On the approximation of moving controls for the wave equation

Nicolae Cîndea

joint work with Carlos Castro and Arnaud Münch
The wave equation with distributed control

We consider the following wave equation:

\[
\begin{align*}
    y_{tt}(x, t) - y_{xx}(x, t) &= v(x, t) \mathbb{1}_{q_T}(x, t), & (x, t) \in Q_T \\
    y(x, t) &= 0, & (x, t) \in \Sigma_T \\
    y(x, 0) &= y_0(x), & y_t(x, 0) = y_1(x), & x \in (0, 1).
\end{align*}
\] (1)

- $Q_T = (0, 1) \times (0, T)$;
- $\Sigma_T = \{0, 1\} \times (0, T)$;
- $q_T = \omega \times (0, T) \subset Q_T$;
- $(y_0, y_1) \in H^1_0(0, 1) \times L^2(0, 1)$.

Controllability problem

We search a control $v \in L^2(q_T)$ such that

\[
y(\cdot, T) = 0, \quad y_t(\cdot, T) = 0.
\] (2)
Some references

  - Hilbert Uniqueness Method (HUM).

  - Geometric Control Condition.

  - Spurious high frequencies issue.
Aim of this talk

For time-dependent control domains $q_T$:

- prove the exact controllability of the wave equation;
- give a constructive method to approach the control of minimal $L^2$-norm;
- discuss the numerical implementation of this method.
Aim of this talk

For time-dependent control domains $q_T$:

- prove the exact controllability of the wave equation;
- give a constructive method to approach the control of minimal $L^2$-norm;
- discuss the numerical implementation of this method.


Proposition (C. Carlos, N.C, A. Münch – 2014)

Assume that \( q_T \subset (0, 1) \times (0, T) \) is a finite union of connected open sets and satisfies the following hypotheses:

*any characteristic line starting at a point \( x \in (0, 1) \) at time \( t = 0 \) and following the optical geometric laws when reflecting at the boundary \( \Sigma_T \) must meet \( q_T \).

Then, there exists \( C > 0 \) such that the following estimate holds:

\[
\|(\varphi(\cdot, 0), \varphi_t(\cdot, 0))\|_H^2 \leq C \left( \|\varphi\|_{L^2(q_T)}^2 + \|L\varphi\|_{L^2(0,T;H^{-1}(0,1))}^2 \right),
\]

for every \( \varphi \in C([0, T], L^2(0, 1)) \cap C^1([0, T], H^{-1}(0, 1)) \) and satisfying \( L\varphi \in L^2(0, T; H^{-1}(0, 1)) \).

**Notation:** \( H = L^2(0, 1) \times H^{-1}(0, 1) \).

\( L\varphi = \varphi_{tt} - \varphi_{xx} \).
We follow the method used by C. Castro in the case of a moving pointwise control:


Some ingredients of the proof:

- D’Alembert formulae;
Observability inequality in time-dependent domain case

Idea of the proof

We follow the method used by C. Castro in the case of a moving pointwise control:

C. Castro, Exact controllability of the 1-D wave equation from a moving interior point, ESAIM COCV., 19 (2013).

Some ingredients of the proof:

▶ D'Alembert formulae;
▶ known observability inequality in the boundary case;
▶ equi-repartition of energy.

\[
\int_{t-\delta}^{t+\delta} |\varphi_x(0, s)|^2 ds \leq \frac{1}{\delta} \int_{\mathcal{U}^\delta(x,t+x)} (|\varphi_x|^2 + |\varphi_t|^2) dy ds
\]

\[
\int_{t-\delta}^{t+\delta} |\varphi_x(0, s)|^2 ds \leq \frac{1}{\delta} \int_{\mathcal{U}^\delta(x,t-x)} (|\varphi_x|^2 + |\varphi_t|^2) dy ds
\]
Observability inequality in time-dependent domain case

Idea of the proof

We follow the method used by C. Castro in the case of a moving pointwise control:

\[\text{C. Castro, } \textit{Exact controllability of the 1-D wave equation from a moving interior point}, \text{ ESAIM COCV., 19 (2013)}.\]

Some ingredients of the proof:

- D’Alembert formulae;
- known observability inequality in the boundary case;
Observability inequality in time-dependent domain case

Idea of the proof

We follow the method used by C. Castro in the case of a moving pointwise control:

C. Castro, Exact controllability of the 1-D wave equation from a moving interior point, ESAIM COCV., 19 (2013).

Some ingredients of the proof:

▶ D'Alembert formulae;
▶ known observability inequality in the boundary case;
▶ equi-repartition of energy.

