Numerical controllability of the wave equation using time-space finite elements

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joint work with
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UNIVERSITE BLAISE PASCAL CLERMONT-FERRAND

Oberwolfach
1 Nov - 7 Nov 2015
Recent Developments on Approximation Methods
for Controlled Evolution Equations

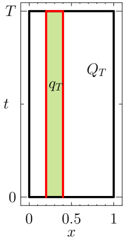
06/11/2015

The wave equation with distributed control

We consider the following wave equation:

$$\begin{cases} y_{tt}(x,t) - \Delta y(x,t) = v(x,t)\,\mathbb{1}_{q_T}(x,t), & (x,t) \in Q_T \\ y(x,t) = 0, & (x,t) \in \Sigma_T \\ y(x,0) = y_0(x), \qquad y_t(x,0) = y_1(x), & x \in \Omega. \end{cases} \tag{1}$$

- $Q_T = \Omega \times (0,T)$;
- $\Sigma_T = \partial\Omega \times (0,T)$;
- $q_T = \omega \times (0,T) \subset Q_T$;
- $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$.



Controllability problem

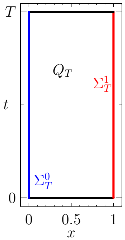
We search a control $v \in L^2(q_T)$ such that

$$y(\cdot,T) = 0, \qquad y_t(\cdot,T) = 0. \tag{2}$$

The wave equation with boundary control

We consider the following wave equation:

$$\begin{cases} y_{tt}(x,t) - \Delta y(x,t) = 0, & (x,t) \in Q_T \\ y(x,t) = 0, & (x,t) \in \Sigma_T^0 \\ y(x,t) = v(x,t), & (x,t) \in \Sigma_T^1 \\ y(x,0) = y_0(x), \qquad y_t(x,0) = y_1(x), & x \in \Omega. \end{cases} \tag{1}$$



- $Q_T = \Omega \times (0,T)$;
- $\Sigma_T^i = \Gamma^i \times (0,T)$;
- $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$.

Controllability problem

We search a control $v \in L^2(\Sigma_T^1)$ such that

$$y(\cdot,T) = 0, \qquad y_t(\cdot,T) = 0. \tag{2}$$

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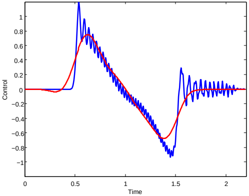
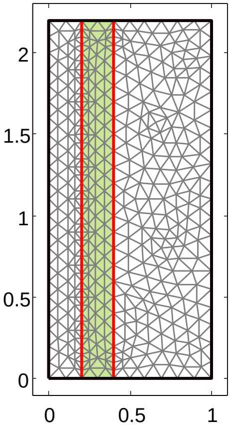


Aim of this talk



Propose a method to approximate the control of minimal L^2-norm for the wave-like equations using a space-time finite element discretization:

- avoid the spurious frequencies issue

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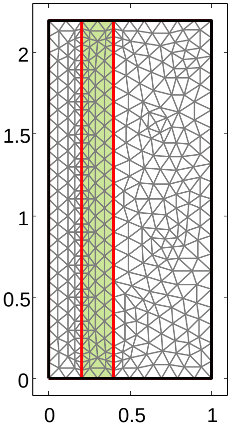

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- easy to implement

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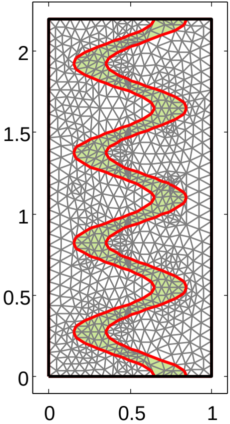

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- avoid the spurious frequencies issue
- easy to implement
- moving controls case

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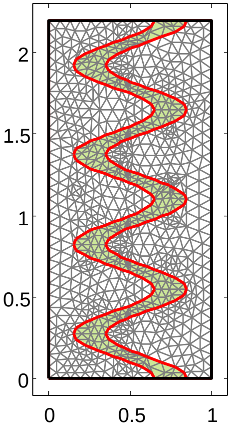

Propose a method to approximate the control of minimal L^2-norm for the wave-like equations using a space-time finite element discretization:

- avoid the spurious frequencies issue
- easy to implement
- moving controls case
- convergence of the controls?

Plan

From a minimization problem to a mixed formulation

Numerical approximation and simulations

Application to wave equations with moving controls

The controllability of the wave equation is reduced to the following minimization problem:

$$\min_{(\varphi_0,\varphi_1)\in \boldsymbol{H}} J^\star(\varphi_0,\varphi_1) = \frac{1}{2}\iint_{q_T} |\varphi|^2 dxdt + \langle \varphi_t(\cdot,0), y_0\rangle_{-1,1} - \langle \varphi(\cdot,0), y_1\rangle_2 \,,$$

where

- $\boldsymbol{H} = L^2(\Omega)\times H^{-1}(\Omega)$
- φ is the solution of the following backward equation:

$$\begin{cases} L\varphi = 0, & \text{in } Q_T \\ \varphi = 0, & \text{on } \Sigma_T \\ (\varphi(T),\varphi_t(T)) = (\varphi_0,\varphi_1), & \text{in } \Omega. \end{cases}$$

- The well posedness of the minimization of $J^\star$ can be deduced from the coercivity of J^*: there is a constant $k_T > 0$ such that for every $(\varphi_0, \varphi_1) \in \boldsymbol{H}$ we have

$$\|(\varphi(\cdot,0), \varphi_t(\cdot,0))\|_{\boldsymbol{H}}^2 \leq k_T \iint_{q_T} |\varphi|^2 dxdt. \qquad \text{(OBS)}$$

- The control of minimal L^2-norm is given by

$$v = -\varphi \mathbb{1}_{q_T}.$$

- The observability inequality (OBS) is, in general, not uniform with respect to the discretization step.

