Numerical approximation of periodic solutions for dissipative hyperbolic equations

Nicolae Cîndea
joint work with S. Micu and J. Morais Pereira

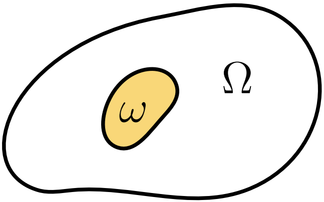
Monastir, 27/05/2015

A dissipative wave equation

$$\left\{\begin{array}{ll} \ddot{w}(t,x) - \Delta w(t,x) + a\dot{w}(t,x) = f(t,x), & \text{in } (0,\infty)\times\Omega \\ w(t,x) = 0, & \text{in } (0,\infty)\times\partial\Omega \\ w(0,x) = w_0(x), \quad \dot{w}(0,x) = w_1(x), & \text{in } \Omega \end{array}\right. \qquad \text{(D)}$$

Hypotheses:

- Ω and $\omega \subset \Omega$ are two open sets in $\mathbb{R}^d$ with C^1 boundaries
- $a \in C^1(\overline{\Omega})$, $a(x) \geq 0, \quad \forall x \in \Omega$
  $a(x) > 0, \quad \forall x \in \omega$
- f is a T-periodic function with $T > 0$



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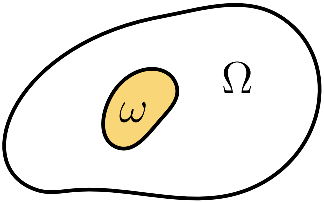

Question

There exists a T-periodic solution w of (D)?

Observability and existence of periodic solutions

We consider the following wave equation:

$$\begin{cases} \ddot{u}(t,x) - \Delta u(t,x) = 0, & (t,x) \in (0,\infty) \times \Omega \\ u(t,x) = 0, & (t,x) \in (0,\infty) \times \partial\Omega \\ u(0,x) = u_0(x),\ \dot{u}(0,x) = u_1(x), & x \in \Omega. \end{cases} \qquad (\star)$$

Inequality of observability

We say that $(\star)$ is observable in time $T_1 > 0$, with the observation $y(t) = a\dot{u}(t)$, if there exists a constant $k_{T_1} > 0$ such that

$$\int_0^{T_1} |a(x)\dot{u}(t,x)|^2 dt \geq k_{T_1} \left(\|u_0\|^2_{H_0^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)} \right)$$

for every $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

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for every $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$.

Theorem

If $(\star)$ is observable in a time $T_1 > 0$ and $f \in C(0,T; H^1_0(\Omega))$ is T-periodic then there exists a unique T-periodic solution w of (D).

Observability $\Rightarrow$ existence of periodic solutions

Sketch of the proof (1)

We denote $\Lambda : H_0^1(\Omega) \times L^2(\Omega) \to H_0^1(\Omega) \times L^2(\Omega)$:

$$\Lambda(w_0, w_1) = (w(T, \cdot), \dot{w}(T, \cdot))$$

where w is solution of

$$\begin{cases} \ddot{w}(t,x) - \Delta w(t,x) + a(x)\dot{w}(t,x) = f(t,x), & (t,x) \in (0,\infty) \times \Omega \\ w(t,x) = 0, & (t,x) \in (0,\infty) \times \partial\Omega \\ w(0,x) = w_0(x), \quad \dot{w}(0,x) = w_1(x), & x \in \Omega. \end{cases}$$

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Idea of the proof

Show that Λ have a fixed point.

By Duhamel's formula we have:

$$\Lambda(w_0, w_1) = \mathbb{S}(T)(w_0, w_1) + \int_0^T \mathbb{S}(T-t)(0, f(t,\cdot))dt$$

- $(\mathbb{S}(t))_{t\geq 0}$ is the semi-group associated to the dissipative wave equation.

In fact, we show that there exists a $n \in \mathbb{N}^*$ such that Λ^n is a contraction:

$$\Lambda^n(w_0, w_1) = \mathbb{S}(nT)(w_0, w_1) + \int_0^{nT} \mathbb{S}(nT-t)(0, f(t,\cdot))dt$$

and

$$\|\Lambda^n(w_0, w_1) - \Lambda^n(z_0, z_1)\|_{H_0^1\times L^2} = \|\mathbb{S}(nT)(w_0 - z_0, w_1 - z_1)\|_{H_0^1\times L^2}.$$

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Observability $\iff$ Stability :
there exist $M > 0$ and $\mu > 0$ such that

$$\|\mathbb{S}(t)(w_0, w_1)\|_{H_0^1 \times L^2} \leq M e^{-\mu t} \|(w_0, w_1)\|_{H_0^1 \times L^2}, \qquad \forall t \geq 0$$

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For n large enough Λ^n is a contraction.
Let $(\widehat{w}_0, \widehat{w}_1)$ be the unique fixed point of Λ^n.
Then $(\widehat{w}_0, \widehat{w}_1)$ is a fixed point for Λ.

