INDEX THEORY AND GROUPOIDS

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Abstract. These lecture notes are mainly devoted to a proof using groupoids and KK-theory of Atiyah-Singer index theorem on compact smooth manifolds. We will present an elementary introduction to groupoids, C^*-algebras, KK-theory and pseudodifferential calculus on groupoids. We will finish by showing that the point of view adopted here generalizes to the case of conical pseudomanifolds.

Contents

Introduction

During this course we intend to give the tools involved in our approach of index theory for singular spaces. The global framework adopted here is Noncommutative Geometry.

Most of the ideas were introduced by A. Connes and G. Skandalis [10, 11, 14] in the 80'th with the study of foliation. The goal was to study the space of leaves $M/\mathcal{F}$ of a smooth foliation on a smooth manifold.

The first idea, due to A. Connes, is to study such a topological space via a C^*-*algebra* similar to the algebra of continuous functions on the space $M/\mathcal{F}$.

This idea is motivated by the Gelfand transform which gives a one to one correspondence between locally compact spaces and commutative C^*-algebras: any commutative algebra A is isomorphic to the algebra $C_0(X)$ of continuous function vanishing at infinity on some locally compact space X.

In order to associate a relevant C^*-algebra to the foliation $\mathcal{F}$, the first step is to find a *groupoid* associated to $\mathcal{F}$: it should be smooth, define the foliation and as small as possible. The answer to this problem is the holonomy groupoid. Next we can

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associate to such a groupoid a C^*-algebra. The definitions and study of groupoids C^*-algebras are due to J. Renault [44, 45].

In the 60'th, M. Atiyah and I. Singer [6] have shown there famous index theorem. Roughly speaking, they showed that given a closed manifold one can associate to any elliptic operator an integer called the *index* which can be described in two different way: one purely analytic and the other one purely topological.
A. Connes with G. Skandalis [14], using the holonomy groupoid, were able to give sense to (and prove) the index theorem for foliations.

Moreover, in both cases of closed manifolds or foliations, the index map can be described with the use of a groupoid of another kind, namely a *deformation groupoid*. This point, already explained by A. Connes in [13], is developed in a joint work of the two authors and V. Nistor [20], in order to produce a statement and a simplified proof of the index theorem which have the great advantage to work for a class of singular spaces (namely, pseudomanifolds).

The contents of this series of lecture is the following.

As mentioned earlier, the first problem in the study of a singular geometrical situation is to associate to it a mathematical object which carries the information that one wants to study and which is regular enough to be analyzed in a reasonable way. In noncommutative geometry we often answer to this problem by looking for a good *groupoid* and construct its *C^*-algebra*. These points will be the subject of the first two sections.
Once the situation is desingularized, say, trough the construction of a groupoid and its C^*-algebra, one may look for invariants which capture the basic properties. Roughly speaking, the *KK-theory* groups are convenient groups of invariants for C^*-algebras and KK-theory comes with powerful tools to make computations. Kasparov's bivariant K-theory will be the main topic of sections 3 to 5.
Then we will go to *index theory*. First, we will briefly explain in section 6 the pseudo-differential calculus on groupoids. Then we will prove in section 7 the Atiyah Singer index theorem for compact manifolds using the language of groupoids and KK-theory. Finally we will see during the last section how one can extend these results to *conical pseudo-manifolds*.

Acknowledgments We would like to thank Georges Skandalis who allowed us to use his several works to make this course, in particular the manuscript of one of his course [46, 47]. In addition, we would like to thank warmly Jorge Plazas for having typewritten a part of this course during the summer school.

Groupoids and there C^*-algebras

This first part will be devoted to the notion of groupoids, especially differentiable groupoids. We will see the definitions and look at standard examples. The interested reader may look for example at [33, 12]. Then we will recall the definition of C^*-algebras and see how one can associate a C^*-algebra to a groupoid. The theory of C^*-algebra of groupoid was initiated by Jean Renault [44]. A really good reference for the construction of groupoid C^*-algebras is [31] from which the end of this section is inspired.

1. Groupoids

1.1. **Definitions and basic examples of groupoids.**

Definition 1. Let G and $G^{(0)}$ be two sets. A structure of *groupoid* on G over $G^{(0)}$ is given by the following homomorphisms:

- An injective map $u : G^{(0)} \to G$. The map u is called the *unit map*. We often identify $G^{(0)}$ with its image in G. The set $G^{(0)}$ is called the *set of units* of the groupoid.
- Two surjective maps: $r, s : G \to G^{(0)}$, which are respectively the *range* and *source* map. They are equal to identity on the space of units.
- An involution:
$$\begin{array}{cccc} i: & G & \to & G \\ & \gamma & \mapsto & \gamma^{-1} \end{array}$$
called the *inverse* map. It satisfies: $s \circ i = r$.
- A map
$$\begin{array}{cccc} p: & G^{(2)} & \to & G \\ & (\gamma_1, \gamma_2) & \mapsto & \gamma_1 \cdot \gamma_2 \end{array}$$
called the *product*, where the set
$$G^{(2)} := \{(\gamma_1, \gamma_2) \in G \times G \mid s(\gamma_1) = r(\gamma_2)\}$$
is the set of *composable pair*. Moreover for $(\gamma_1, \gamma_2) \in G^{(2)}$ we have $r(\gamma_1 \cdot \gamma_2) = r(\gamma_1)$ and $s(\gamma_1 \cdot \gamma_2) = s(\gamma_2)$.

The following properties must be fulfilled:

- The product is associative: for any $\gamma_1,\ \gamma_2,\ \gamma_3$ in G such that $s(\gamma_1) = r(\gamma_2)$ and $s(\gamma_2) = r(\gamma_3)$ the following equality holds
$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3)\ .$$
- For any γ in G: $r(\gamma) \cdot \gamma = \gamma \cdot s(\gamma) = \gamma$ and $\gamma \cdot \gamma^{-1} = r(\gamma)$.

We will often use the following notations:
$$G_A := s^{-1}(A)\ ,\ G^B = r^{-1}(B) \ \text{ and } G_A^B = G_A \cap G^B\ .$$
If x belongs to $G^{(0)}$, the *s-fiber* (resp. *r-fiber*) of G over x is $G_x = s^{-1}(x)$ (resp. $G^x = r^{-1}(x)$).

The groupoid is *topological* when G and $G^{(0)}$ are topological spaces with $G^{(0)}$ Hausdorff, the structural homomorphisms are continuous and i is an homeomorphism. We will often ask our topological groupoids to be *locally compact*. This means that $G \rightrightarrows G^{(0)}$ is a topological groupoid, such that G is second countable, each point γ in G has a compact (Hausdorff) neighborhood, and the map s is open. In this situation the map r is open and the s-fibers of G are Hausdorff.

The groupoid is *smooth* when G and $G^{(0)}$ are second countable smooth manifolds with $G^{(0)}$ Hausdorff, the structural homomorphisms are smooth, u is an embedding, s is a submersion and i is a diffeomorphism.

When $G \rightrightarrows G^{(0)}$ is at least topological, we say that G is *s-connected* when for any $x \in G^{(0)}$, the s-fiber of G over x is connected. The s-connected component of a groupoid G is $\cup_{x \in X} CG_x$ where CG_x is the connected component of the s-fiber G_x which contains the unit $u(x)$.

Examples

1. A space X is a groupoid over itself with $s = r = u = \mathrm{Id}$.

2. A group $G \rightrightarrows \{e\}$ is a groupoid over its unit e, with the usual product and inverse map.

3. A group bundle : $\pi : E \to X$ is a groupoid $E \rightrightarrows X$ with $r = s = \pi$ and algebraic operations given by the structure of group of each fiber E_x, $x \in X$.

4. If $\mathcal{R}$ is an equivalence relation on a space X, then the graph of $\mathcal{R}$:

$$G_{\mathcal{R}} := \{(x, y) \in X \times X \mid x\mathcal{R}y\}$$

admits a structure of groupoid over X, which is given by:

$$u(x) = (x, x)\ ,\ s(x, y) = y\ ,\ r(x, y) = x\ , (x, y)^{-1} = (y, x)\ ,\ (x, y) \cdot (y, z) = (x, z)$$

for $x,\ y,\ z$ in X.

When $x\mathcal{R}y$ for any $x,\ y$ in X, $G_{\mathcal{R}} = X \times X \rightrightarrows X$ is called the *pair groupoid*.

5. If G is a group acting on a space X, the *groupoid of the action* is $G \times X \rightrightarrows X$ with the following structural homomorphisms

$$\begin{gathered} u(x) = (e, x)\ ,\ s(g, x) = x\ ,\ r(g, x) = g \cdot x\ , \\ (g, x)^{-1} = (g^{-1}, g \cdot x)\ ,\ (h, g \cdot x) \cdot (g, x) = (hg, x)\ , \end{gathered}$$

for x in X and $g,\ h$ in G.

6. Let X be a topological space the *homotopy groupoid* of X is

$$\Pi(X) := \{\bar{c} \mid c : [0, 1] \to X \text{ a continuous path}\} \rightrightarrows X$$

where $\bar{c}$ denotes the homotopy class (with fixed endpoints) of c. We let

$$\begin{gathered} u(x) = \overline{c_x} \text{ where } c_x \text{ is the constant path equal to } x,\ s(\bar{c}) = c(0),\ r(\bar{c}) = c(1) \\ \bar{c}^{-1} = \overline{c^{-1}} \text{ where } c^{-1}(t) = c(1 - t), \\ \overline{c_1} \cdot \overline{c_2} = \overline{c_1 \cdot c_2} \text{ where } c_1 \cdot c_2(t) = c_2(2t) \text{ for } t \in [0, \tfrac{1}{2}] \text{ and } c_1 \cdot c_2(t) = c_1(2t - 1) \text{ for } t \in [\tfrac{1}{2}, 1]\ . \end{gathered}$$

When X is a smooth manifold of dimension n, $\Pi(X)$ is naturally endowed with a smooth structure (of dimension $2n$). A neighborhood of $\bar{c}$ being of the form $\{\bar{c_1}\bar{c}\bar{c_0} \mid c_1(0) = c(1),\ c(0) = c_0(1),\ Imc_i \subset U_i\ i = 0, 1\}$ where U_i is a given neighborhood of $c(i)$ in X.

1.2. **Homomorphisms and Morita Equivalences.**

homomorphisms

Let $G \rightrightarrows G^{(0)}$ be a groupoid of source s_G and range r_G and $H \rightrightarrows H^{(0)}$ be a groupoid of source s_H and range r_H. A groupoid *homomorphism* from G to H is given by two maps :

$$f : G \to H \text{ and } f^{(0)} : G^{(0)} \to H^{(0)}$$

such that

- $r_H \circ f = f^{(0)} \circ r_G$,
- $f(\gamma)^{-1} = f(\gamma^{-1})$ for any $\gamma \in G$,
- $f(\gamma_1 \cdot \gamma_2) = f(\gamma_1) \cdot f(\gamma_2)$ for $\gamma_1,\ \gamma_2$ in G such that $s_G(\gamma_1) = r_G(\gamma_2)$.

