On the approximation of moving controls for the wave equation

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joint work with Carlos Castro and Arnaud Münch

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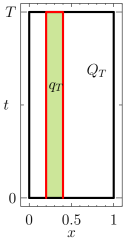
Figure courtesy Fabio Fagnani.

The wave equation with distributed control

We consider the following wave equation:

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) = v(x,t)\,\mathbb{1}_{q_T}(x,t), & (x,t) \in Q_T \\ y(x,t) = 0, & (x,t) \in \Sigma_T \\ y(x,0) = y_0(x), \qquad y_t(x,0) = y_1(x), & x \in (0,1). \end{cases} \tag{1}$$

- $Q_T = (0,1) \times (0,T)$;
- $\Sigma_T = \{0,1\} \times (0,T)$;
- $q_T = \omega \times (0,T) \subset Q_T$;
- $(y_0, y_1) \in H_0^1(0,1) \times L^2(0,1)$.



Controllability problem

We search a control $v \in L^2(q_T)$ such that

$$y(\cdot,T) = 0, \qquad y_t(\cdot,T) = 0. \tag{2}$$

J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués.* Masson, Paris, 1988.

- Hilbert Uniqueness Method (HUM).

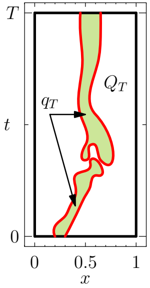
C. Bardos, G. Lebeau, and J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim., 1992.

- Geometric Control Condition.

E. Zuazua, *Propagation, observation, and control of waves approximated by finite difference methods*, Siam Review, 2005.

- spurious high frequencies issue.

Aim of this talk



For time-dependent control domains q_T:

- prove the exact controllability of the wave equation;
- give a constructive method to approach the control of minimal L^2-norm;
- discuss the numerical implementation of this method.

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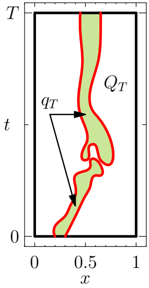

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A. Y. Khapalov, *Controllability of the wave equation with moving point control*, Appl. Math. Optim. (1995).

L. Cui, X. Liu, H. Gao, *Exact controllability for a one-dimensional wave equation in non-cylindrical domains*, J. Math. Anal. Appl. (2013).

C. Castro, *Exact controllability of the 1-D wave equation from a moving interior point*, ESAIM COCV (2013).

Observability inequality in time-dependent domain case

Proposition (C. Carlos, N.C, A. Münch – 2014)

Assume that $q_T \subset (0,1) \times (0,T)$ is a finite union of connected open sets and satisfies the following hypotheses:
any characteristic line starting at a point $x \in (0,1)$ at time $t = 0$ and following the optical geometric laws when reflecting at the boundary Σ_T must meet q_T.
Then, there exists $C > 0$ such that the following estimate holds :

$$\|(\varphi(\cdot,0), \varphi_t(\cdot,0))\|^2_{\boldsymbol{H}} \leq C\left(\|\varphi\|^2_{L^2(q_T)} + \|L\varphi\|^2_{L^2(0,T;H^{-1}(0,1))}\right),$$

for every $\varphi \in C([0,T], L^2(0,1)) \cap C^1([0,T], H^{-1}(0,1))$ and satisfying $L\varphi \in L^2(0,T; H^{-1}(0,1))$.

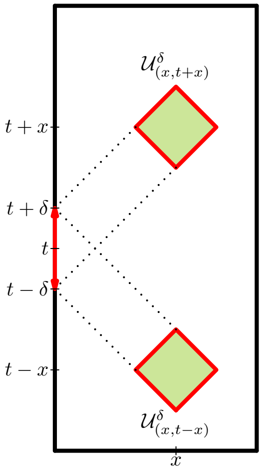
Notation: $\boldsymbol{H} = L^2(0,1) \times H^{-1}(0,1)$.
$L\varphi = \varphi_{tt} - \varphi_{xx}$.