□

Plan

Numerical analysis of the problem

A particular case: monochromatic sources

Numerical results

Perspectives and conclusions

Plan

Some notation

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$$\|\pi_h\varphi - \varphi\|_{H_0^1} \leq C_0 h^\theta \|\varphi\|_{H^2\cap H_0^1}, \qquad (\varphi \in H^2(\Omega)\cap H_0^1(\Omega)),$$
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- discretized equation:

$$\begin{cases} \ddot{w}_h(t) + A_h w_h(t) + B_h B_h^* \dot{w}_h(t) = f_h(t), & (t > 0) \\ w_h(0) = w_{0h}, \quad \dot{w}_h(0) = w_1. \end{cases} \qquad (D_h)$$

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- $(\mathbb{S}_h(t))_{t\geq 0}$ denotes the semi-group associated to the discrete equation (D_h).
- $\Lambda_h : V_h \times V_h \to V_h \times V_h, \quad \Lambda_h(w_{0h}, w_{1h}) = (w_h(T), \dot{w}_h(T))$.

Energy associated to the discrete system

The discrete energy corresponding to (D_h) is defined by

$$E_h(t) = \frac{1}{2}\left(\|A_h^{\frac{1}{2}} w_h\|^2 + \|\dot{w}_h\|^2\right).$$

If $f_h = 0$, then taking the inner product by $\dot{w}_h$ in equation (D_h), we deduce that

$$\frac{dE_h}{dt}(t) = -\|B^*\dot{w}_h(t)\|_U^2 \qquad (t \geq 0).$$

Thus, if $f_h = 0$, the energy E_h is non increasing.

Hypothesis

We shall suppose that the following holds

$$\lim_{t\to\infty} E_h(t) = 0.$$

Existence of periodic solutions of discrete system

Theorem (N.C., S. Micu, J. Morais (2013))

Let $h > 0$. *Assume that* $\lim_{t\to\infty} E_h(t) = 0$ *and that* $f_h \in C([0,\infty); V_h)$ *is* T*-periodic.*
There exists a unique $(\widehat{w}_h^0, \widehat{w}_h^1) \in V_h^2$ *such that the corresponding solution* $(\widehat{w}_h, \dot{\widehat{w}}_h) \in C^1\left([0,\infty); V_h^2\right)$ *of* (D_h) *with initial data* $(\widehat{w}_h^0, \widehat{w}_h^1)$ *is* $T-$*periodic.*

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Idea of the proof:

- fixed-point algorithm
- decay of the discrete energy: for every $h > 0$ there exists constants $M > 0$ and $\omega(h) > 0$ such that

$$E_h(t) \leq M^2 E_h(0) e^{-2\omega(h)t}.$$

□

Some remarks

1. The following are equivalent:
   - $\lim_{t \to \infty} E_h(t) = 0$.
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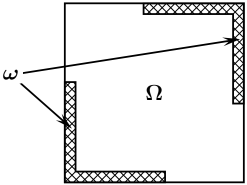

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   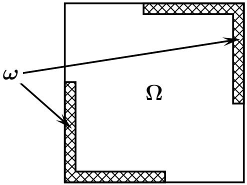

3. In general case $\omega(h) \to 0$ when $h \to 0$.

A discretization with vanishing viscosity

We consider the following discretization of (D):

$$\begin{cases} \ddot{w}_h(t) + A_h w_h(t) + B_h B_h^* \dot{w}_h(t) + \vartheta h^\eta A_h \dot{w}_h(t) = f_h(t) \\ w_h(0) = w_{0h}, \quad \dot{w}_h(0) = w_1. \end{cases} \qquad (D_{h\vartheta})$$

- $\vartheta \in [0, 1]$
- $\eta > 0$
- $(\mathbb{S}_{h\vartheta})_{t \geq 0}$ the associated semi-group
- $\Lambda_{h\vartheta} : V_h \times V_h \to V_h \times V_h$, $\Lambda_{h\vartheta}(w_{0h}, w_{1h}) = (w_h(T), \dot{w}_h(T))$ where w_h is the solution of $(D_{h\vartheta})$.
- if $f_h \equiv 0$ then

$$\frac{dE_h}{dt}(t) = -\|B^* \dot{w}_h(t)\|^2 - \vartheta h^\eta \|A_h^{\frac{1}{2}} \dot{w}_h(t)\|^2 \leq 0.$$