Minimization with respect to φ

We replace the standard minimization problem

$$\min_{(\varphi_0,\varphi_1)\in \boldsymbol{H}} J^\star(\varphi_0,\varphi_1)$$

by the following one

$$\min_{\varphi\in W} J^\star(\varphi) = \frac{1}{2}\iint_{q_T} |\varphi|^2 dxdt + \langle \varphi_t(\cdot,0), y_0\rangle_{-1,1} - \langle \varphi(\cdot,0), y_1\rangle_2\,,$$

with $W = \left\{\varphi \in \Phi \text{ such that } L\varphi = 0 \in L^2(0,T;H^{-1}(\Omega))\right\}$ and

$$\Phi = \left\{ \begin{array}{l} \varphi \in C(0,T;L^2(\Omega)) \ \cap \ C^1(0,T;H^{-1}(\Omega)); \\ L\varphi \in L^2(0,T;H^{-1}(\Omega)). \end{array} \right\}$$

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Φ is a Hilbert space endowed with the scalar product:

$$(\varphi,\overline{\varphi})_\Phi = \iint_{q_T} \varphi\overline{\varphi} dxdt + \eta\int_0^T \langle L\varphi(\cdot,t), L\overline{\varphi}(\cdot,t)\rangle_{-1} dt.$$

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- φ can be obtained as the solution of the following mixed formulation:

$$\left\{ \begin{array}{rcll} a(\varphi,\overline{\varphi}) + b(\overline{\varphi},\lambda) & = & l(\overline{\varphi}), & \forall \overline{\varphi} \in \Phi \\ b(\varphi,\overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in \Lambda. \end{array} \right. \qquad \text{(MF)}$$

Minimization with respect to φ

A mixed formulation

$$\begin{cases} a(\varphi,\overline{\varphi}) + b(\overline{\varphi},\lambda) & = & l(\overline{\varphi}), & \forall \overline{\varphi} \in \Phi \\ b(\varphi,\overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in \Lambda \end{cases} \qquad \text{(MF)}$$

where

$$a : \Phi \times \Phi \to \mathbb{R}, \quad a(\varphi,\overline{\varphi}) = \iint_{q_T} \varphi\,\overline{\varphi}\,dx\,dt$$

$$b : \Phi \times \Lambda \to \mathbb{R}, \quad b(\varphi,\lambda) = \int_0^T \langle L\varphi(\cdot,t), \lambda(\cdot,t)\rangle_{-1,1} dt$$

$$l : \Phi \to \mathbb{R}, \quad l(\varphi) = -\langle \varphi_t(\cdot,0), y_0\rangle_{-1,1} + \int_\Omega \varphi(\cdot,0)\,y_1 dx.$$

Well-posedness of the mixed formulation

Theorem

We assume that there exists $C > 0$ such that for every $\varphi \in \Phi$

$$\|(\varphi(\cdot,0),\varphi_t(\cdot,0))\|^2_{\boldsymbol{H}} \leq C\left(\|\varphi\|^2_{L^2(q_T)} + \|L\varphi\|^2_{L^2(0,T;H^{-1}(\Omega))}\right).$$

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1. *The mixed formulation (MF) is well-posed.*
2. *The unique solution $(\varphi,\lambda) \in \Phi \times \Lambda$ is the unique saddle-point of the Lagrangian*

$$\mathcal{L}(\varphi,\lambda) = \frac{1}{2}a(\varphi,\varphi) + b(\varphi,\lambda) - l(\varphi).$$

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1. *The mixed formulation (MF) is well-posed.*
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$$\mathcal{L}(\varphi, \lambda) = \frac{1}{2}a(\varphi, \varphi) + b(\varphi, \lambda) - l(\varphi).$$
3. *The optimal function φ is the minimizer of $J^\star$ over Φ while the optimal function $\lambda \in \Lambda$ is the state of the controlled wave equation* (1) *in the weak sense (associated to the control $-\varphi\, \mathbb{1}_{q_T}$).*

Well-posedness of the mixed formulation

Idea of the proof

- a continuous over $\Phi \times \Phi$
  symmetric
  positive

$$a(\varphi, \overline{\varphi}) = \iint_{q_T} \varphi \, \overline{\varphi} \, dx \, dt$$

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$$\mathcal{N}(b) = \{\varphi \in \Phi \text{ such that } b(\varphi, \lambda) = 0,\ \forall \lambda \in \Lambda\}$$