We say that f is a *homomorphism over* $f^{(0)}$. When $G^{(0)} = H^{(0)}$ and $f^{(0)} = \mathrm{Id}$ we say that f is a *homomorphism over identity.*

The homomorphism f is an *isomorphism* when the maps $f,\ f^{(0)}$ are bijection and $f^{-1} : H \to G$ is a homomorphism over $(f^{(0)})^{-1}$.

As usual, when we are dealing with topological groupoids we ask f to be continuous and when we are dealing with smooth groupoids we ask f to be smooth.

Morita equivalence

In most situations, the good notion of "isomorphism of locally compact groupoids" is the weaker notion of Morita equivalence.

Definition 2. Two locally compact groupoids $G \rightrightarrows G^{(0)}$ and $H \rightrightarrows H^{(0)}$ are *Morita equivalent* if there exists a locally compact groupoid $P \rightrightarrows G^{(0)} \sqcup H^{(0)}$ such that

- the restrictions of P over $G^{(0)}$ and $H^{(0)}$ are respectively G and H:
$$P_{G^{(0)}}^{G^{(0)}} = G \text{ and } P_{H^{(0)}}^{H^{(0)}} = H$$
- for any $\gamma \in P$ there exists η in $P_{G^{(0)}}^{H^{(0)}} \cup P_{H^{(0)}}^{G^{(0)}}$ such that (γ, η) is a composable pair (ie $s(\gamma) = r(\eta)$).

Examples 1. Let $f : G \to H$ be an isomorphism of locally compact groupoid then the following groupoid defines a Morita equivalence between H and G:

$$P = G \sqcup \tilde{G} \sqcup \tilde{G}^{-1} \sqcup H \rightrightarrows G^{(0)} \sqcup H^{(0)}$$

where with the obvious notations we have

$$G = \tilde{G} = \tilde{G}^{-1}$$

$$s_P = \begin{cases} s_G \text{ on } G \\ s_H \circ f \text{ on } \tilde{G} \\ r_G \text{ on } \tilde{G}^{-1} \\ s_P = s_H \text{ on } H \end{cases}, \ r_P = \begin{cases} r_G \text{ on } G \sqcup \tilde{G} \\ s_H \circ f \text{ on } \tilde{G}^{-1} \\ r_H \text{ on } H \end{cases}, \ u_P = \begin{cases} u_G \text{ on } G^{(0)} \\ u_H \text{ on } H^{(0)} \end{cases}$$

$$i_P(\gamma) = \begin{cases} i_G(\gamma) \text{ on } G \\ i_H(\gamma) \text{ on } H \\ \gamma \in \tilde{G}^{-1} \text{ on } \tilde{G} \\ \gamma \in \tilde{G} \text{ on } \tilde{G}^{-1} \end{cases}, p_P(\gamma_1, \gamma_2) = \begin{cases} p_G(\gamma_1, \gamma_2) \text{ on } G^{(2)} \\ p_H(\gamma_1, \gamma_2) \text{ on } H^{(2)} \\ p_G(\gamma_1, \gamma_2) \in \tilde{G} \text{ for } \gamma_1 \in G, \ \gamma_2 \in \tilde{G} \\ p_G(\gamma_1, f^{-1}(\gamma_2)) \in \tilde{G} \text{ for } \gamma_1 \in \tilde{G}, \ \gamma_2 \in H \\ p_G(\gamma_1, \gamma_2) \in G \text{ for } \gamma_1 \in \tilde{G}, \ \gamma_2 \in \tilde{G}^{-1} \\ f \circ p_G(\gamma_1, \gamma_2) \in H \text{ for } \gamma_1 \in \tilde{G}, \ \gamma_2 \in \tilde{G}^{-1} \end{cases}$$

2. Suppose that $G \rightrightarrows G^{(0)}$ is a locally compact groupoid and $\varphi : X \to G^{(0)}$ is an open surjective map, where X is a locally compact space. The *pull back groupoid* is the groupoid:

$${}^*\varphi^*(G) \rightrightarrows X$$

where

$${}^*\varphi^*(G) = \{(x, \gamma, y) \in X \times G \times X \mid \varphi(x) = r(\gamma) \text{ and } \varphi(y) = s(\gamma)\}$$

with $s(x, \gamma, y) = y$, $r(x, \gamma, y) = x$, $(x, \gamma_1, y)\cdot(y, \gamma_2, z) = (x, \gamma_1\cdot\gamma_2, z)$ and $(x, \gamma, y)^{-1} = (y, \gamma^{-1}, x)$.
ï??¿The groupoid ${}^*\varphi^*(G)$ is naturally endowed with a structure of locally compact groupoid. Moreover the groupoids G and ${}^*\varphi^*(G)$ are Morita equivalent.

To prove this last point, one can put a structure of locally compact groupoid on $P = G \sqcup X \times_r G \sqcup G \times_s X \sqcup {}^*\varphi^*(G)$ over $X \sqcup G^{(0)}$ where $X \times_r G = \{(x, \gamma) \in X \times G \mid \varphi(x) = r(\gamma)\}$ and $G \times_s X = \{(\gamma, x) \in G \times X \mid \varphi(x) = s(\gamma)\}$.

1.3. **The orbits of a groupoid.**

Suppose that $G \rightrightarrows G^{(0)}$ is a groupoid of source s and range r.

Definition 3. The *orbit* of G passing trough x is the following subset of $G^{(0)}$:

$$Or_x = r(G_x) = s(G^x) \ .$$

We let $G^{(0)}/G$ or $Or(G)$ be the *space of orbits*.

The *isotropy group* of G at x is G_x^x which is naturally endowed with a group structure with x as unit. Notice that multiplication induces a free left (resp. right) action of

G_x^x on G^x (resp. G_x). Moreover the orbits space of this action is precisely Or_x and the restriction $s : G^x \to Or_x$ is the quotient map.

Examples and remark 1. In the example 4. below the orbits of $G_{\mathcal{R}}$ correspond exactly to the orbits of the equivalence relation $\mathcal{R}$. In the example 5. the orbits of the groupoid of the action are the orbits of the action.
2. The second assertion in the definition of Morita equivalence means precisely that both $G^{(0)}$ and $H^{(0)}$ meet all the orbits of P. Moreover one can show that the map

$$\begin{array}{ccc} Or(G) & \to & Or(H) \\ Or(G)_x & \mapsto & Or(P)_x \cap H^{(0)} \end{array}$$

is a bijection. In other word, when two groupoids are Morita Equivalent, they have the same orbits space.

Groupoids are often used in Noncommutative Geometry for the study of geometrical singular situations. In many geometrical situations, the topological space which arise is strongly non Hausdorff and the standard tools do not apply. Nevertheless, it is sometimes possible to associate to such a space X a relevant C^*-algebra to take the place of $C_0(X)$. Usually we first associate a groupoid $G \rightrightarrows G^{(0)}$ such that its space of orbits $G^{(0)}/G$ is (equivalent to) X. If the groupoid is enough regular (smooth for example) then we can associate natural C^*-algebras to G. This point will be discussed later.
In other words we desingularize a singular space by viewing it as coming from the action of a nice groupoid on its space of units. To illustrate this point let us look at two examples.

1.4. **Groupoids associated to a foliation.**

Let M be a smooth manifold.

Definition 4. A (regular) smooth *foliation* $\mathcal{F}$ on M of dimension p is a partition $\{F_i\}_I$ of M where each F_i is an immersed sub-manifold of dimension p called a *leaf*. Moreover the manifold M admits charts of the following type:

$$\varphi : U \to \mathbb{R}^p \times \mathbb{R}^q$$

where U is open in M and such that for any connected component P of $F_i \cap U$ where $i \in I$, there is a $t \in \mathbb{R}^q$ such that $\varphi(P) = \mathbb{R}^p \times \{t\}$.

In this situation the *tangent space to the foliation* $T\mathcal{F} := \cup_I TF_i$ is a sub-bundle of TM stable under Lie Bracket.

The *space of leaves* $M/\mathcal{F}$ is the quotient of M by the equivalence relation: being on the same leaf.

Typical example. Take $M = P \times T$ where P and T are connected smooth manifolds with the partition into leaves given by $\{P \times \{t\}\}_{t\in T}$. Every foliation is locally of this type.

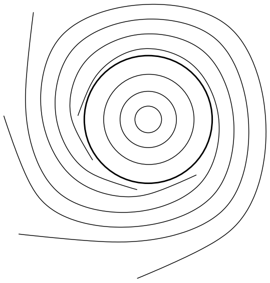
The space of leaves of a foliation is often difficult to study. As it appears in the following two examples:

Examples 1. Let $\tilde{\mathcal{F}}_a$ be the foliation on the plane $\mathbb{R}^2$ by lines $\{y = ax + t\}_{t\in\mathbb{R}}$ where a belongs to $\mathbb{R}$. Take the torus $T = \mathbb{R}^2/\mathbb{Z}^2$ as being the quotient of $\mathbb{R}^2$ by translations of $\mathbb{Z}^2$. We denote by $\mathcal{F}_a$ the foliation induced by $\tilde{\mathcal{F}}_a$ on T. When a is rational the space of leaves is a circle but when a is irrational it is topologically

equivalent to a point (ie: each point is in any neighborhood of any other point).
2. Let $\mathbb{C} \setminus \{(0)\}$ be foliated by:

$$\{S_t\}_{t\in]0,1]} \cup \{D_t\}_{t\in]0,2\pi]}$$

where $S_t = \{z \in \mathbb{C} \mid |z| = t\}$ is the circle of radius t and $D_t = \{z = e^{i(x+t)+x} \mid x \in \mathbb{R}_*^+\}$.



The *holonomy groupoid* is a smooth groupoid which desingularize the space of leaves of a foliation. Precisely, if $\mathcal{F}$ is a smooth foliation on a manifold M its *holonomy groupoid* is the smallest s-connected smooth groupoid $G \rightrightarrows M$ whose orbits are precisely the leaves of the foliation.
Here *smallest* means that if $H \rightrightarrows M$ is another s-connected smooth groupoid whose orbits are the leaves of the foliation then there is a surjective groupoid homomorphism : $H \to G$ over identity.

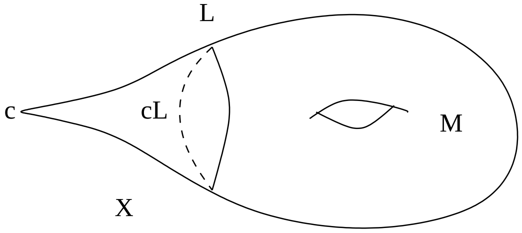
The first naive idea to define such a groupoid is to consider the graph of the equivalence relation being in the same leaf. This does not work: you get a groupoid but it may not be smooth. This fact can be observed on example 2. below. Another idea consists in looking at the *homotopy groupoid*. Let $\Pi(\mathcal{F})$ be the set of homotopy class of smooth paths lying on leaves of the foliation. It is naturally endowed with a groupoid structure similarly to the homotopy groupoid of section 1. example 6. Such a groupoid can be naturally equipped with a smooth structure (of dimension $2p + q$) and the holonomy groupoid is a quotient of this homotopy groupoid. In particular, when the leaves have no homotopy, the holonomy groupoid is the graph of the equivalence relation being in the same leaf.