We follow the method used by C. Castro in the case of a moving pointwise control:

C. CASTRO, *Exact controllability of the 1-D wave equation from a moving interior point*, ESAIM COCV., 19 (2013).

Some ingredients of the proof :

- D'Alembert formulae;



$$\int_{t-\delta}^{t+\delta} |\varphi_x(0,s)|^2 ds \le \frac{1}{\delta} \iint_{\mathcal{U}^{\delta}_{(x,t+x)}} \left(|\varphi_x|^2 + |\varphi_t|^2\right) dyds$$

$$\int_{t-\delta}^{t+\delta} |\varphi_x(0,s)|^2 ds \le \frac{1}{\delta} \iint_{\mathcal{U}^{\delta}_{(x,t-x)}} \left(|\varphi_x|^2 + |\varphi_t|^2\right) dyds$$

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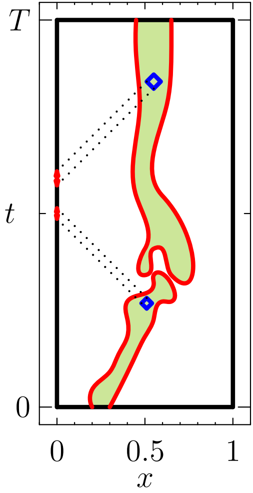
Some ingredients of the proof :

- D'Alembert formulae;
- known observability inequality in the boundary case;



Boundary observability inequality:

$$\|(\varphi(\cdot,0),\varphi_t(\cdot,0))\|_H^2 \le C\int_0^T |\varphi_x(0,t)|^2 dt.$$

combined with the previous estimate gives:

$$\|(\varphi(\cdot,0),\varphi_t(\cdot,0))\|_V^2 \le C\left(\|\varphi_t\|^2_{L^2(q_T)} + \|\varphi_x\|^2_{L^2(q_T)}\right)$$

$$H = L^2(0,1)\times H^{-1}(0,1)$$

$$V = H_0^1(0,1)\times L^2(0,1)$$

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Some ingredients of the proof :

- D'Alembert formulae;
- known observability inequality in the boundary case;
- equi-repartition of energy.

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Some ingredients of the proof :

- D'Alembert formulae;
- known observability inequality in the boundary case;
- equi-repartition of energy.

Remark

The proof of the proposition is specific to the one-dimensional case.

Corollary (C. Castro, N.C., A. Münch – 2014)

Let $T > 0$ and $q_T \subset (0,1) \times (0,T)$ be such that any characteristic line starting at a point $x \in (0,1)$ at time $t = 0$ and following the optical geometric laws when reflecting at the boundary Σ_T must meet q_T.
Then the wave equation is null controllable in time T.

Controllability in time-dependent control domain case

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Proof.

We apply HUM. □

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Proof.

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Numerical approximation :

- usual problems due to the controllability of high frequencies;
- problems due to the controllability domain non-constant in time.

Hilbert Uniqueness Method - a reformulation

N. Cîndea and A. Münch, *A mixed formulation for the direct approximation of the control of minimal L^2-norm for linear type wave equations*, Calcolo, Vol. 52, 2015.

$$\min_{\varphi \in \Phi} \hat{J}^\star(\varphi), \qquad \text{subject to} \quad L\varphi = 0.$$

$$\Phi = \left\{ \begin{array}{l} \varphi \in C([0,T], H_0^1(0,1)) \cap C^1([0,T], L^2(0,1)) \\ \text{such that } L\varphi \in L^2(0,T,H^{-1}(0,1)) \end{array} \right\}.$$

Remark

Φ is an Hilbert space endowed with the inner product

$$(\varphi, \overline{\varphi})_\Phi = \iint_{q_T} \varphi(x,t)\overline{\varphi}(x,t)\, dxdt + \eta \iint_{Q_T} \langle L\varphi, L\overline{\varphi}\rangle_{-1}\, dx\, dt.$$

for any fixed $\eta > 0$.

Idea of the method: step by step

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Idea of the method: step by step

We consider the following mixed formulation : find $(\varphi, \lambda) \in \Phi \times L^2(0, T, H_0^1(0,1))$ solution of

$$\begin{cases} a(\varphi, \overline{\varphi}) + b(\overline{\varphi}, \lambda) & = & l(\overline{\varphi}), & \forall \overline{\varphi} \in \Phi \\ b(\varphi, \overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in L^2(0, T, H_0^1(0,1)), \end{cases}$$

where

$$a : \Phi \times \Phi \to \mathbb{R}, \quad a(\varphi, \overline{\varphi}) = \iint_{q_T} \varphi \overline{\varphi} dx dt + \eta \iint_{Q_T} \langle L\varphi, L\overline{\varphi} \rangle_{-1} \, dx \, dt.$$

$$b : \Phi \times L^2(0, T, H_0^1(0,1)) \to \mathbb{R}, \quad b(\varphi, \lambda) = \int_0^T \langle L\varphi(\cdot, t), \lambda(\cdot, t) \rangle_{-1,1} dt.$$

$$l : \Phi \to \mathbb{R}, \quad l(\varphi) = -\langle \varphi_t(\cdot, 0), y_0 \rangle_{-1,1} + \int_0^1 \varphi(x, 0) y_1(x) dx.$$