  - b satisfies the *inf-sup condition*: there exists $\delta > 0$ such that

$$\inf_{\lambda \in \Lambda} \sup_{\varphi \in \Phi} \frac{b(\varphi, \lambda)}{\|\varphi\|_\Phi \|\lambda\|_\Lambda} \geq \delta.$$

$$a(\varphi, \overline{\varphi}) = \iint_{q_T} \varphi \, \overline{\varphi} \, dx \, dt$$

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An *augmented* Lagrangian strategy

For any $r > 0$ we define the *augmented* Langrangian $\mathcal{L}_r$ by:

$$\mathcal{L}_r(\varphi, \lambda) = \frac{1}{2} a_r(\varphi, \varphi) + b(\varphi, \lambda) - l(\varphi),$$

where $a_r : \Phi \times \Phi \to \mathbb{R}$ is given by

$$a_r(\varphi, \overline{\varphi}) = a(\varphi, \overline{\varphi}) + r \int_0^T \langle L\varphi(\cdot, t), L\overline{\varphi}(\cdot, t) \rangle_{-1} dt.$$

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Remark:

Since $a(\varphi, \varphi) = a_r(\varphi, \varphi)$ for every $\varphi \in W$, the Lagrangians $\mathcal{L}$ and $\mathcal{L}_r$ share the same saddle points.

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Discretization of the mixed formulation

Let Φ_h and Λ_h be two finite dimensional spaces such that for every discretization parameter $h > 0$:

- $\Phi_h \subset \Phi$
- $\Lambda_h \subset \Lambda$.

We introduce the following approximating problems:

$$\left\{ \begin{array}{ll} a_r(\varphi_h, \overline{\varphi}_h) + b(\overline{\varphi}_h, \lambda_h) = l(\overline{\varphi}), & \forall \overline{\varphi} \in \Phi_h \\ b(\varphi_h, \overline{\lambda}_h) = 0, & \forall \overline{\lambda}_h \in \Lambda_h \end{array} \right. \qquad (MF_h)$$

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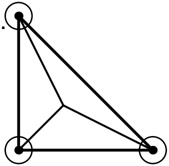
Φ_h must be chosen such that $L\varphi_h \in L^2(0, T; H^{-1}(\Omega))$

For instance, Φ_h could be chosen as a finite element space of class C^1. E.g. *Hsieh-Clough-Tocher (HCT)* finite element space.



Well-posedness of the discrete mixed formulation

For a fixed $h > 0$ the mixed formulation (MF_h) is well-posed as a consequence of the following two properties:

- a_r is coercive on the subset $\mathcal{N}_h(b) \subset \Phi_h \subset \Phi$;
- discrete inf-sup condition: there exists $\delta_h > 0$ such that

$$\inf_{\lambda_h \in \Lambda_h} \sup_{\varphi_h \in \Phi_h} \frac{b(\varphi_h, \lambda_h)}{\|\varphi_h\|_\Phi \|\lambda_h\|_\Lambda} \geq \delta_h.$$

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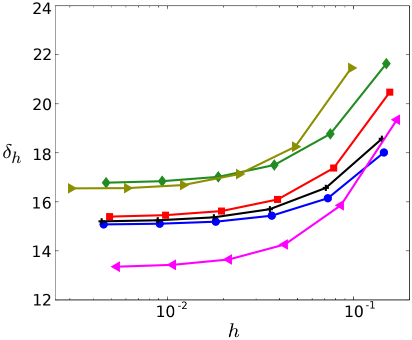
Remark:

The constant δ_h may go to zero when h goes to zero...
If this is the case then (φ_h, λ_h) may not converge to (φ, λ) in $\Phi \times \Lambda$ when $h \to 0$.

- For this choice of spaces Φ_h and Λ_h, there exists $\delta > 0$ such that

$$\delta_h \geq \delta, \qquad \forall h > 0?$$



Some difficulties

Some tricky terms appear in the mixed formulation

How can we implement numerically the following terms?

- $\displaystyle\int_0^T \langle L\varphi(\cdot,t), L\overline{\varphi}(\cdot,t)\rangle_{-1} dt$
- $\displaystyle\int_0^T \langle L\varphi(t), \lambda(t)\rangle_{-1,1} dt$

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- $\displaystyle\int_0^T \langle L\varphi(\cdot,t), L\overline{\varphi}(\cdot,t)\rangle_{-1} dt \approx C_0 h^\alpha \iint_{Q_T} L\varphi L\overline{\varphi}\,dxdt$
- $\displaystyle\int_0^T \langle L\varphi(t), \lambda(t)\rangle_{-1,1} dt \approx (C_0 h^\alpha)^{\frac{1}{2}} \iint_{Q_T} L\varphi\lambda\,dxdt$

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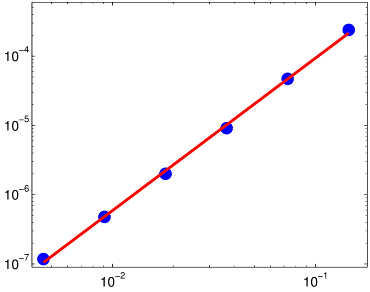
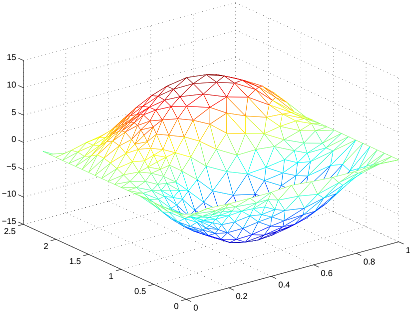