Theorem (N.C., S. Micu, J. Morais (2013))

Let $h \in (0, h^*)$, $\vartheta > 0$ and $\eta = \theta$.
Furthermore, assume that $f_h \in C([0,\infty); V_h)$ is a T-periodic function.
Then there exists a unique $(\widehat{w}_h^0, \widehat{w}_h^1) \in V_h \times V_h$ such that the corresponding solution $(\widehat{w}_h, \dot{\widehat{w}}_h) \in C^1([0,\infty); V_h \times V_h)$ of $(D_{h\vartheta})$ with initial data $(\widehat{w}_h^0, \widehat{w}_h^1)$ is $T-$periodic.

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$$E_h(t) \leq M^2 E_h(0) e^{-2\omega t}. \quad \square$$

Some error estimates — non-viscous case

Theorem (N.C., S. Micu, J. Morais (2013))

Let f be a T-periodic function such that $f_{|_{[0,T]}} \in W^{1,1}(0,T;H_0^1)$. Assume that $BB^* \in \mathcal{L}(H^2 \cap H_0^1, H_0^1)$ and $\lim_{t\to\infty} E_h(t) = 0$.

Let $\widehat{U}^0$ and $\widehat{U}_h^0$ be the unique fixed points of Λ and Λ_h. Then there exists a constant $C > 0$ such that, for each $n \geq 1$ and $h < h^*$, the following estimate holds

$$\|\widehat{U}^0 - \widehat{U}_h^0\|_X \leq C \left(n\, h^\theta + \frac{q^n}{1-q} + \frac{q_h^n}{1-q_h} \right) \|f\|_{W^{1,1}(0,T;H_0^1)},$$

where $q = e^{-\omega T}$ and $q_h = e^{-\omega(h)T}$.

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where $q = e^{-\omega T}$ and $q_h = e^{-\omega(h)T}$.

Remark: This result does not ensure the convergence of $(\widehat{U}_h^0)_{h>0}$ to $\widehat{U}^0$. Indeed, since $q_h = e^{-\omega(h)T}$ may tend to 1, the terms nh^θ and $\frac{q_h^n}{1-q_h}$ may not tend simultaneously to zero as h does.

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Some error estimates — viscous case

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Let f be a T-periodic function such that $f_{|_{[0,T]}} \in W^{1,1}(0,T;H_0^1)$. Assume that $BB^* \in \mathcal{L}(H^2 \cap H_0^1, H_0^1)$, $\vartheta > 0$ and $\eta = \theta$. Let $\widehat{U}^0$ and $\widehat{U}_h^0$ be the unique fixed points of Λ and $\Lambda_{h\vartheta}$. Then there exists a constant $C > 0$ such that, for each $n \geq 1$ and $h < h^*$, the following estimate holds

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where $q = e^{-\omega T}$ and $r = e^{-\omega(h,\vartheta)T}$.

Remark: This result does ensures the convergence of $(\widehat{U}_h^0)_{h>0}$ to $\widehat{U}^0$. Indeed, Indeed, by taking $n = \left[\frac{\theta}{|\ln(\max\{q,r\})|} \ln\left(\frac{1}{h}\right) \right] + 1$, we obtain $\|\widehat{U}^0 - \widehat{U}_h^0\|_X \leq C h^\theta \, \|f\|_{W^{1,1}(0,T;H_0^1)} \qquad (n \geq 1)$.

Plan

A particular case: monochromatic source terms

We suppose that the nonhomogeneous periodic term f has the following particular form

$$f(t,x) = e^{i\varsigma t} g(x),$$

where $\varsigma \in \mathbb{R}$ and $g \in L^2(\Omega)$.
Evidently, these functions are periodic of period $T = \frac{2\pi}{\varsigma}$ and are usually called *monochromatic*.
They appear in many important applications including acoustic, electromagnetic and geophysical wave propagation.

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They appear in many important applications including acoustic, electromagnetic and geophysical wave propagation.
For instance, the wave equation

$$w_{tt}(t,x) - \Delta w(t,x) = e^{i\varsigma t} g(x), \qquad (x \in \Omega,\ t > 0)$$

has a periodic solution $w = e^{i\varsigma t} u$ if and only if u verifies the Helmholtz's equation

$$(\varsigma^2 + \Delta) u(x) = -g(x), \qquad (x \in \Omega).$$

C. Bardos and J. Rauch.
Variational algorithms for the Helmholtz equation using time evolution and artificial boundaries.
Asymptotic Anal., 9 (1994), pp. 101-117.