Idea of the method: step by step

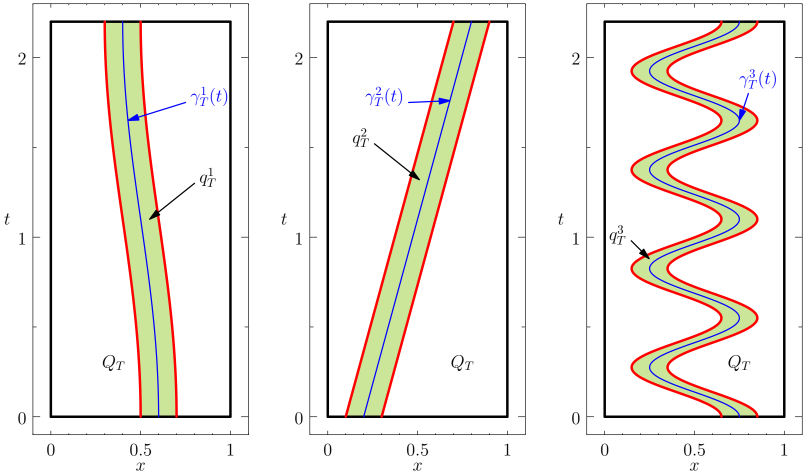
1. write the minimization of J^* as a saddle-point problem for an associated Lagrangian.
2. write the optimality conditions for the Lagrangian as a mixed-formulation in φ and λ.
3. use the generalized observability inequality in order to prove that this mixed formulation is well-posed:
   - φ is the dual variable
   - λ is the controlled solution.

Idea of the method: step by step

1. write the minimization of J^* as a saddle-point problem for an associated Lagrangian.
2. write the optimality conditions for the Lagrangian as a mixed-formulation in φ and λ.
3. use the generalized observability inequality in order to prove that this mixed formulation is well-posed:
   - φ is the dual variable
   - λ is the controlled solution.
4. discretize the mixed formulation and prove that the discrete controls converge to the exact continuous controls:
   - C^1 finite elements for φ
   - P_1 finite elements for λ.