1.5. **The noncommutative tangent space of a conical pseudomanifold.**

It may happens that the underlying topological space which is under study is a nice compact space which is "almost" smooth. This is the case of pseudo-manifolds [24, 34, 51], for a review on the subject see [9, 28]. In such a situation we can desingularize the tangent space [19, 18]. Let us see how it works in the case of a conical pseudomanifold with one singularity.

Let M be an m-dimensional compact manifold with a compact boundary L. We attach to L the cone $cL = L \times [0, 1]/L \times \{0\}$, using the obvious map $L \times \{1\} \to L \subset \partial M$. The new space $X = cL \cup M$ is a compact pseudomanifold with a singularity [24]. In general, there is no manifold structure around the vertex c of the cone.

We will use the following notations: $X^\circ = X \setminus \{c\}$ is the *regular part*, X^+ denotes $M \setminus L = X \setminus cL$, $\overline{X_+} = M$ its closure in X and $X^- = L\times]0, 1[$. If y is a point of the cylindrical part of $X \setminus \{c\}$, we will write $y = (y_L, k_y)$ where $y_L \in L$ and $k_y \in]0, 1]$ are the tangential and radial coordinates. We extend the map k on M to a smooth defining function for its boundary; in particular, $k^{-1}(1) = L = \partial M$ and $k(M) \subset [1, +\infty[$.



Let us consider $T\overline{X^+}$, the restriction to $\overline{X^+}$ of the tangent bundle of X°. As a $\mathcal{C}^\infty$ vector bundle, it is a smooth groupoid with unit space $\overline{X^+}$. We define the groupoid $T^{\mathsf{S}}X$ as the disjoint union:

$$T^{\mathsf{S}}X = X^- \times X^- \ \cup \ T\overline{X^+} \underset{r}{\overset{s}{\rightrightarrows}} X^\circ,$$

where $X^- \times X^- \rightrightarrows X^-$ is the pair groupoid.

In order to endow $T^{\mathsf{S}}X$ with a smooth structure, compatible with the usual smooth structure on $X^- \times X^-$ and on $T\overline{X^+}$, we have to take care of what happen around points of $T\overline{X^+}|_{\partial\overline{X^+}}$.

Let τ be a smooth positive function on $\mathbb{R}$ such that $\tau^{-1}(\{0\}) = [1, +\infty[$. We let $\tilde{\tau}$ be the smooth map from X° to $\mathbb{R}^+$ given by $\tilde{\tau}(y) = \tau \circ k(y)$.

Let (U, ϕ) be a local chart for X° around $z \in \partial\overline{X^+}$. Setting $U^- = U \cap X^-$ and $\overline{U^+} = U \cap \overline{X^+}$, we define a local chart of G by:

$$\tilde{\phi}: \quad U^- \times U^- \cup \ T\overline{U^+} \quad \longrightarrow \quad \mathbb{R}^m \times \mathbb{R}^m$$

$$\tilde{\phi}(x,y) = (\phi(x), \frac{\phi(y) - \phi(x)}{\tilde{\tau}(x)}) \text{ if } (x,y) \in U^- \times U^- \text{ and} \tag{1.1}$$

$$\tilde{\phi}(x,V) = (\phi(x), (\phi)_*(x,V)) \text{ elsewhere.}$$

We define in this way a structure of smooth groupoid on $T^{\mathsf{S}}X$. Note that at the topological level, the space of orbits of $T^{\mathsf{S}}X$ is equivalent to X: there is a canonical isomorphism between the algebras $C(X)$ and $C(X^\circ/T^{\mathsf{S}}X)$.

The smooth groupoid $T^{\mathsf{S}}X \rightrightarrows X^\circ$ is called the *noncommutative tangent space* of X.

1.6. **Lie Theory for smooth groupoids.** Let us go into the more specific world of smooth groupoids. Similarly to Lie groups which admit Lie algebras, any smooth groupoids has a *Lie algebroid* [41, 40].

Definition 5. A *Lie algebroid* $\mathcal{A} = (p : \mathcal{A} \to TM, [\ ,\ ]_{\mathcal{A}})$ on a smooth manifold M is a vector bundle $\mathcal{A} \to M$ equipped with a bracket $[\ ,\ ]_{\mathcal{A}} : \Gamma(\mathcal{A}) \times \Gamma(\mathcal{A}) \to \Gamma(\mathcal{A})$ on the module of sections of $\mathcal{A}$ together with a homomorphism of fiber bundle $p : \mathcal{A} \to TM$ from $\mathcal{A}$ to the tangent bundle TM of M called the *anchor*, such that:

i) the bracket $[\ ,\ ]_{\mathcal{A}}$ is $\mathbb{R}$-bilinear, antisymmetric and satisfies the Jacobi identity,

ii) $[X, fY]_{\mathcal{A}} = f[X,Y]_{\mathcal{A}} + p(X)(f)Y$ for all $X,\ Y \in \Gamma(\mathcal{A})$ and f a smooth function of M,

iii) $p([X,Y]_{\mathcal{A}}) = [p(X), p(Y)]$ for all $X,\ Y \in \Gamma(\mathcal{A})$.

Each Lie groupoid admits a Lie algebroid. Let us recall this construction.

Let $G \overset{s}{\underset{r}{\rightrightarrows}} G^{(0)}$ be a Lie groupoid. We denote by T^sG the subbundle of TG of s-vertical tangent vectors. That is T^sG is the kernel of the differential Ts of s.

For all γ in G let $R_\gamma : G_{r(\gamma)} \to G_{s(\gamma)}$ be the right multiplication by γ. A tangent vector field Z on G is *right invariant* if it satisfies:

- Z is s-vertical: $Ts(Z) = 0$.
- For all (γ_1, γ_2) in $G^{(2)}$, $Z(\gamma_1 \cdot \gamma_2) = TR_{\gamma_2}(Z(\gamma_1))$.

Note that if Z is a right invariant vector field and h^t its flow then for all t, the local diffeomorphism h^t is a *local left translation* of G that is $h^t(\gamma_1 \cdot \gamma_2) = h^t(\gamma_1) \cdot \gamma_2$ when it makes sense.

The Lie algebroid $\mathcal{A}G$ of G is defined in the following way:

- The fiber bundle $\mathcal{A}G \to G^{(0)}$ is the restriction of T^sG to $G^{(0)}$. In other words: $\mathcal{A}G = \cup_{x \in G^{(0)}} T_x G_x$ is the union of the tangent spaces to the s-fiber at the corresponding unit.
- The *anchor* $p : \mathcal{A}G \to TG^{(0)}$ is the restriction of the differential Tr of r to $\mathcal{A}G$.
- If $Y : U \to \mathcal{A}G$ is a local section of $\mathcal{A}G$, where U is an open subset of $G^{(0)}$, we define the local *right invariant vector field* Z_Y *associated* with Y by

  $$Z_Y(\gamma) = TR_\gamma(Y(r(\gamma))) \text{ for all } \gamma \in G^U .$$

  The Lie bracket is then defined by:

  $$\begin{array}{rccc} [\ ,\ ] : & \Gamma(\mathcal{A}G) \times \Gamma(\mathcal{A}G) & \longrightarrow & \Gamma(\mathcal{A}G) \\ & (Y_1, Y_2) & \mapsto & [Z_{Y_1}, Z_{Y_2}]_{G^{(0)}} \end{array}$$

  where $[Z_{Y_1}, Z_{Y_2}]$ denotes the s-vertical vector field obtained with the usual bracket and $[Z_{Y_1}, Z_{Y_2}]_{G^{(0)}}$ is the restriction of $[Z_{Y_1}, Z_{Y_2}]$ to $G^{(0)}$.

Example If $\Pi(\mathcal{F})$ is the homotopy groupoid (or the holonomy groupoid) of a smooth foliation, its Lie algebroid is the tangent space $T\mathcal{F}$ to the foliation. The anchor is the inclusion. In particular the Lie algebroid of the pair groupoid $M \times M$ on a smooth manifold M is TM, the anchor being the identity map.

Lie theory for groupoids is much more tricky than for groups. For a long time people thought that, as for Lie algebras, every Lie algebroid integrates into a Lie groupoid [42]. In fact this assertion, named *Lie's third theorem for Lie algebroid* is false. This was pointed out by a counter example given by P. Molino and R. Almeida in [1]. Since that time, a lot of works has been done around this problem. A few years ago M. Crainic and R.L. Fernandes [15] have completely solved this question by giving a necessary and sufficient condition for the integrability of Lie algebroids.

1.7. **Example of groupoids involved in Index theory.** Index theory is a part of non commutative geometry where groupoids may play a crucial role. Index theory will be treated later in this course but we want to present here some of the groupoids which will arise.

Definition 6. A smooth groupoid G is called a *deformation groupoid* if:

$$G = G_1 \times \{0\} \cup G_2 \times]0,1] \rightrightarrows G^{(0)} = M \times [0,1] ,$$

where G_1 and G_2 are smooth groupoids with unit space M. That is, G is obtained by gluing $G_2\times]0,1] \rightrightarrows M\times]0,1]$ which is the groupoid G_2 parametrized by $]0,1]$ with the groupoid $G_1 \times \{0\} \rightrightarrows M \times \{0\}$.

Example Let G be a smooth groupoid and let $\mathcal{A}G$ be its Lie algebroid. The *adiabatic groupoid* of G [13, 36, 37] is a deformation of G on its Lie algebroid:

$$G_{ad} = \mathcal{A}G \times \{0\} \cup G\times]0,1] \rightrightarrows G^{(0)} \times [0,1] \ .$$

Here, the vector bundle $\pi : \mathcal{A}G \to G^{(0)}$ is considered as a groupoid in the obvious way. One can put a natural smooth structure on G_{ad}.

The tangent groupoid

A special example of adiabatic groupoid is the *tangent groupoid* of A. Connes [13]. Consider the pair groupoid $M \times M$ on a smooth manifold M. We have seen that its Lie algebroid is TM. In this situation, the adiabatic groupoid is called the *tangent groupoid* and is given by:

$$\mathcal{G}^t_M := TM \times \{0\} \sqcup M \times M\times]0,1] \rightrightarrows M \times [0,1] \ .$$

The Lie algebroid is the bundle $\mathcal{A}(\mathcal{G}^t_M) := TM \times [0,1] \to M \times [0,1]$ with anchor $p : (x, V, t) \in TM \times [0,1] \mapsto (x, tV, t, 0) \in TM \times T[0,1]$.

Choose a riemannian metric on M, the smooth structure on $\mathcal{G}^t_M$ is such that the following map :

$$\begin{array}{ccc} \mathcal{U} \subset TM \times [0,1] & \to & \mathcal{G}^t_M \\ (x, V, t) & \mapsto & \left\{ \begin{array}{l} (x, V, 0) \text{ if } t = 0 \\ (x, exp_x(-tV), t) \text{ elsewhere} \end{array} \right. \end{array}$$

is a smooth diffeomorphism on its image, where $\mathcal{U}$ is an open neighborhood of $TM \times \{0\}$.

