^c$

1.b.iii) If $n \geq 11$, then $(n, n, 3)$ (with cube $= (-1)^\delta 7^\gamma$) + level lowering (to $N_0 \mid 3 \cdot 5 \cdot 7^2$ or $N_0 = 3^4 \cdot N_1$ with $N_1 \mid 5 \cdot 7^2$, if $\alpha \geq 3$ or $\alpha = 1$ respectively) $\Rightarrow \alpha = 1$ and $n = 11$.
$(n, n, n) \rightsquigarrow 23 \nmid z$ and $67 \nmid z$.
Sieve modulo 23 and 67 $\rightsquigarrow$ contradiction.

2. Assume $n = 5 \rightsquigarrow$ Thue-Mahler equation $z^5 - 3^a 5^b y^5 = 7^c$
Hambrook's Magma implementation (2012) of de Weger and Tzanakis' algorithm for solving Thue-Mahler equations (1992)

1.b.iii) If $n \geq 11$, then $(n, n, 3)$ (with cube $= (-1)^\delta 7^\gamma$) + level lowering (to $N_0 \mid 3 \cdot 5 \cdot 7^2$ or $N_0 = 3^4 \cdot N_1$ with $N_1 \mid 5 \cdot 7^2$, if $\alpha \geq 3$ or $\alpha = 1$ respectively) $\Rightarrow \alpha = 1$ and $n = 11$.
$(n, n, n) \rightsquigarrow 23 \nmid z$ and $67 \nmid z$.
Sieve modulo 23 and 67 $\rightsquigarrow$ contradiction.

2. Assume $n = 5 \rightsquigarrow$ Thue-Mahler equation $z^5 - 3^a 5^b y^5 = 7^c$
Hambrook's Magma implementation (2012) of de Weger and Tzanakis' algorithm for solving Thue-Mahler equations (1992) $\rightsquigarrow$ solutions :

$$3 \cdot 5^3 - 7^3 = 5^2 + 7 = 3^4 - 7^2 = 2^5.$$

Main Theorem 1

The only primitive solutions to equation $x + y = z^n$ with $S = \{3, 5, 7\}$ and $x > |y| > 0$ are given by

$$(x, y) = (3, 1),\ (5, -1),\ (5, 3),\ (7, -3),\ (7, 1),\ (9, -5),\ (9, -1),\ (9, 7),\ (15, -7),\ (15, 1),\ (21, -5),\ (21, 15),\ (25, -21),\ (25, -9),\ (25, 7),\ (27, 5),\ (35, -27),\ (35, -3),\ (35, 1),\ (49, -45),\ (49, 15),\ (63, 1),\ (81, -49),\ (105, -5),\ (125, 3),\ (135, -35),\ (135, -7),\ (147, -3),\ (175, 21),\ (175, 81),\ (189, -125),\ (189, 7),\ (225, -9),\ (343, -243),\ (375, -343),\ (405, -5),\ (441, -225),\ (625, -49),\ (675, 1),\ (729, -245),\ (1029, -5),\ (1225, -225),\ (1323, -27),\ (1875, -147),\ (3375, 2401),\ (3969, -1225),\ (3969, -125),\ (9375, 1029),\ (10125, -125),\ (15625, -1701),\ (50625, -3969),\ (59535, 1),\ (540225, -2401),\ (688905, -5),\ (4782969, 4375) \text{ and } (24310125, -10125).$$

Main results (continued)

Main Theorem 2

The only primitive solutions to equation $x + y = z^n$ with $S = \{2, 3\}$ and, say, $x \geq |y| > 0$ and $z > 0$ are given by the following infinite families

$$(x, y, z, n) = (2, -1, 1, n),\ (3, -2, 1, n),\ (4, -3, 1, n),\ (9, -8, 1, n),$$
$$(2^{n-1}, 2^{n-1}, 2, n),\ (3 \cdot 2^{n-2}, 2^{n-2}, 2, n),\ (3 \cdot 2^{n-1}, -2^{n-1}, 2, n),$$
$$(2 \cdot 3^{n-1}, 3^{n-1}, 3, n),\ (2^2 \cdot 3^{n-1}, -3^{n-1}, 3, n),$$
$$(2^3 \cdot 3^{n-2}, 3^{n-2}, 3, n), \quad \text{all with } n \geq 2,$$
$$(x, y, z, n) = (3^2 \cdot 2^{n-3}, -2^{n-3}, 2, n) \quad \text{for } n \geq 3$$

and by

$$(x, y, z, n) = (16, 9, 5, 2),\ (18, -2, 4, 2),\ (24, 1, 5, 2),$$
$$(27, -2, 5, 2),\ (81, -32, 7, 2),\ (48, 1, 7, 2),\ (128, -3, 5, 3),$$
$$(288, 1, 17, 2) \text{ and } (486, -2, 22, 2).$$