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Convergence of fixed points

Theorem (N. C., S. Micu, J. Morais (2013))

Let $\varsigma \in \mathbb{R}$, $T = \frac{2\pi}{\varsigma}$, $g \in H_0^1$ *and let* $f \in W^{1,1}(0,T;H_0^1)$ *given by*

$$f(t,x) = e^{i\varsigma t} g(x).$$

Assume that $BB^* \in \mathcal{L}(H^2 \cap H_0^1, H_0^1)$ *and that*

$$\lim_{t\to\infty} E_h(t) = 0.$$

By taking $f_h(t) = e^{i\varsigma t}\pi_h g$, *let* $\widehat{U}^0$ *and* $\widehat{U}_h^0$ *be the unique fixed points of* Λ *and* Λ_h, *respectively. Then there exist* $h_1 > 0$ *and* $K > 0$, *such that for every* $h < h_1$ *and* $n \geq 1$

$$\|\widehat{U}^0 - \widehat{U}_h^0\|_X \leq K \left(nh^\theta + \frac{q^n}{1-q} + r_1^n \right) \|f\|_{W^{1,1}(0,T;H_0^1)}.$$

Lemma

Let $\varsigma \in \mathbb{R}$. There exists $h_0 > 0$ with the property that, for every $h < h_0$, there exist two subspaces W_h^1 and W_h^2 of V_h such that

1. *V_h may be written as*

$$V_h = W_h^1 \oplus W_h^2$$

2. *There exist two positive constants M_1 and ω_1, independent of h, such that for every $t \geq 0$*

$$\|\mathbb{S}_h(t)U_h^0\|_X^2 \leq M_1 e^{-\omega_1 t}\|U_h^0\|_X \qquad (U_h^0 \in W_h^1 \times W_h^1)$$

3. *There exists a constant $C > 0$, independent of h, such that*

$$\left\|(i\,\varsigma\, I - \mathbb{A}_h)^{-1}U_h^0\right\|_X \leq C h^{\theta}\|U_h^0\|_X \qquad (U_h^0 \in W_h^2 \times W_h^2).$$

We denote

- $(\phi_h^n)_{1\leq n\leq N(h)}$ eivenvectors of $A_{0h}^{\frac{1}{2}}$
- $(\lambda_h^n)_{1\leq n\leq N(h)}$ the corresponding eivenvalues of $A_{0h}^{\frac{1}{2}}$.

For a fixed value of $\delta > 0$, we take

$$W_h^1 = \text{Span}\{\phi_h^n \mid \lambda_h^n \leq \frac{\delta}{h^\theta}\}$$
$$W_h^2 = [W_h^1]^\perp.$$

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$$W_h^2 = [W_h^1]^\perp.$$

On note $\mathbb{A}_h^1 = \begin{bmatrix} 0 & I \\ -A_h & 0 \end{bmatrix}$, $\qquad \mathbb{A}_h = \begin{bmatrix} 0 & I \\ A_h & -B_h B_h^* \end{bmatrix}$ and let (Φ_h^n) be the eigenvectors of skew-adjoint operator $\mathbb{A}_h^1$:

$$\Phi_h^n = \frac{1}{\sqrt{2}} \begin{bmatrix} \phi_h^{|n|} \\ i\text{sgn}(n)\lambda_h^{|n|}\phi_h^{|n|} \end{bmatrix}, \qquad (1 \leq |n| \leq N(h)).$$

We take

$$U_h^0 = \sum_{\lambda_{h\,|n|} > \frac{\delta}{h^\theta}} a_n \Phi_{hn} \in \left(W_h^2\right)^2$$

and we remark that

$$\left\|(i\varsigma I - \mathbb{A}_h)^{-1} U_h^0 - (i\varsigma I - \mathbb{A}_h^1)^{-1} U_h^0\right\|_X \leq$$

$$\left\|(i\varsigma I - \mathbb{A}_h)^{-1}\right\|_{\mathcal{L}(V_h)} \left\|U_h^0 - (i\varsigma I - \mathbb{A}_h)(i\varsigma I - \mathbb{A}_h^1)^{-1} U_h^0\right\|_X \leq$$

$$\left\|(i\varsigma I - \mathbb{A}_h)^{-1}\right\|_{\mathcal{L}(V_h)} \left\|\mathbb{B}_h (i\varsigma I - \mathbb{A}_h^1)^{-1} U_h^0\right\|_X,$$

where $\mathbb{B}_h = \begin{bmatrix} 0 & 0 \\ 0 & -B_h B_h^* \end{bmatrix}$ and, hence,

$$\left\|(i\varsigma I - \mathbb{A}_h)^{-1} U_h^0\right\|_X \leq C \|(i\varsigma I - \mathbb{A}_h^1)^{-1} U_h^0\|_X.$$