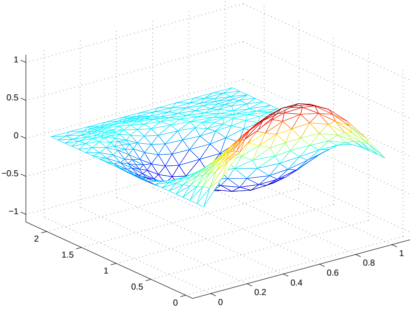
- red line: $C_0 h^\alpha$
- $C_0 \approx 1.48 \times 10^{-2}$
- $\alpha \approx 2.1993$
- blue dots: γ_h

$$y_0(x) = \sin(\pi x), \qquad y_1(x) = 0, \quad (x \in (0,1))$$

$$T = 2.2, \qquad q_T = \left(\frac{1}{5},\ \frac{2}{5}\right) \times (0,\ T).$$



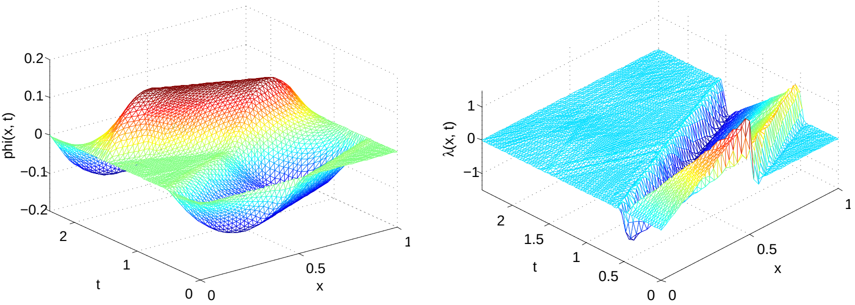
φ_h



λ_h

$$y_0(x) = 4x\mathbb{1}_{(0,\frac{1}{2})}(x), \quad y_1(x) = 0, \qquad (x \in (0,1))$$

$$T = 2.4, \qquad \Sigma_T^0 = \{0\} \times (0,T), \quad \Sigma_T^1 = \{1\} \times (0,T)$$



φ_h $\qquad$ λ_h

An example with boundary control

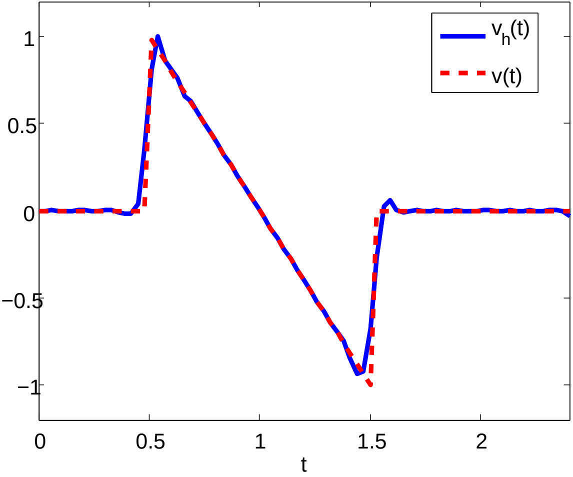


Figure : Exact control vs. approximated control

An example with boundary control

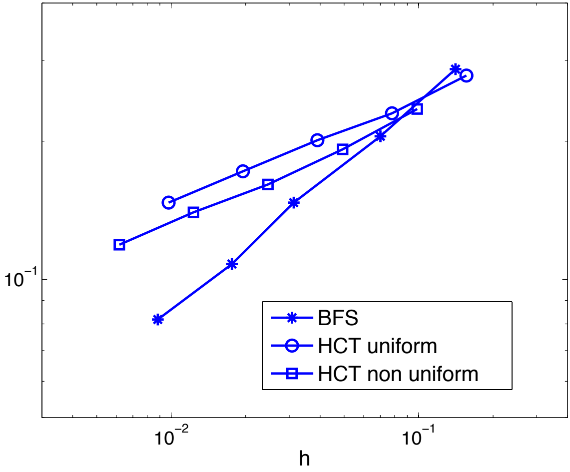
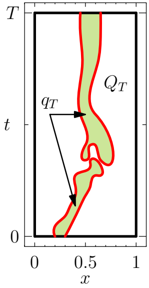


Figure : Evolution of $\|v - v_h\|_{L^2(0,T)}$ w.r.t. h for BFS finite element ($\star$), HCT-uniform mesh ($\circ$) and HCT- non uniform mesh ($\square$); $r = 1$.

Plan

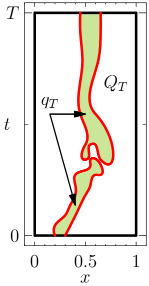
Time-dependent control domains q_T case



For time-dependent control domains q_T:

- prove the exact controllability of the wave equation;
- give a constructive method to approach the control of minimal L^2-norm;
- discuss the numerical implementation of this method.

Time-dependent control domains q_T case



For time-dependent control domains q_T:

- prove the exact controllability of the wave equation;
- give a constructive method to approach the control of minimal L^2-norm;
- discuss the numerical implementation of this method.