The previous construction of the tangent groupoid of a compact manifold generalize to the case of conical manifold. When X is a conical manifold, its tangent groupoid is a deformation of the pair groupoid over X° into the groupoid $T^{\mathsf{S}}X$. This deformation has a nice description at the level of Lie algebroids. Indeed, with the notation of 1.5, the Lie algebroid of $\mathcal{G}^t_X$ is the (unique) Lie algebroid given by the fiber bundle $\mathcal{A}\mathcal{G}^t_X = [0,1] \times \mathcal{A}(T^{\mathsf{S}}X) = [0,1] \times TX^\circ \to [0,1] \times X^\circ$, with anchor map

$$\begin{array}{rccc} p_{\mathcal{G}^t_X} : & \mathcal{A}\mathcal{G}^t_X = [0,1] \times TX^\circ & \longrightarrow & T([0,1] \times X^\circ) = T[0,1] \times TX^\circ \\ & (\lambda, x, V) & \mapsto & (\lambda, 0, x, (\tilde{\tau}(x) + \lambda)V) \ . \end{array}$$

Such a Lie algebroid is almost injective, thus it is integrable [15, 17].
Moreover, it integrates into the *tangent groupoid* which is defined by:

$$\mathcal{G}^t_X = X^\circ \times X^\circ\times]0,1] \ \cup \ T^{\mathsf{S}}X \times \{0\} \rightrightarrows X^\circ \times [0,1].$$

Once again one can equip such a groupoid with a smooth structure compatible with the usual one on each pieces: $X^\circ \times X^\circ\times]0,1]$ and $T^{\mathsf{S}}X \times \{0\}$ [19].

The Thom groupoid

Another important deformation groupoid for our purpose is the *Thom groupoid* [20].

Let $\pi : E \to X$ be a *conical vector* bundle. This means that X is a conical manifolds (or a smooth manifold without vertexes) and we have a smooth vector bundle $\pi^\circ : E^\circ \to X^\circ$ which restriction to $X^- = L\times]0,1[$ is equal to $E_L\times]0,1[$

where $E_L \to L$ is a smooth vector bundle. If $E^+ \to X^+$ denotes the bundle E° restricted to X^+, then E is the conical manifold: $E = cE_L \cup E^+$.

When X is a smooth manifold (with no conical point), it is just the usual notion of smooth vector bundle.

From the definition, π restricts to a smooth vector bundle map $\pi^\circ : E^\circ \to X^\circ$. We let $\pi_{[0,1]} = \pi^\circ \times id : E^\circ \times [0,1] \to X^\circ \times [0,1]$.

We consider the tangent groupoids $\mathcal{G}^t_X \rightrightarrows X^\circ$ for X and $\mathcal{G}^t_E \rightrightarrows E^\circ$ for E equipped with a smooth structure constructed using the same gluing function τ (in particular $\tilde{\tau_X} \circ \pi = \tilde{\tau_E}$). We denote by ${}^*\pi^*_{[0,1]}(\mathcal{G}^t_X) \rightrightarrows E^\circ \times [0,1]$ the pull back of $\mathcal{G}^t_X$ by $\pi_{[0,1]}$.

We first associate to the conical vector bundle E a deformation groupoid $\mathcal{T}^t_E$ from ${}^*\pi^*_{[0,1]}(\mathcal{G}^t_X)$ to $\mathcal{G}^t_E$. More precisely, we define:

$$\mathcal{T}^t_E := \mathcal{G}^t_E \times \{0\} \sqcup {}^*\pi^*_{[0,1]}(\mathcal{G}^t_X) \times]0,1] \rightrightarrows E^\circ \times [0,1] \times [0,1].$$

Once again, one can equip $\mathcal{T}^t_E$ with a smooth structure [20] and the restriction of $\mathcal{T}^t_E$ to $E^\circ \times \{0\} \times [0,1]$ leads to a smooth groupoid:

$$\mathcal{H}_E = T^{\mathsf{S}}E \times \{0\} \sqcup {}^*\pi^*(T^{\mathsf{S}}X) \times]0,1] \rightrightarrows E^\circ \times [0,1],$$

called a *Thom groupoid* associated to the conical vector bundle E over X.

The following example explains what these constructions become if there are no singularities.

Example Suppose that $p : E \to M$ is a smooth vector bundle over the smooth manifold M. Then $\mathcal{G}^t_E = TE \times \{0\} \sqcup E \times E \times]0,1] \rightrightarrows E \times [0,1]$ and $\mathcal{G}^t_M = TM \times \{0\} \sqcup M \times M \times]0,1] \rightrightarrows M \times [0,1]$ are the usual tangent groupoids. In this example the groupoid $\mathcal{T}^t_E$ will be given by

$$\mathcal{T}^t_E = TE \times \{0\} \times \{0\} \sqcup {}^*p^*(TM) \times \{0\} \times]0,1] \sqcup E \times E \times]0,1] \times [0,1] \rightrightarrows E \times [0,1] \times [0,1]$$

and is smooth. Similarly, the Thom groupoid will be given by: $\mathcal{H}_E := TE \times \{0\} \sqcup {}^*p^*(TM) \times]0,1] \rightrightarrows E \times [0,1]$.

1.8. **Haar systems.** A locally compact groupoid $G \rightrightarrows G^{(0)}$ can be viewed as a family of locally compact spaces:

$$G_x = \{\gamma \in G \mid s(\gamma) = x\}$$

parametrized by $x \in G^{(0)}$. Moreover, right translations act on these spaces. Precisely, to any $\gamma \in G$ is associated the homeomorphism

$$\begin{array}{cccc} R_\gamma : & G_y & \to & G_x \\ & \eta & \mapsto & \eta \cdot \gamma \ . \end{array}$$

This picture enables to define the good analogue of Haar measure on locally compact groups to locally compact groupoids, namely *Haar systems*. The following definition is due to J. Renault [44].

Definition 7. A *Haar system* on G is a collection $\nu = \{\nu_x\}_{x \in G^{(0)}}$ of positive regular Borel measure on G satisfying the following conditions:

(1) *Support:* For every $x \in G^{(0)}$, the support of ν_x is contained in G_x.

(2) *Invariance:* For all $\gamma \in G$, the right-translation operator $R_\gamma : G_y \to G_x$ is measure-preserving. That is, for all $f \in C_c(G)$:
$$\int f(\eta) d\nu_y(\eta) = \int f(\eta \cdot \gamma) d\nu_x(\eta) \ .$$
(3) *Continuity:* For all $f \in C_c(G)$, the map
$$\begin{array}{ccc} G^{(0)} & \to & \mathbb{C} \\ x & \mapsto & \int f(\gamma) d\nu_x(\gamma) \end{array}$$
is continuous.

Contrary to the case of locally compact groups, Haar systems on groupoids may not exist. Moreover, when such a Haar system exists , it may not be unique. In the special case of smooth groupoid a Haar system always exists [38, 43] and any two Haar systems $\{\nu_x\}$ and $\{\mu_x\}$ differ by a continuous and positive function f on $G^{(0)}$: $\nu_x = f(x)\mu_x$ for all $x \in G^{(0)}$.

Example: When the source and range map are local homeomorphisms, a possible choice for ν_x is the counting measure on G_x.

2. C^*-ALGEBRAS OF GROUPOIDS

We will start this second part with the definition of a C^*-algebra together with some results. Then we will see how are constructed the maximal and minimal C^*-algebras associated to a groupoid. We will compute explicit examples.

2.1. C^*-algebras - Basic definitions.

In this chapter we introduce the terminology and we give some examples and properties of C^*-algebras. One can look at [21, 39, 3] for a more complete overview on this subject.

Definition 8. A *C^*-algebra A* is a complex Banach algebra with an involution $x \mapsto x^*$ such that:

(1) $(\lambda x + \mu y)^* = \bar{\lambda} x^* + \bar{\mu} y^*$ for $\lambda,\ \mu \in \mathbb{C}$ and $x,\ y \in A$,

(2) $(xy)^* = y^* x^*$ for $x,\ y \in A$, and

(3) $\|x^* x\| = \|x\|^2$ for $x \in A$.

Note that it follows from the definition that $*$ is isometric.
The element x in A is *self-adjoint* if $x^* = x$, *normal* if $xx^* = x^* x$. When 1 belongs to A, x is *unitary* if $xx^* = x^* x = 1$.
Given two C^*-algebras A, B, a homomorphism respecting the involution is a called a $*$-homomorphism.

Examples 1. Let $\mathcal{H}$ be an Hilbert space. The algebra $\mathcal{L}(\mathcal{H})$ of all continuous linear transformation of $\mathcal{H}$ is a C^*-algebra. The involution of $\mathcal{L}(\mathcal{H})$ is given by the usual adjunction of bounded linear operators.
2. $\mathcal{K}(\mathcal{H})$ is the norm closure of finite rank operators on $\mathcal{H}$. It is a sub C^*-algebra of $\mathcal{L}(\mathcal{H})$.ï??¿
3. The algebra $M_n(\mathbb{C})$ is a C^*-algebra. It is a special example of the previous kind, when $dim(\mathcal{H}) = n$.
4. Let X be a locally compact, Hausdorff, topological space. The algebra $C_0(X)$ of continuous functions vanishing at ∞ endowed with the supremum norm and the

involution $f \mapsto \bar{f}$ is a commutative C^*-algebra. When X is compact, 1 belongs to $C(X) = C_0(X)$.

Conversely every commutative C^*-algebra A is isomorphic to $C_0(X)$ for some locally compact space X (and it is compact precisely when A is unital). Precisely, a *character* $\mathcal{X}$ of A is a continuous homomorphism of algebras $\mathcal{X} : A \to \mathbb{C}$. The set X of characters of A, called the *spectrum* of A, can be endowed with a locally compact space topology. The *Gelfand transform* $\mathcal{F} : A \to C_0(X)$ given by $\mathcal{F}(x)(\mathcal{X}) = \mathcal{X}(x)$ is the desired $*$-isomorphism.

Let A be a C^*-algebra and $\mathcal{H}$ a Hilbert space.

Definition 9. A $*$-*representation* of A in $\mathcal{H}$ is a $*$-homomorphism $\pi : A \to \mathcal{L}(\mathcal{H})$. The representation is *faithful* if π is injective.

Theorem 10. *(Gelfand-Naimark) If A is a C^*-algebra, there exists a Hilbert space $\mathcal{H}$ and a faithful representation $\pi : a \to \mathcal{L}(\mathcal{H})$.*

In other words any C^*-algebra is isomorphic to a norm-closed involutive subalgebra of $\mathcal{L}(\mathcal{H})$. Moreover, when A is separable, $\mathcal{H}$ can be taken to be the (unique up to isometry) separable Hilbert space of infinite dimension.

Enveloping algebra
Given a Banach $*$-algebra A, consider the family π_α of all continuous $*$-representations for A. The Hausdorff completion of A for the seminorm $\|x\| = sup_\alpha(\|\pi_\alpha(x)\|)$ is a C^*-algebra called the *enveloping C^*-algebra* of A.