Remark that, at the same time, h_0 and δ can be chosen such that

$$|\varsigma| < \frac{\delta}{2h^\theta} \text{ for every } h < h_0$$

and, hence the operator $(i\varsigma I - \mathbb{A}_h^1)^{-1}$ is well defined in $\mathcal{L}((W_h^2)^2)$. Moreover, we have that

$$\left\|(i\varsigma I - \mathbb{A}_h^1)^{-1} U_h^0\right\|_X = \left\| \sum_{\Phi_{hn} \in (W_h^2)^2} \frac{a_n}{i\varsigma - i\lambda_{h\,|n|}} \Phi_{hn} \right\|_X \leq \max_{\Phi_{hn} \in (W_h^2)^2} \frac{1}{|\varsigma - \lambda_{h\,|n|}|} \|U_h^0\|_X .$$

Since $\Phi_{hn} \in (W_h^2)^2$ implies that $\lambda_{hn} > \frac{\delta}{h^\theta}$, we deduce that there exists a constant $C_2 > 0$ such that the following inequality holds

$$\left\|(i\varsigma I - \mathbb{A}_h^1)^{-1} U_h^0\right\|_X \leq C_2 h^\theta \|U_h^0\|_X \qquad (U_h^0 \in \left(W_h^2\right)^2, \ h < h_0).$$

□

Let $U^0 \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)$ such that $\Pi_h U^0 := U^0_h \in (W^1_h)^2$. We have that

$$\Lambda_h^n U_h^0 = \mathbb{S}_h(nT)(U_h^0 - \widehat{U}_h^0) + \widehat{U}_h^0$$

and $\widehat{U}_h^0$ satisfies $(i\varsigma - \mathbb{A}_h)\widehat{U}_h^0 = G_h$, where $G_h = \begin{bmatrix} 0 \\ \pi_h g \end{bmatrix}$.
By using the Lemma we deduce that there exist two unique elements $g_h^1 \in W_h^1$ and $g_h^2 \in W_h^2$ such that $\pi_h g = g_h^1 + g_h^2$. Let us denote by $G_h^i = \begin{bmatrix} 0 \\ g_h^i \end{bmatrix}$, $i = 1, 2$.

$$\begin{aligned} \|\Lambda_h^n U_h^0 - \widehat{U}_h^0\|_X &= \|\mathbb{S}_h(nT)(U_h^0 - \widehat{U}_h^0)\|_X \\ &\leq \left\|\mathbb{S}_h(nT)(U_h^0)\right\|_X + \left\|\mathbb{S}_h(nT)(i\varsigma - \mathbb{A}_h)^{-1}(G_h)\right\|_X. \end{aligned}$$

Since $U_h^0 \in (W_h^1)^2$, from Lemma we deduce that

$$\left\|\mathbb{S}_h(nT)(U_h^0)\right\|_X \leq M_1 e^{-\omega_1 nT} \|U_h^0\|_X \qquad (n \geq 0). \qquad (1)$$

On the other hand, by writing $G_h = G_h^1 + G_h^2$ and by using properties of the spaces W_h^1 and W_h^2, we deduce that

$$\begin{aligned}\left\|\mathbb{S}_h(nT)(i\varsigma - \mathbb{A}_h)^{-1}(G_h)\right\|_X &\leq \left\|\mathbb{S}_h(nT)(i\varsigma - \mathbb{A}_h)^{-1}(G_h^1)\right\|_X + \left\|\mathbb{S}_h(nT)(i\varsigma - \mathbb{A}_h)^{-1}(G_h^2)\right\|_X = \\ &= \left\|(i\varsigma - \mathbb{A}_h)^{-1}\mathbb{S}_h(nT)(G_h^1)\right\|_X + \left\|\mathbb{S}_h(nT)(i\varsigma - \mathbb{A}_h)^{-1}(G_h^2)\right\|_X \leq \\ &\leq M_1 e^{-\omega_1 nT} \left\|(i\varsigma - \mathbb{A}_h)^{-1}\right\|_{\mathcal{L}(V_h)} \|G_h^1\|_X + C h^\theta \|G_h^2\|_X.\end{aligned}$$

□

Plan

Numerical analysis of the problem

A particular case: monochromatic sources

Numerical results

Perspectives and conclusions

One dimensional wave equation

Consider the following one-dimensional wave equation

$$\begin{cases} \ddot{w}(t,x) - \dfrac{\partial^2 w}{\partial x^2}(t,x) + a(x)\dot{w}(t,x) = f(t,x), \quad t > 0,\ x \in (0,1), \\ w(t,0) = w(t,1) = 0, \quad t > 0 \end{cases}$$

One dimensional wave equation

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- $a : [0,1] \to \mathbb{R}$ is a nonnegative regular function which is strictly positive in a subdomain $\omega \subset (0,1)$.
- $f \in \mathcal{C}([0,\infty); L^2(0,1))$ is a periodic function of period T such that $f_{|(0,T)} \in W^{1,1}(0,T; H_0^1(0,1))$.