A. Y. Khapalov, *Controllability of the wave equation with moving point control*, Appl. Math. Optim. (1995).

L. Cui, X. Liu, H. Gao, *Exact controllability for a one-dimensional wave equation in non-cylindrical domains*, J. Math. Anal. Appl. (2013).

C. Castro, *Exact controllability of the 1-D wave equation from a moving interior point*, ESAIM COCV (2013).

Observability inequality in time-dependent domain case

Proposition (C. Carlos, N.C, A. Münch – 2014)

Assume that $q_T \subset (0,1) \times (0,T)$ is a finite union of connected open sets and satisfies the following hypotheses:
any characteristic line starting at a point $x \in (0,1)$ at time $t = 0$ and following the optical geometric laws when reflecting at the boundary Σ_T must meet q_T.
Then, there exists $C > 0$ such that the following estimate holds :

$$\|(\varphi(\cdot,0), \varphi_t(\cdot,0))\|_{\boldsymbol{H}}^2 \le C\left(\|\varphi\|_{L^2(q_T)}^2 + \|L\varphi\|_{L^2(0,T;H^{-1}(0,1))}^2\right),$$

for every $\varphi \in C([0,T], L^2(0,1)) \cap C^1([0,T], H^{-1}(0,1))$ and satisfying $L\varphi \in L^2(0,T; H^{-1}(0,1))$.

Notation: $\boldsymbol{H} = L^2(0,1) \times H^{-1}(0,1)$.
$L\varphi = \varphi_{tt} - \varphi_{xx}$.

Observability inequality in time-dependent domain case

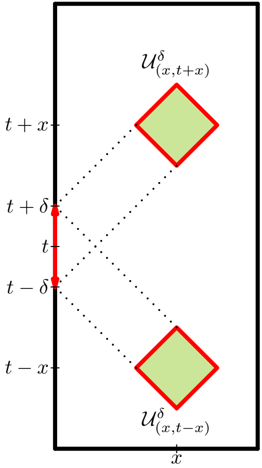
Idea of the proof

We follow the method used by C. Castro in the case of a moving pointwise control:

C. Castro, *Exact controllability of the 1-D wave equation from a moving interior point*, ESAIM COCV., 19 (2013).

Some ingredients of the proof :

- D'Alembert formulae;



$$\int_{t-\delta}^{t+\delta} |\varphi_x(0,s)|^2 ds \leq \frac{1}{\delta} \iint_{\mathcal{U}^{\delta}_{(x,t+x)}} \left(|\varphi_x|^2 + |\varphi_t|^2\right) dyds$$

$$\int_{t-\delta}^{t+\delta} |\varphi_x(0,s)|^2 ds \leq \frac{1}{\delta} \iint_{\mathcal{U}^{\delta}_{(x,t-x)}} \left(|\varphi_x|^2 + |\varphi_t|^2\right) dyds$$

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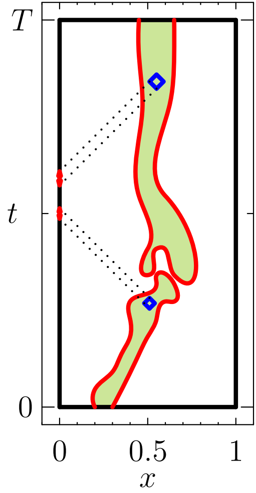
Some ingredients of the proof :

- D'Alembert formulae;
- known observability inequality in the boundary case;



Boundary observability inequality:

$$\|(\varphi(\cdot,0),\varphi_t(\cdot,0))\|_H^2 \le C\int_0^T |\varphi_x(0,t)|^2 dt.$$

combined with the previous estimate gives:

$$\|(\varphi(\cdot,0),\varphi_t(\cdot,0))\|_V^2 \le C\left(\|\varphi_t\|_{L^2(q_T)}^2 + \|\varphi_x\|_{L^2(q_T)}^2\right)$$

$$H = L^2(0,1)\times H^{-1}(0,1)$$

$$V = H_0^1(0,1)\times L^2(0,1)$$

Observability inequality in time-dependent domain case

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We follow the method used by C. Castro in the case of a moving pointwise control:

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Some ingredients of the proof :

- D'Alembert formulae;
- known observability inequality in the boundary case;
- equi-repartition of energy.

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Some ingredients of the proof :

- D'Alembert formulae;
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- equi-repartition of energy.

Remark

The proof of the proposition is specific to the one-dimensional case.

Controllability in time-dependent control domain case

Corollary (C. Castro, N.C., A. Münch – 2014)

Let $T > 0$ and $q_T \subset (0,1) \times (0,T)$ be such that
any characteristic line starting at a point $x \in (0,1)$ at time $t = 0$ and following the optical geometric laws when reflecting at the boundary Σ_T must meet q_T.
Then the wave equation is null controllable in time T.

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Proof.

We apply HUM. □

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Then the wave equation is null controllable in time T.

Proof.

We apply HUM. □

Numerical approximation :

- usual problems due to the controllability of high frequencies;
- problems due to the controllability domain non-constant in time.