Units
A C^*-algebra may or may not have a unit, but it can always be embedded into a unital C^*-algebra $\tilde{A}$:

$$\tilde{A} := \{x + \lambda \mid x \in A,\ \lambda \in \mathbb{C}\}$$

with the obvious product and involution. The norm on $\tilde{A}$ is given for all $x \in \tilde{A}$ by: $\|x\|^\sim = Sup\{\|xy\|,\ y \in A\ ;\ \|y\| = 1\}$. On A we have $\|\cdot\| = \|\cdot\|^\sim$. The algebra A is a closed two sided ideal in $\tilde{A}$ and $\tilde{A}/A = \mathbb{C}$.

Functional calculus
Let A be a C^*-algebra. If x belongs to A, the *spectrum* of x in A is the compact set:

$$Sp(x) = \{\lambda \in \mathbb{C} \mid x - \lambda \textit{ is not invertible in } \tilde{A}\}$$

The *spectral radius* of X is the number:

$$\nu(x) = sup\{|\lambda|\ ;\ \lambda \in Sp(x)\}\ .$$

We have:

$$\begin{gathered} Sp(x) \subset \mathbb{R} \text{ when } x \text{ is self-adjoint } (x^* = x), \\ Sp(x) \subset \mathbb{R}_+ \text{ when } x \text{ is positive } (x = y^*y \text{ with } y \in A), \\ Sp(x) \subset U(1) \text{ when } x \text{ is unitary } (x^*x = xx^* = 1)\ . \end{gathered}$$

When x is *normal*: $x^*x = xx^*$, these conditions on the spectrum are equivalences.

When x is normal, $\nu(x) = \|x\|$. This enables to show that for any polynomial $P \in \mathbb{C}[x]$, $\|P(x)\| = sup\{P(t) \mid t \in Sp(x)\}$ (using that $Sp(P(x)) = P(Sp(x))$). We can then define $f(x) \in A$ for every continuous function $f : Sp(x) \to \mathbb{C}$. Precisely, according to Weirstrass' theorem, there is a sequence (P_n) of polynomials which converges uniformly to f on $Sp(x)$. We simply define $f(x) = limP_n(x)$.

2.2. **The reduced and maximal C^*-algebra of a groupoid.** We will restrict our study to the case of Hausdorff locally compact groupoids, for the non Hausdorff case, one can look at [13, 11, 31] .

From now, $G \rightrightarrows G^{(0)}$ is a locally compact Hausdorff groupoid equipped with a fixed Haar system $\nu = \{\nu_x\}_{x\in G^{(0)}}$. We let $C_c(G)$ be the space of complex valued functions with compact support on G. It is provided with a structure of involutive algebra as follows. If f and g belong to $C_c(G)$ we define
the *involution* by

$$\text{for } \gamma \in G,\ f^*(\gamma) = \overline{f(\gamma^{-1})},$$

the *convolution product* by

$$\text{for } \gamma \in G,\ f * g(\gamma) = \int_{\eta\in G_x} f(\gamma\eta^{-1})g(\eta)d\nu_x(\eta),$$

where $x = s(\gamma)$. The 1-norm on $C_c(G)$ is defined by

$$\|f\|_1 = \sup_{x\in G^{(0)}} \max\left(\int_{G_x} |f(\gamma)|d\nu_x(\gamma), \int_{G_x} |f(\gamma^{-1})|d\nu_x(\gamma)\right).$$

The *full groupoid* C^*-algebra $C^*(G,\nu)$ is defined to be the enveloping C^*-algebra of the Banach $*$-algebra $\overline{C_c(G)}^{\|\cdot\|_1}$ obtained by completion of $C_c(G)$ with respect to the norm $\|\cdot\|_1$.

Given x in $G^{(0)}$, $f \in C_c(G)$, $\xi \in L^2(G_x, \nu_x)$ and $\gamma \in G_x$, we set

$$\pi_x(f)(\xi)(\gamma) = \int_{\eta\in G_x} f(\gamma\eta^{-1})\xi(\eta)d\nu_x(\eta)\ .$$

One can show that π_x define a $*$-representation of $C_c(G)$ on the Hilbert space $L^2(G_x, \nu_x)$. Moreover we have for any $f \in C_c(G)$ the inequality $\|\pi_x(f)\| \le \|f\|_1$. The *reduced norm* on $C_c(G)$ is

$$\|f\|_r = \mathop{sup}_{x\in G^{(0)}} \{\|\pi_x(f)\|\}$$

which is a C^*-norm. The *reduced* C^*-algebra $C_r(G,\nu)$ is defined to be the C^*-algebra obtained by completion of A with respect to $\|\cdot\|_r$.

When G is smooth, the reduced and maximal C^*-algebras of the groupoid G do not depend up to isomorphism on the choice of the Haar system ν, in the general case they do not depend on ν up to Morita equivalence [44]. When there is no ambiguity on the Haar system, we will only denote $C^*(G)$ and $C^*_r(G)$ the maximal and reduced C^*-algebras.

The identity map on $C_c(G)$ induces a surjective homomorphism from $C^*(G)$ to $C^*_r(G)$. Thus $C^*_r(G)$ is a quotient of $C^*(G)$.
For a quite large class of groupoids, *amenable* groupoids [2], the reduced and maximal C^*-algebras are equal. This will be the case for all the groupoids we will meet in the last part of this course devoted to index theory.

Examples 1. When $X \rightrightarrows X$ is a locally compact space, $C^*(X) = C^*_r(X) = C_0(X)$.
2. When $G \rightrightarrows e$ is a group and Λ a Haar measure on G, we recover the usual notion of reduced and maximal C^*-algebra of a group.

3. Let M be a smooth manifold and $TM \rightrightarrows M$ the tangent bundle. Provide the vector bundle TM with a euclidean structure. The Fourier transform:

$$f \in C_c(TM),\ (x,w) \in T^*M, \quad \hat{f}(x,w) = \frac{1}{(2\pi)^{n/2}} \int_{X \in T_xM} e^{-iw(X)} f(X) dX$$

gives an isomorphism between $C^*(TM) = C^*_r(TM)$ and $C_0(T^*M)$.
4. Let X be a locally compact space, with μ a measure on X and consider the pair groupoid $X \times X \rightrightarrows X$. If $f,\ g$ belongs to $C_c(X \times X)$, the convolution product is given by:

$$f * g(x,y) = \int_{z \in X} f(x,z) g(z,y) d\mu(z)$$

and a representation of $C_c(X \times X)$ is given by

$$\pi : C_c(X \times X) \to \mathcal{L}(L^2(X,\mu));\ \pi(f)(\xi)(x) = \int_{z \in X} f(x,z)\xi(z) d\mu(z)$$

when $f \in C_c(X \times X), \xi \in L^2(X,\mu)$ and $x \in X$.
It turns out that $C^*(X \times X) = C^*_r(X \times X) \simeq \mathcal{K}(L^2(X,\mu))$.
5. Let M be a compact smooth manifold and $\mathcal{G}^t_M \rightrightarrows M \times [0,1]$ its tangent groupoid. In this situation $C^*(\mathcal{G}^t_M) = C^*_r(\mathcal{G}^t_M)$ is a continuous field $(A_t)_{t \in [0,1]}$ of C^*-algebras ([21]) with $A_0 \simeq C_0(T^*M)$ being a commutative C^*-algebra and for any $t \in]0,1]$, $A_t \simeq \mathcal{K}(L^2(M))$ [13].

In the sequel we will need the two following properties of C^*-algebras of groupoids.

Properties 1. Let G_1 and G_2 be two locally compact groupoids equipped with Haar systems and suppose for instance that G_1 is amenable. Then according to [2], $C^*(G_1) = C^*_r(G_1)$ is *nuclear* - which implies that for any C^*-algebra B there is only one tensor product C^*-algebra $C^*(G_1) \otimes B$. The groupoid $G_1 \times G_2$ is a locally compact and

$$C^*(G_1 \times G_2) \simeq C^*(G_1) \otimes C^*(G_2) \ \text{ and } C^*_r(G_1 \times G_2) \simeq C^*(G_1) \otimes C^*_r(G_2)\ .$$

2. Let $G \rightrightarrows G^{(0)}$ be a locally compact groupoid with a Haar system ν.
An open subset $U \subset G^{(0)}$ is *saturated* if U is a union of orbits of G, in other words $U = s(r^{-1}(U)) = r(s^{-1}(U))$. The set $F = G^{(0)} \setminus U$ is a closed saturated subset of $G^{(0)}$. The Haar system ν can be restricted to the restrictions $G|_U := G^U_U$ and $G|_F := G^F_F$ and we have the following exact sequence of C^*-algebras [27, 43]:

$$0 \to C^*(G|_U) \xrightarrow{i} C^*(G) \xrightarrow{r} C^*(G|_F) \to 0$$

where $i : C_c(G|_U) \to C_c(G)$ is given by the extension of functions by 0 while $r : C_c(G) \to C_c(G|_F)$ is given by the restriction of functions.

KK-Theory

This part on KK-theory starts with some historical introduction. Then in order to motivate our purpose we will list most of the properties of the functor KK. Then from section 4 up to section 5 we will define in great details all the ingredients involved in KK-theory. In order to make this course we have made an intensive use of the following references [47, 26, 46, 52]. Moreover a significant part of this chapter has been written by Jorge Plazas from the lectures and we would like to thank him for his big help.

3. Introduction to KK-theory

3.1. **Historical comments.**

The story begins with several studies of M. Atiyah [4, 5].

Firstly, recall that if X is a compact set, the *K-theory* of X is constructed in the following way: let $\mathcal{E}v$ be the set of isomorphism classes of continuous vector bundles over X. Thanks to the direct sum of bundles, the set $\mathcal{E}v$ is naturally endowed with a structure of abelian semi-group. One can then symetrize $\mathcal{E}v$ in order to get a group, this gives the K-theory group of X:

$$K^0(X) = \{[E] - [F] \ ; \ [E], [F] \in \mathcal{E}v\}$$

For example the K-theory of a point is $\mathbb{Z}$ since a vector bundle on a point is just a vector space and vector spaces are classified, up to isomorphism, by their dimension. When $\mathcal{H}$ is an infinite dimensional separable Hilbert space, the set $\mathcal{F}(\mathcal{H})$ of *Fredholm operators* on $\mathcal{H}$ is the open subset of $\mathcal{L}(\mathcal{H})$ made of bounded operators T on $\mathcal{H}$ such that the dimension of the kernel and cokernel of T are finite. The set $\mathcal{F}(\mathcal{H})$ is stable under the composition. We set

$$[X, \mathcal{F}(\mathcal{H})] = \{\text{homotopy classes of continuous maps: } X \to \mathcal{F}(\mathcal{H})\}$$

The set $[X, \mathcal{F}(\mathcal{H})]$ is naturally endowed with a structure of semi-group. M. Atiyah and independently Janish, showed that $[X, \mathcal{F}(\mathcal{H})]$ is actually (a group) isomorphic to $K^0(X)$ [4]. The idea of the proof is the following. If $f : X \to \mathcal{F}(\mathcal{H})$ is continuous, we can make a compact perturbation of f, in order to get $\tilde{f}$ such that $\tilde{f}$ is in the same homotopy class as f and $\mathrm{Ker}\tilde{f} := \cup_{x\in} ker(\tilde{f}_x)$ together with $\mathrm{CoKer}\tilde{f} := \cup_{x\in} ker(\tilde{f}_x)$ are vector bundles (i.e. of constant dimension) on X. The isomorphism is then given by:

$$\begin{array}{ccc} \mathcal{E} & \to & K^0(X) \\ \bar{f} & \mapsto & [\mathrm{Ker}\tilde{f}] - [\mathrm{CoKer}\tilde{f}] \end{array}$$

Thus K-theory can be expressed in terms of Fredholm operators.