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- $f \in \mathcal{C}([0,\infty); L^2(0,1))$ is a periodic function of period T such that $f_{|(0,T)} \in W^{1,1}(0,T; H_0^1(0,1))$.
- $\mathcal{I}_h$ a mesh of the interval $(0,1)$ formed by N equidistant points and we denote $(h = 1/(N+1))$
- $V_h = \left\{\varphi \in C(0,1) \,|\, \varphi_{|I} \in P_2(I)\ \forall\ I \in \mathcal{I}_h, \varphi(0) = \varphi(1) = 0\right\}.$

- the source term

$$f(t,x) = (-k^2 + \pi^2)\sin(\pi x)\cos(kt) - ka(x)\sin(\pi x)\sin(kt).$$

with

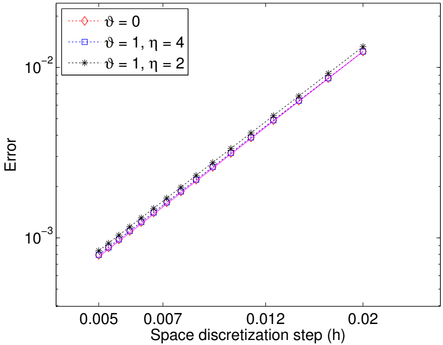
$$k = \frac{2\pi}{T}$$

- the T-periodic solution:

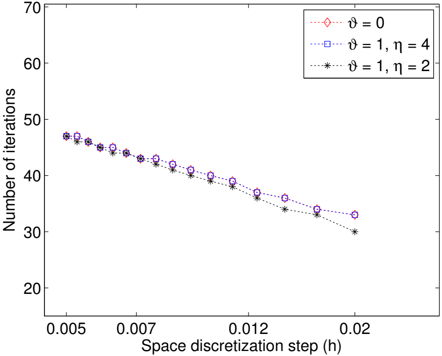
$$w(t,x) = \sin(\pi x)\cos(kt)$$

- the fixed point of the operator Λ:

$$\widehat{U}_0 = \begin{bmatrix} \sin(\pi x) \\ 0 \end{bmatrix}$$



(a)



(b)

Figure : (a) Error for a fixed period $T = \frac{\pi}{2}$. (b) The number of iterations necessary to achieve a precision $\epsilon = h^3$ in the fixed point algorithm.

Period T	0.10	0.15	0.30	0.45	0.60	0.80	1.00
$n(h)$ for $\vartheta = 0$	10000	4203	1518	794	491	238	95
$n(h)$ for $\vartheta = 1$, $\eta = 4$	10000	4183	1510	791	490	237	94
$n(h)$ for $\vartheta = 1$, $\eta = 2$	660	448	233	155	108	90	71
Error	0.0873	0.0370	0.0096	0.0043	0.0026	0.0015	0.0011

Table : Number of iterations $n(h)$ and error $\|\widehat{U}_0 - \Lambda_{h\vartheta}^{n(h)} U_0\|_X$ for different values of T.

- the source term

$$\begin{aligned} f(t,x) =& \alpha t(T-t)\left(6(T-t)^2 - 18t(T-t) + 6t^2\right) x^3(1-x)^3 \\ &- \alpha\left(1 + t^3(T-t)^3 x(1-x)\right)\left(6(1-x)^2 - 18x(1-x) + 6x^2\right) \\ &+ \alpha 3t^2(T-t)^2(T-2t)a(x)x^3(1-x)^3, \end{aligned}$$

- the T-periodic solution:

$$w(t,x) = \alpha\left(1 + t^3(T-t)^3\right) x^3(1-x)^3$$

- the fixed point of the operator Λ:

$$\widehat{U}^0 = \begin{bmatrix} \alpha x^3(1-x)^3 \\ 0 \end{bmatrix}.$$

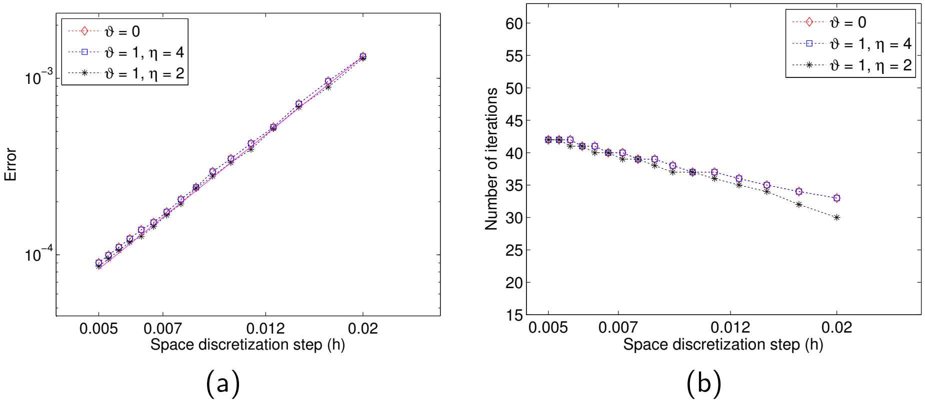


Figure : (a) Error for a period $T = 1.5$. (b) The number of iterations necessary to achieve a precision $\epsilon = h^3$ in the fixed point algorithm.

Consider the following two-dimensional wave equation

$$\begin{cases} \ddot{w}(t,x) - \Delta w(t,x) + a(x)\dot{w}(t,x) = f(t,x), \quad t > 0,\ x \in \Omega, \\ w(t,x) = 0, \quad t > 0, \quad x \in \partial\Omega \end{cases}$$

Two dimensional wave equation

Consider the following two-dimensional wave equation

$$\begin{cases} \ddot{w}(t,x) - \Delta w(t,x) + a(x)\dot{w}(t,x) = f(t,x), \quad t > 0,\ x \in \Omega, \\ w(t,x) = 0, \quad t > 0, \quad x \in \partial\Omega \end{cases}$$

- $a : \Omega \to \mathbb{R}$ is a nonnegative regular function which is strictly positive in a subdomain $\omega \subset \Omega$.
- $f \in \mathcal{C}([0,\infty); L^2(\Omega))$ is a periodic function of period T such that $f_{|(0,T)} \in W^{1,1}(0,T; H_0^1(\Omega))$.

Two dimensional wave equation

Consider the following two-dimensional wave equation

$$\begin{cases} \ddot{w}(t,x) - \Delta w(t,x) + a(x)\dot{w}(t,x) = f(t,x), & t>0,\ x\in\Omega, \\ w(t,x) = 0, \quad t>0, \quad x\in\partial\Omega \end{cases}$$

- $a : \Omega \to \mathbb{R}$ is a nonnegative regular function which is strictly positive in a subdomain $\omega \subset \Omega$.
- $f \in \mathcal{C}([0,\infty); L^2(\Omega))$ is a periodic function of period T such that $f_{|(0,T)} \in W^{1,1}(0,T; H_0^1(\Omega))$.
- $\mathcal{T}_h$ a triangular mesh of Ω
- $V_h = \left\{\varphi \in C(\omega) \,|\, \varphi_{|T} \in P_1(T)\ \forall\ T \in \mathcal{T}_h, \varphi = 0\text{on } \partial\Omega\right\}$.

- the source term

$$
\begin{aligned}
f(t,x,y) =& \alpha\big(6t(T-t)^3 - 18t^2(T-t)^2 + 6t^3(T-t)\big)x^3(1-x)^3y^3(1-y)^3 \\
&- \alpha\left(1+t^3(T-t)^3\right)\left(6x(1-x)^3 - 18x^2(1-x)^2 + 6x^3(1-x)\right)y^3(1-y)^3 \\
&- \alpha\left(1+t^3(T-t)^3\right)\left(6y(1-y)^3 - 18y^2(1-y)^2 + 6y^3(1-y)\right)x^3(1-x)^3 \\
&+ \alpha\big(3t^2(T-t)^3 - 3t^3(T-t)^2\big)a(x,y)x^3(1-x)^3y^3(1-y)^3,
\end{aligned}
$$

- the T-periodic solution:

$$
w(t,x,y) = \alpha\left(1+t^3(T-t)^3\right)x^3(1-x)^3y^3(1-y)^3
$$

- the fixed point of the operator Λ:

$$
\widehat{U}_0 = \begin{bmatrix} \alpha x^3(1-x)^3y^3(1-y)^3 \\ 0 \end{bmatrix}
$$

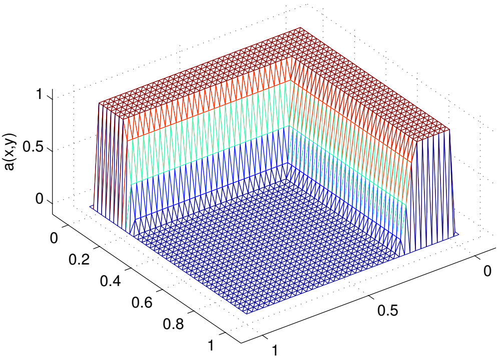


Figure : The function $a \in C^1(\overline{\Omega})$.