Hilbert Uniqueness Method - a reformulation

N. Cîndea and A. Münch, *A mixed formulation for the direct approximation of the control of minimal L^2-norm for linear type wave equations*, Calcolo, Vol. 52, 2015.

$$\min_{\varphi \in \Phi} \hat{J}^\star(\varphi), \qquad \text{subject to} \quad L\varphi = 0.$$

$$\Phi = \left\{ \begin{array}{l} \varphi \in C([0,T], H_0^1(0,1)) \cap C^1([0,T], L^2(0,1)) \\ \text{such that } L\varphi \in L^2(0,T,H^{-1}(0,1)) \end{array} \right\}.$$

Remark

Φ *is an Hilbert space endowed with the inner product*

$$(\varphi, \overline{\varphi})_\Phi = \iint_{q_T} \varphi(x,t)\overline{\varphi}(x,t)\, dxdt + \eta \iint_{Q_T} \langle L\varphi, L\overline{\varphi}\rangle_{-1}\, dx\, dt.$$

for any fixed $\eta > 0$.

Idea of the method: step by step

1. write the minimization of J^* as a saddle-point problem for an associated Lagrangian.

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2. write the optimality conditions for the Lagrangian as a mixed-formulation in φ and λ.

Idea of the method: step by step

We consider the following mixed formulation : find $(\varphi, \lambda) \in \Phi \times L^2(0, T, H_0^1(0,1))$ solution of

$$\begin{cases} a(\varphi, \overline{\varphi}) + b(\overline{\varphi}, \lambda) & = & l(\overline{\varphi}), & \forall \overline{\varphi} \in \Phi \\ b(\varphi, \overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in L^2(0, T, H_0^1(0,1)), \end{cases}$$

where

$$a : \Phi \times \Phi \to \mathbb{R}, \quad a(\varphi, \overline{\varphi}) = \iint_{q_T} \varphi \overline{\varphi} dx dt + \eta \iint_{Q_T} \langle L\varphi, L\overline{\varphi} \rangle_{-1} \, dx \, dt.$$

$$b : \Phi \times L^2(0, T, H_0^1(0,1)) \to \mathbb{R}, \quad b(\varphi, \lambda) = \int_0^T \langle L\varphi(\cdot, t), \lambda(\cdot, t) \rangle_{-1,1} dt.$$

$$l : \Phi \to \mathbb{R}, \quad l(\varphi) = -\langle \varphi_t(\cdot, 0), y_0 \rangle_{-1,1} + \int_0^1 \varphi(x, 0) y_1(x) dx.$$

Idea of the method: step by step

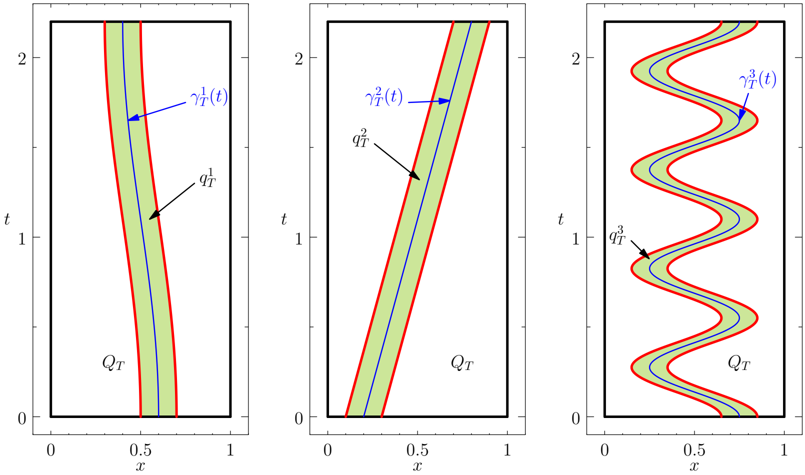
1. write the minimization of J^* as a saddle-point problem for an associated Lagrangian.
2. write the optimality conditions for the Lagrangian as a mixed-formulation in φ and λ.
3. use the generalized observability inequality in order to prove that this mixed formulation is well-posed:
   - φ is the dual variable
   - λ is the controlled solution.

Idea of the method: step by step

1. write the minimization of J^* as a saddle-point problem for an associated Lagrangian.
2. write the optimality conditions for the Lagrangian as a mixed-formulation in φ and λ.
3. use the generalized observability inequality in order to prove that this mixed formulation is well-posed:
   - φ is the dual variable
   - λ is the controlled solution.
4. discretize the mixed formulation and prove that the discrete controls converge to the exact continuous controls:
   - C^1 finite elements for φ
   - P_1 finite elements for λ.