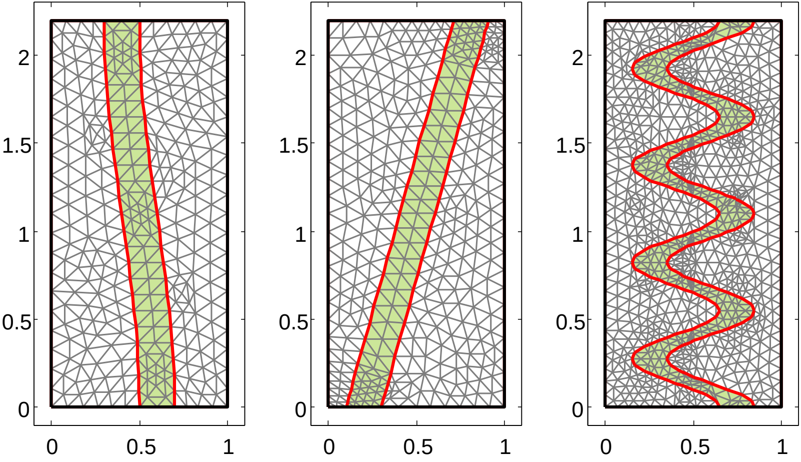
Numerical examples

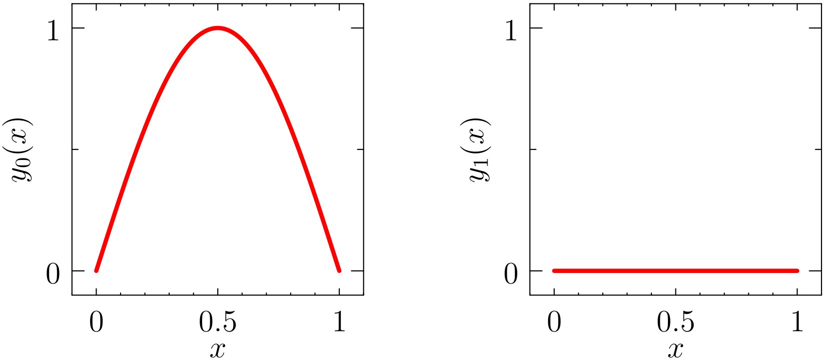
Some controllability domains



Numerical examples

Some controllability domains – and associated meshes





$$y_0(x) = \sin(\pi x).$$

$$y_1(x) = 0.$$

A first numerical example

Results

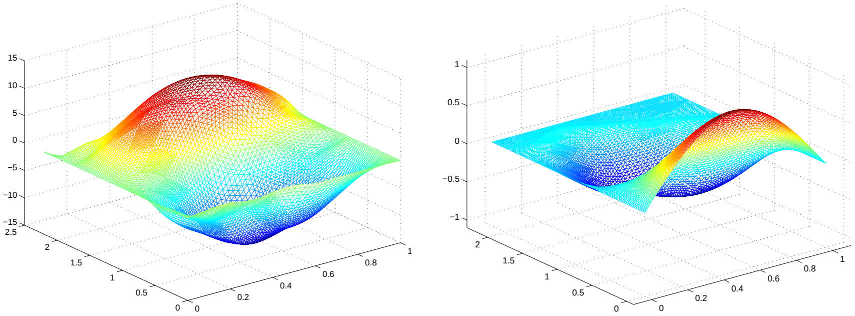


Figure : $q_T = q^1_{2.2}$: Functions φ_h (**Left**) and λ_h (**Right**).

A first numerical example

Results

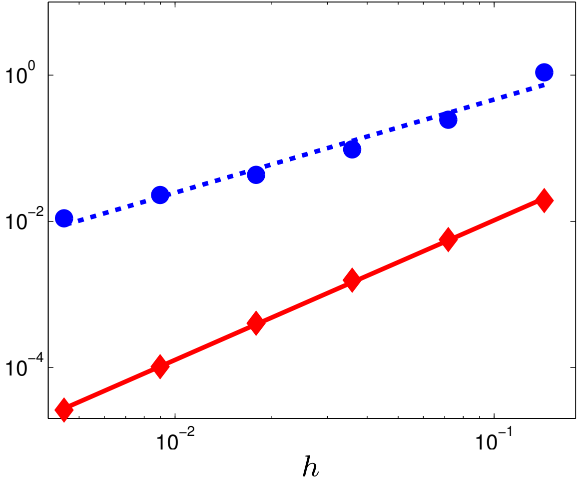
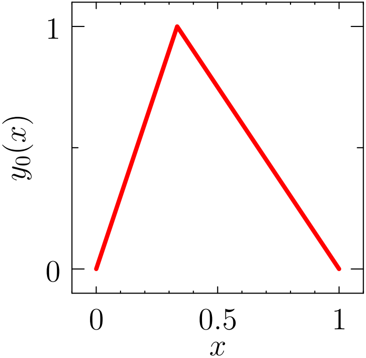
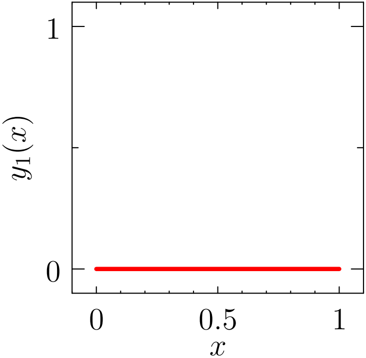


Figure : Norms $\|v - v_h\|_{L^2(q_T)}$ (●) and $\|y - \lambda_h\|_{L^2(Q_T)}$ (♦) vs. h.

A second numerical example

Initial data to control





$$y_0(x) = 3x\mathbb{1}_{0,1/3}(x) + \frac{3(1-x)}{2}\mathbb{1}_{(1/3,1)}(x).$$

$$y_1(x) = 0.$$

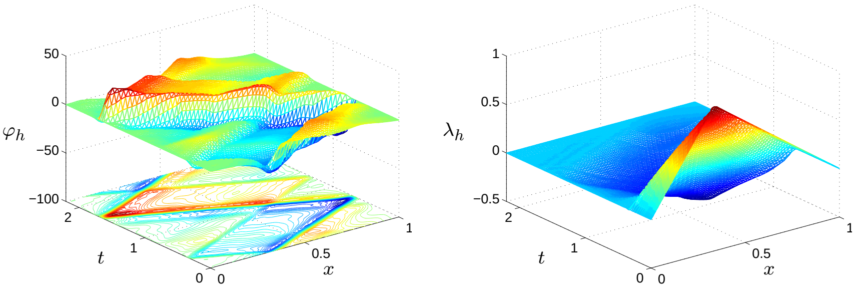


Figure : Functions φ_h (**Left**) and λ_h (**Right**).