Later, M. Atiyah looked for a description of the dual functor : the K-homology of X, with the help of Fredholm operators. This gave rise to $Ell(X)$. The cycles are triples (H, π, F) where:

- $H = H_0 \oplus H_1$ is a $\mathbb{Z}_2$ graded Hilbert space,
- $\pi : C(X) \to \mathcal{L}(H)$ is a representation by operators of degree 0 (this means that $\pi(f) = \begin{pmatrix} \pi_0(f) & 0 \\ 0 & \pi_1(f) \end{pmatrix}$),
- F belongs to $\mathcal{L}(H)$, is of degree 1 (thus it is of the form $F = \begin{pmatrix} 0 & G \\ T & 0 \end{pmatrix}$) and satisfies

  $$F^2 - 1 \in \mathcal{K}(H) \text{ and } [\pi, F] \in \mathcal{K}(H) \ .$$

  In particular G is an inverse of T modulo compact operators.

M. Atiyah also defined the following pairing between $K^0(X)$ and $Ell(X)$:

$$\begin{array}{ccc} K^0(X) \times Ell(X) & \to & \mathbb{Z} \\ ([E], (H, \pi, F)) & \mapsto & \mathrm{Index}(F_E) \end{array}$$

where $\mathrm{Index}(F_E) = \dim(Ker(F_E)) - \dim(CoKer(F_E))$ is the *index* of a Fredholm operator associated to a vector bundle E on X and a cycle (H, π, F) as follows. Let E' be a vector bundle on X such that $E \oplus E' \simeq \mathbb{C}^N \times X$ and let e be the projection of $\mathbb{C}^N \times X$ onto E. We can identify $C(X, \mathbb{C}^N) \underset{C(X)}{\otimes} H$ with H^N. Let $\tilde{e}$ be the image of $e \otimes 1$ under this identification. We define $F_E := \tilde{e}F^N|_{\tilde{e}(H^N)}$ where F^N is the diagonal operator with F in each diagonal entry. The operator F_E is the desired Fredholm operator on $\tilde{e}(H^N)$.

On the other hand, to any C^*-algebra A is associated a group $K_0(A)$. When A is unital, it can be defined as follows:

$$\begin{aligned} K_0(A) = \{[\mathcal{E}] - [\mathcal{F}] \ ; \ [\mathcal{E}], [\mathcal{F}] &\text{are isomorphism classes of} \\ &\text{finitely generated projective } A\text{-modules}\} \ . \end{aligned}$$

Recall that a A-module $\mathcal{E}$ is finitely generated and projective if there exists another A-module $\mathcal{G}$ such that $\mathcal{E} \oplus \mathcal{G} \simeq A^N$ for some integer N.

When X is compact $K^0(X) = K_0(C(X))$ (Swan-Serre theorem).

During the years $79 \sim 80$ 'th G. Kasparov has defined with a great success a bivariant theory, the KK-theory, which generalizes both K-theory and K-homology and is defined for any pair of C^*-algebras [29]. Moreover in many cases $KK(A, B)$ contains all the morphisms from $K_0(A)$ to $K_0(B)$. Before going to the definitions we will end this introduction by listing most of the properties of the bi-functor KK.

3.2. Abstract properties of $KK(A, B)$.

Let A an B be two C^*-algebras. In order to simplify our purpose, we will suppose that A and B are separable. Here is the list of the most important properties of the KK functor.

- $KK(A, B)$ is an abelian group.

- **Functorial properties** The functor KK is covariant in B and contravariant in A: if $f : B \to C$ and $g : A \to D$ are two homomorphisms of C^*-algebras, there exist two homomorphisms of groups:

$$f_* : KK(A, B) \to KK(A, C) \text{ and } g^* : KK(D, B) \to KK(A, B) \ .$$

In particular $id_* = id$ and $id^* = id$.

- Each *-morphism $f : A \to B$ defines an element, denoted by $[f]$, in $KK(A, B)$. We will denote $1_A := [id_A] \in KK(A, A)$.

- **Homotopy invariance** $KK(A, B)$ is homotopy invariant.

Recall that the C^*-algebras A and B are *homotopic*, if there exist two *-morphisms $f : A \to B$ and $g : B \to A$ such that $f \circ g$ is homotopic to id_B and $g \circ f$ is homotopic to id_A.

Two homomorphisms $F, G : A \to B$ are homotopic when there exists a $*$-morphism $H : A \to C([0, 1], B)$ such that $H(a)(0) = F(a)$ and $H(a)(1) = G(a)$ for any $a \in A$.

- If $\mathcal{K}$ is the algebra of compact operators on a Hilbert space there are isomorphisms:

$$KK(A, B \otimes \mathcal{K}) \simeq KK(A \otimes \mathcal{K}, B) \simeq KK(A, B) \ .$$

More generally, the bifunctor KK is invariant under *Morita equivalence*.

- **Suspension** If E is a C^*-algebra there exists an homomorphism

$$\tau_E : KK(A, B) \to KK(A \otimes E, B \otimes E)$$

which satisfies $\tau_E \circ \tau_D = \tau_{E \otimes D}$ for any C^*-algebra D.

- **Kasparov product** There is a well defined bilinear coupling:

$$\begin{array}{ccc} KK(A,D)\times KK(D,B) & \to & KK(A,B) \\ (x,y) & \mapsto & x\otimes y \end{array}$$

called the *Kasparov product.* It is associative, covariant in B and contravariant in A: if $f:C\to A$ and $g:B\to E$ are two homomorphisms of C^*-algebras then

$$f^*(x\otimes y)=f^*(x)\otimes y \text{ and } g_*(x\otimes y)=x\otimes g_*(y).$$

If $g:D\to C$ is another *-morphism, $x\in KK(A,D)$ and $z\in KK(C,B)$ then

$$h_*(x)\otimes z=x\otimes h^*(z)\ .$$

Moreover, the following equalities hold:

$$f^*(x)=[f]\otimes x\ ,\ g_*(z)=z\otimes[g] \text{ and } [f\circ h]=[h]\otimes[f]\ .$$

In particular

$$x\otimes 1_D=1_A\otimes x=x\ .$$

The Kasparov product behaves well with suspensions. If E is a C^*-algebra:

$$\tau_E(x\otimes y)=\tau_E(x)\otimes\tau_E(y)\ .$$

This enables to extend the Kasparov product:

$$\begin{array}{cccc} \underset{D}{\otimes}: & KK(A,B\otimes D)\times kk(D\otimes C,E) & \to & KK(A\otimes C,B\otimes E) \\ & (x,y) & \mapsto & x\underset{D}{\otimes}y:=\tau_C(x)\otimes\tau_B(y) \end{array}$$

- The Kasparov product $\underset{\mathbb{C}}{\otimes}$ is commutative.
- **Higher groups** For any $n\in\mathbb{N}$, let

$$KK_n(A,B):=KK(A,C_0(\mathbb{R}^n)\otimes B)\ .$$

An alternative definition, leading to isomorphic groups, is

$$KK_n(A,B):=KK(A,C_n\otimes B),$$

where C_n is the Clifford algebra of $\mathbb{C}^n$. This will be explained later. The functor KK satisfies the **Bott periodicity**: there is an isomorphism

$$KK_2(A,B)\simeq KK(A,B)\ .$$

- **Exact sequences** Consider the following exact sequence of C^*-algebras:

$$0\to J\xrightarrow{i}A\xrightarrow{p}Q\to 0$$

and let B be another C^*-algebra. Under some more assumptions (for example all the C^*-algebras are nuclear or K-nuclear, or the exact sequence above admits a completely positive norm decreasing cross section [48]) we have the following two periodic exact sequences

$$\begin{tikzcd}
KK(B,J) \arrow[r, "i_*"] & KK(B,A) \arrow[r, "p_*"] & KK(B,Q) \arrow[d, "\delta"] \\
KK_1(B,Q) \arrow[u, "\delta"] & KK_1(B,A) \arrow[l, "p_*"'] & KK_1(B,J) \arrow[l, "i_*"']
\end{tikzcd}$$

$$\begin{tikzcd}
KK(Q,B) \arrow[r, "p^*"] & KK(A,B) \arrow[r, "i^*"] & KK(J,B) \arrow[d, "\delta"] \\
KK_1(J,B) \arrow[u, "\delta"] & KK_1(A,B) \arrow[l, "i^*"'] & KK_1(Q,B) \arrow[l, "p^*"']
\end{tikzcd}$$

where the connecting homomorphisms δ are given by Kasparov products.

- **Final remark** Let us go back to the end of the introduction, to make it more precise.

As a result, we recover the usual K-theory:

$$KK(\mathbb{C},B) \simeq K_0(B),$$

while the *K-homology* of a C^*-algebra A is defined by

$$K^0(A) = KK(A,\mathbb{C}) \ .$$

Any $x \in KK(A,B)$ induces a homomorphism of group:

$$\begin{array}{ccc} KK(\mathbb{C},A) \simeq K_0(A) & \rightarrow & K_0(B) \simeq KK(\mathbb{C},B) \\ \alpha & \mapsto & \alpha \otimes x \end{array}$$

In most situations, the induced homomorphism $KK(A,B) \to Mor(K_0(A), K_0(B))$ is surjective. Thus one can think to KK-elements as homomorphisms between K groups.

When X is a compact space, one has $K^0(X) \simeq K_0(C(X)) \simeq KK(\mathbb{C}, C(X))$ and we will see that $K^0(C(X)) = KK(C(X),\mathbb{C})$ is a quotient of the set $Ell(X)$ defined by M. Atiyah. Moreover the pairing $K^0(X) \times Ell(X) \to \mathbb{Z}$ coincides with the Kasparov product $KK(\mathbb{C}, C(X)) \times KK(C(X),\mathbb{C}) \to KK(\mathbb{C},\mathbb{C}) \simeq \mathbb{Z}$.

4. Hilbert Modules

We review the main properties of Hilbert modules over C^*-algebras necessary to a correct understanding of bivariant K-theory. We follows closely the presentation given by G. Skandalis [46]. Most of proofs given below come from his lectures on the subject. We are indebted to him for allowing us to use his lectures notes. Some of the material given below can also be found in [52], where the reader will find a guide to the literature and a more detailed presentation.

4.1. Basic definitions and examples.

Let A be a C^*-algebra and E be a A-right module.