Boundary observability inequality:

$$\| (\varphi(\cdot, 0), \varphi_t(\cdot, 0)) \|^2_H \leq C \int_0^T |\varphi_x(0, t)|^2 dt.$$  

combined with the previous estimate gives:

$$\| (\varphi(\cdot, 0), \varphi_t(\cdot, 0)) \|^2_V \leq C \left( \| \varphi_t \|^2_{L^2(qT)} + \| \varphi_x \|^2_{L^2(qT)} \right)$$

\[ H = L^2(0, 1) \times H^{-1}(0, 1) \]

\[ V = H^1_0(0, 1) \times L^2(0, 1) \]
We follow the method used by C. Castro in the case of a moving pointwise control:


Some ingredients of the proof:

- D’Alembert formulae;
- known observability inequality in the boundary case;
- equi-repartition of energy.
We follow the method used by C. Castro in the case of a moving pointwise control:


Some ingredients of the proof:
- D’Alembert formulae;
- known observability inequality in the boundary case;
- equi-repartition of energy.

Remark

*The proof of the proposition is specific to the one-dimensional case.*
Corollary (C. Castro, N.C., A. M"unch – 2014)

Let $T > 0$ and $q_T \subset (0, 1) \times (0, T)$ be such that any characteristic line starting at a point $x \in (0, 1)$ at time $t = 0$ and following the optical geometric laws when reflecting at the boundary $\Sigma_T$ must meet $q_T$.

Then the wave equation is null controllable in time $T$. 
Corollary (C. Castro, N.C., A. Münch – 2014)

Let $T > 0$ and $q_T \subset (0, 1) \times (0, T)$ be such that any characteristic line starting at a point $x \in (0, 1)$ at time $t = 0$ and following the optical geometric laws when reflecting at the boundary $\Sigma_T$ must meet $q_T$.

Then the wave equation is null controllable in time $T$.

Proof.

We apply HUM.
Corollary (C. Castro, N.C., A. Münch – 2014)

Let $T > 0$ and $q_T \subset (0, 1) \times (0, T)$ be such that any characteristic line starting at a point $x \in (0, 1)$ at time $t = 0$ and following the optical geometric laws when reflecting at the boundary $\Sigma_T$ must meet $q_T$.

Then the wave equation is null controllable in time $T$.

Proof.

We apply HUM.

Numerical approximation :

- usual problems due to the controllability of high frequencies;
- problems due to the controllability domain non-constant in time.
Hilbert Uniqueness Method - a reformulation


\[
\min_{\varphi \in \Phi} \hat{J}^*(\varphi), \quad \text{subject to} \quad L\varphi = 0.
\]

\[\Phi = \left\{ \varphi \in C([0, T], H^1_0(0, 1)) \cap C^1([0, T], L^2(0, 1)) \right\} \text{ such that } L\varphi \in L^2(0, T, H^{-1}(0, 1)) \].

**Remark**

**$\Phi$ is an Hilbert space endowed with the inner product**

\[
(\varphi, \varphi)_\Phi = \iint_{Q_T} \varphi(x, t)\varphi(x, t) \, dxdt + \eta \iint_{Q_T} \langle L\varphi, L\varphi \rangle_{-1} \, dx \, dt.
\]

for any fixed $\eta > 0$. 
Idea of the method: step by step

1. write the minimization of $J^*$ as a saddle-point problem for an associated Lagrangian.
Idea of the method: step by step

1. write the minimization of $J^*$ as a saddle-point problem for an associated Lagrangian.

2. write the optimality conditions for the Lagrangian as a mixed-formulation in $\varphi$ and $\lambda$. 
Idea of the method: step by step

We consider the following mixed formulation: find

$$\begin{align*}
(\varphi, \lambda) \in \Phi \times L^2(0, T, H_0^1(0, 1)) \text{ solution of }
\left\{ 
\begin{array}{ll}
a(\varphi, \varphi) + b(\varphi, \lambda) = l(\varphi), & \forall \varphi \in \Phi \\
b(\varphi, \lambda) = 0, & \forall \lambda \in L^2(0, T, H_0^1(0, 1)),
\end{array}
\right.
\end{align*}$$

where

$$\begin{align*}
a : \Phi \times \Phi &\to \mathbb{R}, \quad a(\varphi, \varphi) = \int\int_{Q_T} \varphi \overline{\varphi} dx dt + \eta \int\int_{Q_T} \langle L\varphi, \overline{L\varphi} \rangle_{-1} dx dt. \\
b : \Phi \times L^2(0, T, H_0^1(0, 1)) &\to \mathbb{R}, \quad b(\varphi, \lambda) = \int_0^T \langle L\varphi(\cdot, t), \lambda(\cdot, t) \rangle_{-1,1} dt. \\
l : \Phi &\to \mathbb{R}, \quad l(\varphi) = -\langle \varphi_t(\cdot, 0), y_0 \rangle_{-1,1} + \int_0^1 \varphi(x, 0)y_1(x) dx.
\end{align*}$$
Idea of the method: step by step

1. write the minimization of $J^*$ as a saddle-point problem for an associated Lagrangian.

2. write the optimality conditions for the Lagrangian as a mixed-formulation in $\varphi$ and $\lambda$.