Two dimensional wave equation

$\Omega = (0,1)^2$

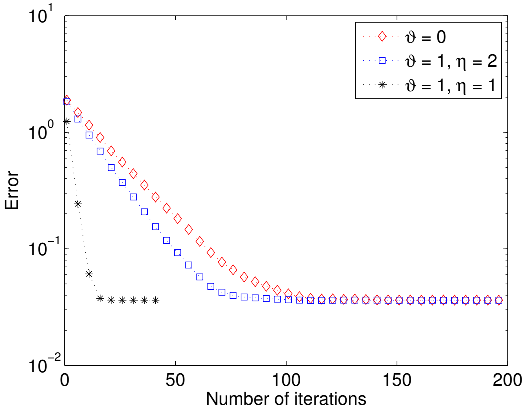
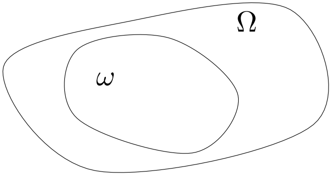
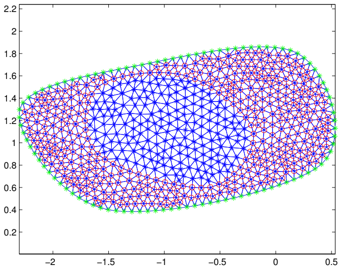


Figure : (a) Evolution of the error in the fixed point algorithm as a function of the iteration's number.





(a) (b)

Figure : (a) Domains Ω and ω. (b) Triangulation of the domain Ω: by circles we design the points in $\Omega \setminus \omega$, and by stars the points in ω.

We consider the following periodic function $f \in C([0,\infty); H_0^1(\Omega))$

$$f(t,x) = \psi(x)\cos\left(\frac{2\pi t}{T}\right),$$

where T is the period and ψ is the solution of the following elliptic problem

$$\begin{cases} \Delta\psi(x) = 1, & (x \in \Omega) \\ \psi(x) = 0, & (x \in \partial\Omega). \end{cases}$$

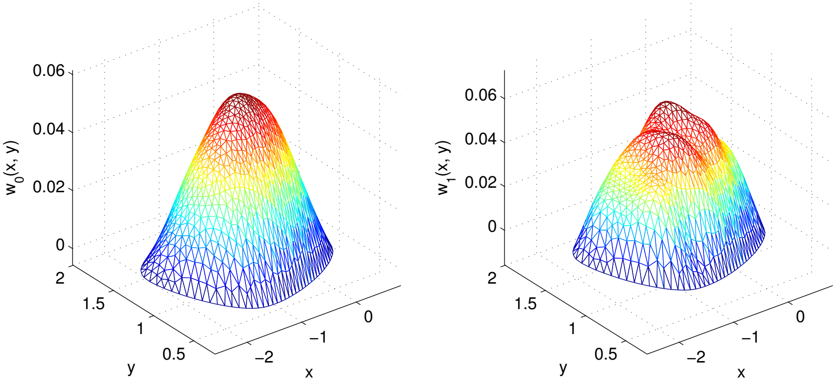


Figure : The fixed point of operator $\Lambda_{h\vartheta}$

Plan

Perspective – the boundary dissipation case?

One can consider the system

$$\begin{cases} \ddot{u}(x,t) - u_{xx}(x,t) = 0, & \text{in } (0,1)\times(0,\infty) \\ u(0,t) = 0, & \text{on } (0,\infty) \\ \ddot{u}(1,t) + u_x(1,t) + \alpha\dot{u}(1,t) = f(t), & \text{on } (0,\infty). \end{cases}$$

- $\alpha > 0$
- $f(t+T) = f(t), \qquad t > 0.$

N. C., S. Micu and A. Pazoto.
Periodic solutions for a weakly dissipated hybrid system.
Journal of Mathematical Analysis and Applications, Vol. 385 (1), p. 399-413, 2012.

Some conclusion

- Existence of periodic solution
- Convergence of the discrete periodic solutions
- Mono-chromatic case and application to Helmholtz equation
- Everything can be extend to plate equations and elasticity.

Thank you!

N. C., S. Micu and J. Morais
Approximation of periodic solutions for a dissipative hyperbolic equation. Numerische Mathematik. Volume 124, Issue 3 (2013), Page 559-601.