A sesquilinear form $(\cdot,\cdot) : E \times E \to A$ is *positive* if for all $x \in E$, $(x,x) \in A_+$. Here A_+ denotes the set of positives element in A. It is *positive definite* if moreover $(x,x) = 0$ if and only if $x = 0$.

Let $(\cdot,\cdot) : E \times E \to A$ be a positive sesquilinear form and set $Q(x) = (x,x)$. By the polarization identity:

$$\forall x,y \in E, \quad (x,y) = \frac{1}{4}\left(Q(x+y) - iQ(x+iy) - Q(x-y) + iQ(x-iy)\right)$$

we get:

$$\forall x,y \in E, \quad (x,y) = (y,x)^*$$

Definition 11. A *pre-hilbert A-module* is a right A-module E with a positive definite sesquilinear map $(\cdot,\cdot) : E \times E \to A$ such that $y \mapsto (x,y)$ is A-linear.

Proposition 12. *Let* $(E, (\cdot,\cdot))$ *be a pre-Hilbert* A*-module. The following:*

$$\forall x \in E, \quad \|x\| = \sqrt{(x,x)} \tag{4.1}$$

defines a norm on E*.*

Proof. The only non trivial fact is the triangle inequality, which results from:

Lemma 13. *(Cauchy-Schwarz inequality)*

$$\forall x, y \in E, \quad (x,y)^*(x,y) \leq \|x\|^2 (y,y)$$

In particular: $\|(x,y)\| \leq \|x\|\|y\|$.

Set $a = (x,y)$. We have for all $t \in \mathbb{R}$: $(xa+ty, xa+ty) \geq 0$, thus:

$$2ta^*a \leq a^*(x,x)a + t^2(y,y) \tag{4.2}$$

Since $(x,x) \geq 0$, we have: $a^*(x,x)a \leq \|x\|^2 a^*a$ (it uses the equivalence: $z^*z \leq w^*w$ if and only if $\|zx\| \leq \|wx\|$ for all $x \in A$) and choosing $t = \|x\|^2$ in (4.2) gives the result. ☐

Definition 14. A *Hilbert* A*-module* is a pre-Hilbert A-module which is complete for the norm defined in (4.1).
A *Hilbert* A*-submodule* of a Hilbert A-module is a closed A-submodule provided with the restriction of the A-valued scalar product.

When there is no ambiguity about the base C^*-algebra A, we simply say pre-Hilbert module and Hilbert module.

Let $(E, (\cdot,\cdot))$ be a pre-Hilbert A-module. Using the continuity of the sesquilinear form $(\cdot,\cdot) : E \times E \to A$ and of the right multiplication $E \to E, x \mapsto xa$ for any $a \in A$, we get that the completion of E for the norm (4.1) is a Hilbert A-module.

Remark 15. In the definition of a pre-Hilbert A-module, one can remove the hypothesis $(\cdot,\cdot)$ is *definite*. In that case, (4.1) defines a semi-norm and one checks that the Hausdorff completion of a pre-Hilbert A-module, in this extended sense, is a Hilbert A-module.

We continue this paragraph with classical examples.
1. The algebra A is a Hilbert A-module with its obvious right A-module structure and:

$$(a,b) := a^*b \ .$$

2. For any positive integer n, A^n is a Hilbert A-module with its obvious right A-module structure and:

$$((a_i),(b_i)) := \sum_{i=1}^{n} a_i^* b_i \ .$$

Observe that $\sum_{i=1}^{n} a_i^* a_i$ is a sum of positive elements in A, which implies that

$$\|(a_i)\| = \sqrt{\|\sum_{i=1}^{n} a_i^* a_i\|} \geq \|a_k\|$$

for all k. It follows that if $(a_1^m, \ldots, a_n^m)_m$ is a Cauchy sequence in A^n, the sequences $(a_k^m)_m$ are Cauchy in A, thus convergent and we conclude that A^n is complete.

3. Example **2.** can be extended to the direct sum of n Hilbert A-modules $E_1, \ldots, E_n$ with the Hilbertian product:

$$((x_i),(y_i)) := \sum_{i=1}^{n} (x_i, y_i)_{E_i}$$

4. If F is a closed A-submodule of a Hilbert A-module E then F is a Hilbert A-module. For instance, a closed right ideal in A is a Hilbert A-module.

5. The standard Hilbert A-module

Let

$$\mathcal{H}_A = \{x = (x_k)_{k\in\mathbb{N}} \mid \sum_{k\in\mathbb{N}} x_k^* x_k \text{ converges } \} \tag{4.3}$$

The right A-module structure is given by $(x_k)a = (x_k a)$ and the Hilbertian A-valued product is:

$$((x_k),(y_k)) = \sum_{k=0}^{+\infty} x_k^* y_k \tag{4.4}$$

This sum converges for elements of $\mathcal{H}_A$, indeed for all $q > p \in \mathbb{N}$:

$$\begin{aligned} \|\sum_{k=p}^{q} x_k^* y_k\| &= \| \left((x_k)_p^q, (y_k)_p^q\right)_{A^{q-p}} \| \\ &\leq \|(x_k)_p^q\|_{A^{q-p}} \|(y_k)_p^q\|_{A^{q-p}} \quad \text{(Cauchy Schwarz inequality in} A^{q-p}\text{)} \\ &= \sqrt{\|\sum_{k=p}^{q} x_k^* x_k\|}\sqrt{\|\sum_{k=p}^{q} y_k^* y_k\|} \end{aligned}$$

This implies that $\sum_{k\geq 0} x_k^* y_k$ satisfies the Cauchy criterion, thus converges, and (4.4) makes sense. Since for all (x_k), (y_k) in $\mathcal{H}_A$:

$$\sum_{k\geq 0} (x_k + y_k)^*(x_k + y_k) = \sum_{k\geq 0} x_k^* x_k + \sum_{k\geq 0} y_k^* x_k + \sum_{k\geq 0} x_k^* y_k + \sum_{k\geq 0} y_k^* y_k$$

is the sum of four convergent series, we find that $(x_k) + (y_k) = (x_k + y_k)$ is in $\mathcal{H}_A$. We also have, as before, that for all $a \in A$ and $(x_k) \in \mathcal{H}_A$:

$$\|\sum_{k=0}^{+\infty} (x_k a)^*(x_k a)\| \leq \|a\|^2 \|\sum_{k=0}^{+\infty} x_k^* x_k\|$$

Hence, $\mathcal{H}_A$ is a pre-Hilbert A-module, and we need to check that it is complete. Let $(u_n)_n = ((u_i^n))_n$ be a Cauchy sequence in $\mathcal{H}_A$. We get, as in example **2.**, that for all $i \in \mathbb{N}$, the sequence $(u_i^n)_n$ is Cauchy in A, thus converges to an element denoted v_i. Let us check that (v_i) belongs to $\mathcal{H}_A$.

Let $\varepsilon > 0$. Choose n_0 such that

$$\forall p > q \geq n_0,\ \|u_q - u_p\|_{\mathcal{H}_A} \leq \varepsilon/2\ .$$

Choose i_0 such that

$$\forall k > j \geq i_0,\ \|\sum_{i=j}^{k} u_i^{n_0 *} u_i^{n_0}\|^{1/2} \leq \varepsilon/2\ .$$

Then thanks to the triangle inequality in A^{k-j} we get for all $p, q \geq n_0$ and $j, k \geq i_0$:

$$\| \sum_{i=j}^{k} u_i^{p*} u_i^p \|^{1/2} \quad \leq \quad \| \sum_{i=j}^{k} (u_i^p - u_i^{n_0})^* (u_i^p - u_i^{n_0}) \|^{1/2} + \| \sum_{i=j}^{k} u_i^{n_0 *} u_i^{n_0} \|^{1/2} \leq \varepsilon$$

Taking the limit $p \to +\infty$, we get: $\| \sum_{i=j}^{k} v_i^* v_i \|^{1/2} \leq \varepsilon$ for all $j, k \geq i_0$ which implies that $(v_i) \in \mathcal{H}_A$. It remains to check that $(u_n)_n$ converges to $v = (v_i)$ in $\mathcal{H}_A$. With the notations above:

$$\forall p, q \geq n_0, \ \forall I \in \mathbb{N}, \quad \| \sum_{i=0}^{I} (u_i^p - u_i^q)^* (u_i^p - u_i^q) \|^{1/2} \leq \varepsilon,$$

taking the limit $p \to +\infty$:

$$\forall q \geq n_0, \ \forall I \in \mathbb{N}, \quad \| \sum_{i=0}^{I} (v_i - u_i^q)^* (v_i - u_i^q) \|^{1/2} \leq \varepsilon,$$

taking the limit $I \to +\infty$:

$$\forall q \geq n_0, \quad \|v - u_q\| \leq \varepsilon,$$

which ends the proof. □

The standard Hilbert space H_A is maybe the most important one. Indeed, Kasparov proved:

Theorem 16. *Let E be a countably generated Hilbert A-module. Then E and $E \oplus H_A$ are isomorphic.*

The proof can be found in [52]. This means that there exists a A-linear unitary map $U : E \oplus H_A \to E$. The notion of unitary uses the notion of adjoint, which will be explained later.

Remark 17. 1. The algebraic sum $\underset{\mathbb{N}}{\oplus} A$ is dense in $\mathcal{H}_A$.
2. We can replace in $\mathcal{H}_A$ the summand A by any sequence of Hilbert A-modules $(E_i)_{i \in \mathbb{N}}$ and the Hilbertian A-valued product by:

$$((x_k), (y_k)) = \sum_{k=0}^{+\infty} (x_k, y_k)_{E_k}$$

If $E_i = E$ for all $i \in \mathbb{N}$, the resulting Hilbert A-module is denoted $l^2(\mathbb{N}, E)$.
3. We can generalize the construction to any family $(E_i)_{i \in I}$ using summable families instead of convergent series.

We end with two concrete examples.

a. Let X be a locally compact space and E an hermitian vector bundle. The space $C_0(X, E)$ of continuous sections of E vanishing at infinity is a Hilbert $C_0(X)$-module with the module structure given by:

$$\xi.a(x) = \xi(x)a(x), \ \xi \in C_0(X, E), \ a \in C_0(X)$$

and the product given by:

$$(\xi, \eta)(x) = (\xi(x), \eta(x))_{E_x}$$

b. Let G be a locally compact groupoid with a Haar system λ and E a hermitian vector bundle over $G^{(0)}$. Then

$$f, g \in C_c(G, r^*E), \quad (f, g)(\gamma) = \int_{G_{s(\gamma)}} (f(\eta\gamma^{-1}), g(\eta))_{E_{r(\eta)}} d\lambda^{s(\gamma)}(\eta) \tag{4.5}$$

gives a positive definite sesquilinear $C_c(G)$-valued form which has the correct behavior with respect to the right action of $C_c(G)$ on $C_c(G, r^*E)$. This leads to two norms $\|f\| = \|(f,f)\|^{1/2}_{C^*(G)}$ and $\|f\|_r = \|(f,f)\|^{1/2}_{C^*_r(G)}$ and two completions of $C_c(G, r^*E)$, denoted $C^*(G, r^*E)$ and $C^*_r(G, r^*E)$ which are Hilbert modules respectively over $C^*(G)$ and $C^*_r(G)$.