Table: $q_T = q^2_{T=2.2}$.

$\sharp$ Mesh	1	2	3	4	5
h	7.18×10^{-2}	3.59×10^{-2}	1.79×10^{-2}	8.97×10^{-3}	4.49×10^{-3}
$\Vert v_h \Vert_{L^2(q_T)}$	5.350	5.263	5.195	5.172	5.165
$\Vert v - v_h \Vert_{L^2(q_T)}$	1.3571	9.78×10^{-1}	6.91×10^{-1}	5.13×10^{-1}	3.69×10^{-1}
$\Vert y - \lambda_h \Vert_{L^2(Q_T)}$	7.12×10^{-3}	3.23×10^{-3}	1.19×10^{-3}	4.82×10^{-4}	2.12×10^{-4}

- v – control of minimal L^2-norm supported on q_T;
- y – controlled solution by control v.

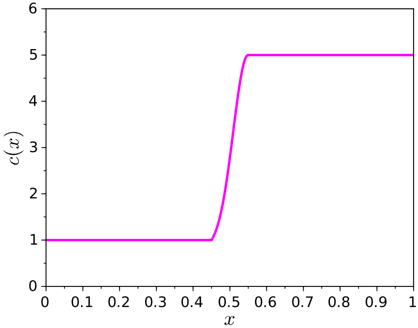
A wave with variable speed of propagation

We consider the following wave equation

$$\begin{cases} y_{tt}(x,t) - (c(x) y_x(x,t))_x = v(x,t)\, \mathbb{1}_{q_T}(x), & (x,t) \in Q_T \\ y(x,t) = 0, & (x,t) \in \Sigma_T \\ y(x,0) = y_0(x), \qquad y_t(x,0) = y_1(x), & x \in (0,1). \end{cases}$$

We take the propagation speed $c \in C^\infty(0,1)$ given by

$$c(x) = \begin{cases} 1, & x \in [0, 0.45] \\ 5, & x \in [0.55, 1]. \end{cases}$$



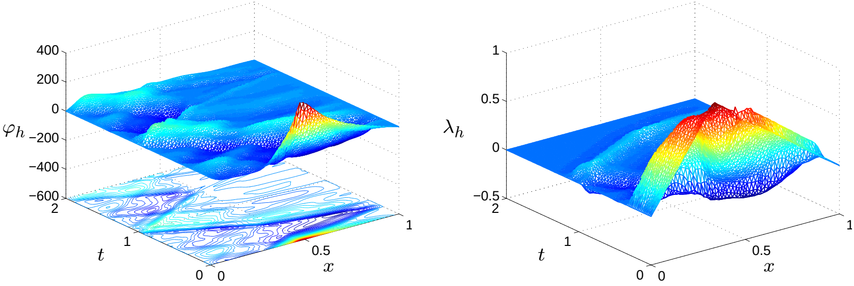


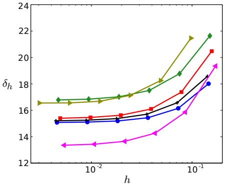
Figure : $q_T = q_2^2$ for a non-constant velocity of propagation
Function φ_h (**Left**) and λ_h (**Right**).

Conclusion

- We proved the exact controllability of the one-dimensional wave equation with a distributed control supported on a non-cylindrical domain;
- We developed a constructive method to compute the control of minimal L^2-norm supported in non-cylindrical domains.
- Numerical results indicate that the computed controls converge to the exact control.

Some perspectives

- $\|v_h - v\|_{L^2(q_T)} \to ch^\theta$?
- Prove a uniform "inf-sup" discrete condition.
- Optimization of the control's support.



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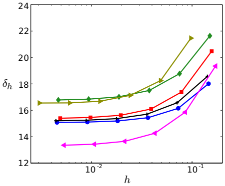

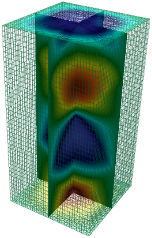

- What about higher dimensional wave equations?

- C. Castro, N. C., A. Münch, *Controllability of the linear 1D wave equation with inner moving forces*, SICON (2014).
- N. C., E. Fernández-Cara, A. Münch, *Numerical controllability of the wave equation through primal methods and Carleman estimates*, ESAIM COCV (2013).
- N. C., A. Münch, *A mixed formulation for the direct approximation of the control of minimal L^2-norm for linear type wave equations*, Calcolo (2015).

Merci!

Thank you!