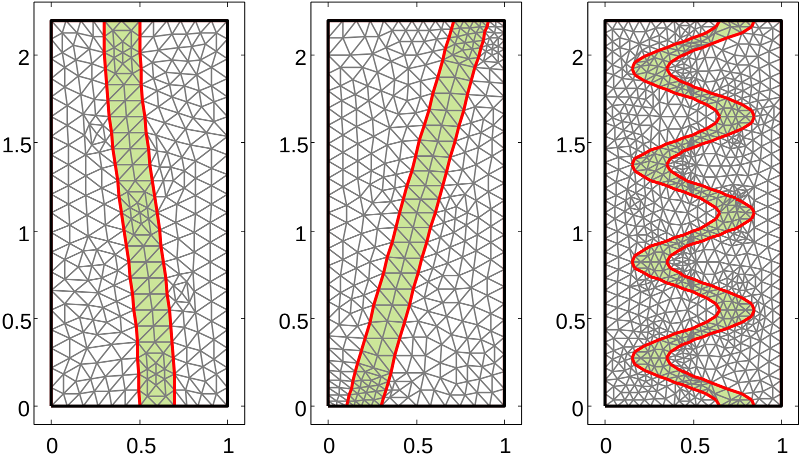
Numerical examples

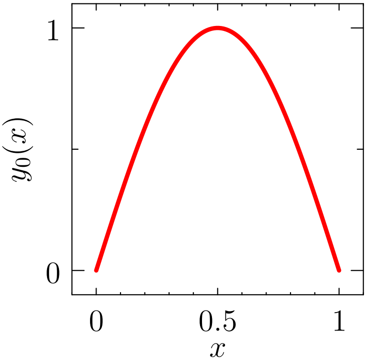
Some controllability domains

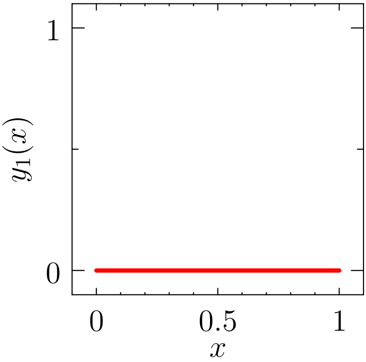


Numerical examples

Some controllability domains – and associated meshes

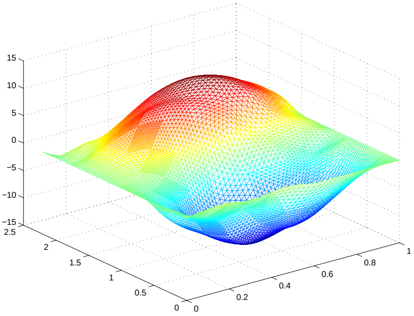






$$y_0(x) = \sin(\pi x).$$

$$y_1(x) = 0.$$



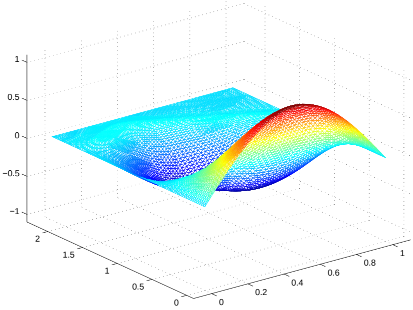


Figure : $q_T = q^1_{2.2}$: Functions φ_h (**Left**) and λ_h (**Right**).

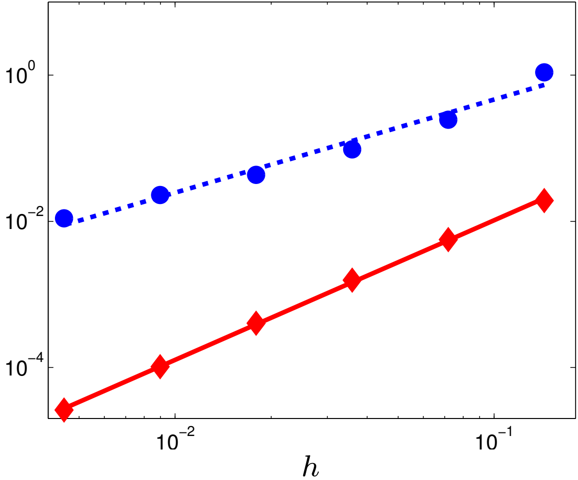
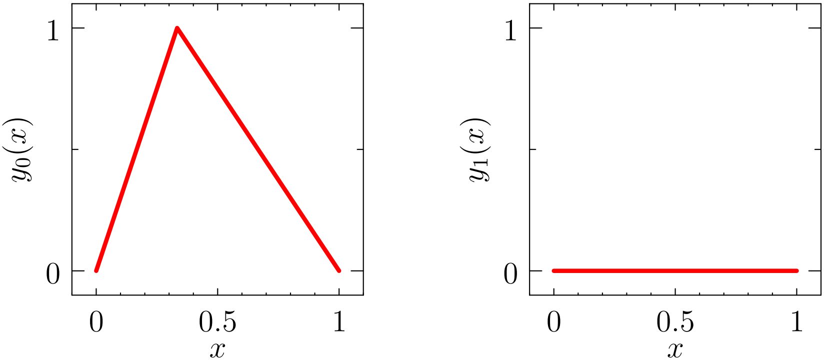


Figure : Norms $\|v - v_h\|_{L^2(q_T)}$ (●) and $\|y - \lambda_h\|_{L^2(Q_T)}$ (♦) vs. h.