3. use the generalized observability inequality in order to prove that this mixed formulation is well-posed:
   - $\varphi$ is the dual variable
   - $\lambda$ is the controlled solution.
Idea of the method: step by step

1. write the minimization of $J^*$ as a saddle-point problem for an associated Lagrangian.

2. write the optimality conditions for the Lagrangian as a mixed-formulation in $\varphi$ and $\lambda$.

3. use the generalized observability inequality in order to prove that this mixed formulation is well-posed:
   - $\varphi$ is the dual variable
   - $\lambda$ is the controlled solution.

4. discretize the mixed formulation and prove that the discrete controls converge to the exact continuous controls:
   - $C^1$ finite elements for $\varphi$
   - $P_1$ finite elements for $\lambda$. 
Numerical examples
Some controllability domains
Numerical examples
Some controllability domains – and associated meshes

Nicolae Cîndea
Approximation of moving controls for the wave equation
A first numerical test

Initial data to control

\[ y_0(x) = \sin(\pi x). \]

\[ y_1(x) = 0. \]
A first numerical example

Results

Figure: \( q_T = q_{1.2} \): Functions \( \varphi_h \) (Left) and \( \lambda_h \) (Right).
A first numerical example

Results

Figure: Norms $\|u - u_h\|_{L^2(q_T)}$ (○) and $\|y - \lambda h\|_{L^2(Q_T)}$ (♦) vs. $h$. 
A second numerical example

Initial data to control

\[ y_0(x) = 3x \mathbb{1}_{0,1/3}(x) + \frac{3(1-x)}{2} \mathbb{1}_{(1/3,1)}(x). \]

\[ y_1(x) = 0. \]
A second numerical example

Results

\[ \varphi_h \text{ (Left)} \quad \lambda_h \text{ (Right)}. \]
A second numerical example

Results

Table: \( q_T = q_T^2 = 2.2 \).

\[
\begin{array}{|c|ccccc|}
\hline
\# \text{ Mesh} & 1 & 2 & 3 & 4 & 5 \\
\hline
h & 7.18 \times 10^{-2} & 3.59 \times 10^{-2} & 1.79 \times 10^{-2} & 8.97 \times 10^{-3} & 4.49 \times 10^{-3} \\
\|v_h\|_{L^2(q_T)} & 5.350 & 5.263 & 5.195 & 5.172 & 5.165 \\
\|v - v_h\|_{L^2(q_T)} & 1.3571 & 9.78 \times 10^{-1} & 6.91 \times 10^{-1} & 5.13 \times 10^{-1} & 3.69 \times 10^{-1} \\
\|y - \lambda_h\|_{L^2(Q_T)} & 7.12 \times 10^{-3} & 3.23 \times 10^{-3} & 1.19 \times 10^{-3} & 4.82 \times 10^{-4} & 2.12 \times 10^{-4} \\
\hline
\end{array}
\]

- \( v \) – control of minimal \( L^2 \)-norm supported on \( q_T \);
- \( y \) – controlled solution by control \( v \).
A wave with variable speed of propagation

We consider the following wave equation

\[
\begin{aligned}
y_{tt}(x, t) - (c(x)y_x(x, t))_x &= v(x, t) \mathbb{1}_{QT}(x), \\
y(x, t) &= 0, \\
y(x, 0) = y_0(x), & \quad y_t(x, 0) = y_1(x),
\end{aligned}
\]

\[(x, t) \in QT \quad (x, t) \in \Sigma_T \quad x \in (0, 1).\]

We take the propagation speed \(c \in C^\infty(0, 1)\) given by

\[
c(x) = \begin{cases} 
1, & x \in [0, 0.45] \\
5, & x \in [0.55, 1].
\end{cases}
\]
A wave with variable speed of propagation

Numerical results

Figure: $q_T = q_2^2$ for a non-constant velocity of propagation. Function $\varphi_h$ (Left) and $\lambda_h$ (Right).
Conclusion

- We proved the exact controllability of the one-dimensional wave equation with a distributed control supported on a non-cylindrical domain;
- We developed a constructive method to compute the control of minimal $L^2$-norm supported in non-cylindrical domains.
- Numerical results indicate that the computed controls converge to the exact control.
Some perspectives

- \[ \| v_h - v \|_{L^2(q_T)} \rightarrow ch^\theta ? \]
- Prove a uniform “inf-sup” discrete condition.
- Optimization of the control’s support.
Some perspectives

- $\| v_h - v \|_{L^2(q_T)} \to c h^\theta$?
- Prove a uniform “inf-sup” discrete condition.
- Optimization of the control’s support.

- What about higher dimensional wave equations?

\[ h \quad \delta_h \]

\[ 10^{-2} \quad 10^{-1} \]
C. Castro, N. C., A. Münch, Controllability of the linear 1D wave equation with inner moving forces, SICON (2014).


N. C., A. Münch, A mixed formulation for the direct approximation of the control of minimal $L^2$-norm for linear type wave equations, Calcolo (2015).

Merci!
Thank you!