A second numerical example

Initial data to control



$$y_0(x) = 3x\mathbb{1}_{0,1/3}(x) + \frac{3(1-x)}{2}\mathbb{1}_{(1/3,1)}(x).$$

$$y_1(x) = 0.$$

A second numerical example

Results

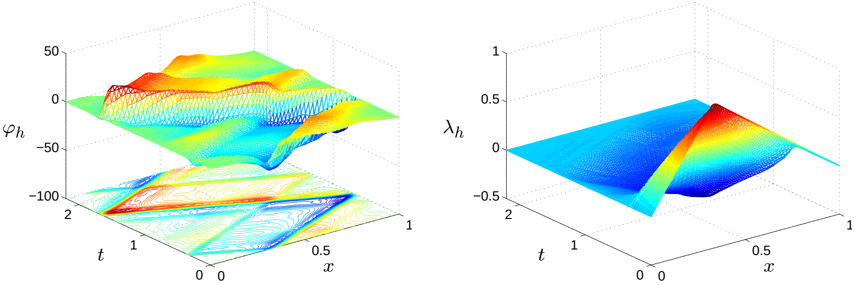


Figure : Functions φ_h (**Left**) and λ_h (**Right**).

Table: $q_T = q^2_{T=2.2}$.

$\sharp$ Mesh	1	2	3	4	5
h	7.18×10^{-2}	3.59×10^{-2}	1.79×10^{-2}	8.97×10^{-3}	4.49×10^{-3}
$\Vert v_h \Vert_{L^2(q_T)}$	5.350	5.263	5.195	5.172	5.165
$\Vert v - v_h \Vert_{L^2(q_T)}$	1.3571	9.78×10^{-1}	6.91×10^{-1}	5.13×10^{-1}	3.69×10^{-1}
$\Vert y - \lambda_h \Vert_{L^2(Q_T)}$	7.12×10^{-3}	3.23×10^{-3}	1.19×10^{-3}	4.82×10^{-4}	2.12×10^{-4}

- v – control of minimal L^2-norm supported on q_T;
- y – controlled solution by control v.

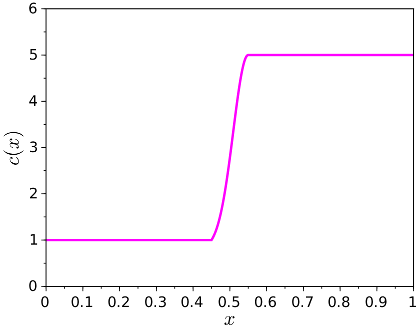
A wave with variable speed of propagation

We consider the following wave equation

$$\begin{cases} y_{tt}(x,t) - (c(x)y_x(x,t))_x = v(x,t)\,\mathbb{1}_{q_T}(x), & (x,t) \in Q_T \\ y(x,t) = 0, & (x,t) \in \Sigma_T \\ y(x,0) = y_0(x), \qquad y_t(x,0) = y_1(x), & x \in (0,1). \end{cases}$$

We take the propagation speed $c \in C^\infty(0,1)$ given by

$$c(x) = \begin{cases} 1, & x \in [0, 0.45] \\ 5, & x \in [0.55, 1]. \end{cases}$$



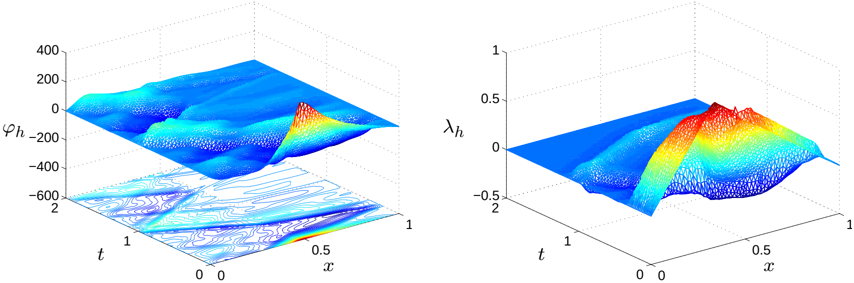


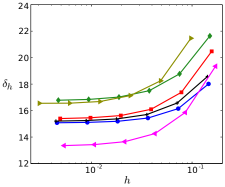
Figure : $q_T = q_2^2$ for a non-constant velocity of propagation
Function φ_h (**Left**) and λ_h (**Right**).

Conclusion

- We developed a constructive method to compute the distributed (and boundary) control of minimal L^2-norm (eventually supported in non-cylindrical domains);
- We proved the exact controllability of the one-dimensional wave equation with a distributed control supported on a non-cylindrical domain;
- Numerical results indicate that the computed controls converge to the exact control.
- A similar method can be used for the dual inverse problem. . .

Some perspectives and open questions

- $\|v_h - v\|_{L^2(q_T)} \to ch^\theta$?
- uniform "inf-sup" discrete condition?
- Optimization of the control's support.



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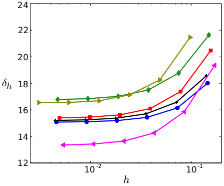

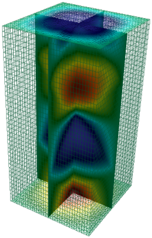

- What about the approximation of controls for higher dimensional wave equations?

C. Castro, N. C., A. Münch, *Controllability of the linear 1D wave equation with inner moving forces*, SICON (2014).

N. C., E. Fernández-Cara, A. Münch, *Numerical controllability of the wave equation through primal methods and Carleman estimates*, ESAIM COCV (2013).

N. C., A. Münch, *A mixed formulation for the direct approximation of the control of minimal L^2-norm for linear type wave equations*, Calcolo (2015).

Thank you!