4.2. **Homomorphisms of Hilbert A-modules.** Let E, F be Hilbert A-modules. We will need the orthogonality in Hilbert modules:

Lemma 18. *Let S be a subset of E. The orthogonal of S:*

$$S^\perp = \{x \in E \mid \forall y \in S,\ (y, x) = 0\}$$

is a Hilbert A-submodule of E.

4.2.1. *Adjoints.* Let $T : E \to F$ be a map. T is *adjointable* if there exists a map $S : F \to E$ such that:

$$\forall (x, y) \in E \times F, \quad (Tx, y) = (x, Sy) \tag{4.6}$$

Definition 19. Adjointable maps are called *homomorphisms of Hilbert A-modules.* The set of adjointable maps from E to F is denoted $\operatorname{Mor}(E, F)$, and $\operatorname{Mor}(E) = \operatorname{Mor}(E, E)$.

The vocabulary will be clear after the next proposition.

Proposition 20. *Let $T \in \operatorname{Mor}(E, F)$.*

- (a) *The operator satisfying (4.6) is unique. It is denoted T^* and called the adjoint of T. One has $T^* \in \operatorname{Mor}(F, E)$ and $(T^*)^* = T$.*
- (b) *T is linear, A-linear and continuous.*
- (c) *$\|T\| = \|T^*\|$, $\|T^*T\| = \|T\|^2$, $\operatorname{Mor}(E, F)$ is a closed subspace of $\mathcal{L}(E, F)$. In particular $\operatorname{Mor}(E)$ is a C^*-algebra.*
- (d) *If $S \in \operatorname{Mor}(E, F)$ and $T \in \operatorname{Mor}(F, G)$ then $TS \in \operatorname{Mor}(E, G)$ and $(TS)^* = S^*T^*$.*

Proof. (a) Let R, S be two maps satisfying (4.6) for T. Then:

$$\forall x \in E, y \in F, \quad (x, Ry - Sy) = 0$$

and taking $x = Ry - Sy$ yields $Ry - Sy = 0$. The remaining is obvious.
(b) $\forall x, y \in E, z \in F, \lambda \in \mathbb{C}$,

$$(T(x + \lambda y), z) = (x + \lambda y, T^*z) = (x, T^*z) + \overline{\lambda}(y, T^*z) = (Tx, z)(\lambda Ty, z)$$

thus $T(x + \lambda y) = Tx + \lambda Ty$ and T is linear. Moreover:

$$\forall x \in E, y \in F, a \in A, \quad (T(xa), y) = (xa, T^*y) = a^*(x, T^*y) = ((Tx)a, y),$$

which gives the A-linearity. Consider the set

$$S = \{(-T^*y, y) \in E \times F \mid y \in F\}\ .$$

Then

$$\begin{aligned}(x_0,y_0)\in S^{\perp} \quad &\Leftrightarrow \quad \forall y\in F,\ (x_0,-T^*y)+(y_0,y)=0\\ &\Leftrightarrow \quad \forall y\in F,\ (y_0-Tx_0,y)=0\end{aligned}$$

Thus $G(T)=\{(x,y)\in E\times F\mid y=Tx\}=S^{\perp}$ is closed and the closed graph theorem implies that T is continuous.
(c) We have:

$$\|T\|^2=\sup_{\|x\|\leq 1}\|Tx\|^2=\sup_{\|x\|\leq 1}(x,T^*Tx)\leq\|T^*T\|\leq\|T^*\|\|T\|\ .$$

Thus $\|T\|\leq\|T^*\|$ and switching T and T^* gives the equality.
One has also proved:

$$\|T\|^2\leq\|T^*T\|\leq\|T^*\|\|T\|=\|T\|^2$$

thus $\|T^*T\|=\|T\|^2$ and the norm of $\mathrm{Mor}(E)$ satisfies the C^*-algebraic equation.

Let $(T_n)_n$ be a sequence in $\mathrm{Mor}(E,F)$, which converges to $T\in\mathcal{L}(E,F)$. Since $\|T\|=\|T^*\|$ and since $T\to T^*$ is (anti-)linear, the sequence $(T_n^*)_n$ is a Cauchy sequence, thus converges to an operator $S\in\mathcal{L}(F,E)$. It is then immediate that S is the adjoint of T. This proves that $\mathrm{Mor}(E,F)$ is closed, in particular $\mathrm{Mor}(E)$ is a C^*-algebra.
(d) Easy. □

Remark 21. There exist continuous linear and A-linear maps $T:E\to F$ which do not have an adjoint. For instance, take $A=C([0,1])$, $J=C_0(]0,1])$ and $T:J\hookrightarrow A$ the inclusion. Assuming that T is adjointable, a one line computation proves that $T^*1=1$. But 1 does not belong to J. Thus $J\hookrightarrow A$ has no adjoint.
One can also take $E=C([0,1])\oplus C_0(]0,1])$ and $T:E\to E, x+y\mapsto y+0$ to produce an example of $T\in\mathcal{L}(E)$ and $T\notin\mathrm{Mor}(E)$.

One can characterize the self-adjoints and positive elements in the C^*-algebra $\mathrm{Mor}(E)$ as follows.

Proposition 22. *Let $T\in\mathrm{Mor}(E)$.*
(a) $T=T^*\Leftrightarrow\forall x\in E,\ (x,Tx)=(x,Tx)^*$
(b) $T\geq 0\Leftrightarrow\forall x\in E,\ (x,Tx)\geq 0$

Proof. (a) The implication ($\Rightarrow$) is obvious. Conversely, set $Q_T(x)=(x,Tx)$. Using the polarization identity:

$$(x,Ty)=\frac{1}{4}\left(Q_T(x+y)-iQ_T(x+iy)-Q_T(x-y)+iQ_T(x-iy)\right)$$

one easily get $(x,Ty)=(Tx,y)$ for all $x,y\in E$, thus T is self-adjoint.
(b) If T is positive, there exists $S\in\mathrm{Mor}(E)$ such that $T=S^*S$. Then $(x,Tx)=(Sx,Sx)$ is positive for all x. Conversely, if $(x,Tx)\geq 0$ for all x then T is self-adjoint using (a) and there exists positive elements T_+,T_- such that:

$$T=T_+-T_-,\ T_+T_-=T_-T_+=0$$

It follows that:

$$\begin{aligned}\forall x\in E,\ (x,T_+x)\quad&\geq\quad(x,T_-x)\\ \forall z\in E,\ (T_-z,T_+T_-z)\quad&\geq\quad(T_-z,T_-T_-z)\\ \forall z\in E,\ (z,(T_-)^3z)\quad&\leq\quad 0\end{aligned}$$

Since T_- is positive, T_-^3 is also positive and the last line above implies $T_-^3 = 0$. It follows that $T_- = 0$ and then $T = T_+ \geq 0$. $\square$

4.2.2. *Orthocompletion.* Recall that for any subset S of E, $S^\perp$ is a Hilbert submodule of E. Remark also that any orthogonal submodules: $F \perp G$ of E are direct summands.

The following properties are left as an exercise:

Proposition 23. *Let F, G be A-submodules of E.*

- $E^\perp = \{0\}$ *and* $\{0\}^\perp = E$.
- $F \subset G \Rightarrow G^\perp \subset F^\perp$.
- $F \subset F^{\perp\perp}$.
- *If $F \perp G$ and $F \oplus G = E$ then $F^\perp = G$ and $G^\perp = F$. In particular F and G are Hilbert submodules.*

Definition 24. A Hilbert A-submodule F of E is said to be *orthocomplemented* if $F \oplus F^\perp = E$.

Remark 25. A Hilbert submodule is not necessarily orthocomplemented, even if it can be topologically complemented. For instance consider $A = C([0,1])$ and $J = C_0(]0,1])$ as a Hilbert A-submodule of A. One easily check that $J^\perp = \{0\}$, thus J is not orthocomplemented. On the other hand: $A = J \oplus \mathbb{C}$.

Lemma 26. *Let $T \in \operatorname{Mor}(E)$. Then*

- $\ker T^* = (\operatorname{Im} T)^\perp$
- $\overline{\operatorname{Im} T} \subset (\ker T^*)^\perp$

The proof is obvious. Note the difference in the second point with the case of bounded operators on Hilbert spaces (where equality always occurs). Thus, in general, $\ker T^* \oplus \overline{\operatorname{Im} T}$ is not the whole of E. It happens precisely when $\overline{\operatorname{Im} T}$ is orthocomplemented.

To emphasize, we can have T^* injective without having $\operatorname{Im} T$ dense in E (for instance: $T : C[0,1] \to C[0,1], f \mapsto tf$). Nevertheless, we have:

Theorem 27. *Let $T \in \operatorname{Mor}(E, F)$. The following assertions are equivalent:*

1. $\operatorname{Im} T$ *closed,*
2. $\operatorname{Im} T^*$ *closed,*
3. 0 *is isolated in* T^*T,
4. 0 *is isolated in* TT^*,

and in that case $\operatorname{Im} T$, $\operatorname{Im} T^*$ *are orthocomplemented.*

Thus, under the assumption of the theorem $\ker T^* \oplus \operatorname{Im} T = F$, $\ker T \oplus \operatorname{Im} T^* = E$. We gather some technical preliminaries into a lemma:

Lemma 28. *Let $T \in \operatorname{Mor}(E, F)$. Then*

1. $T^*T \geq 0$. *We set* $|T| = \sqrt{T^*T}$.
2. $\overline{\operatorname{Im} T^*} = \overline{\operatorname{Im} |T|} = \overline{\operatorname{Im} T^*T}$
3. *Assume that $T(E_1) \subset F_1$ for some Hilbert submodules E_1, F_1. Then $T|_{E_1} \in \operatorname{Mor}(E_1, F_1)$.*
4. *If T is onto then TT^* is invertible (in $\operatorname{Mor}(F)$) and $E = \ker T \oplus \operatorname{Im} T^*$.*

Proof of the lemma. (1) is obvious using lemma 22.
(2) On has $T^*T(E) \subset T^*(F)$. Conversely:

$$T^* = \lim T^*(1/n + TT^*)^{-1}TT^* \ .$$

This is a convergence in norm because:

$$\|T^*(1/n + TT^*)^{-1}TT^* - T^*\| = \|\frac{1}{n}T^*(\frac{1}{n} + TT^*)^{-1}\| = O(1/\sqrt{n}).$$

It follows that $T^*(F) \subset \overline{T^*T(E)}$ and thus $\overline{\operatorname{Im} T^*} = \overline{\operatorname{Im} T^*T}$. Replacing T by $|T|$ we get the other equality.
(3) Easy.
(4) By the open mapping theorem, there exists a positive real number $k > 0$ such that each $y \in F$ has a preimage x_y by T with $\|y\| \geq k\|x_y\|$. Using Cauchy-Schwarz for T^*y and x_y, we get:

$$(*) \quad \|T^*y\| \geq k\|y\| \quad \forall y \in F \ .$$

Recall that in a C^*-algebra, the inequality $a^*a \leq b^*b$ is equivalent to: $\|ax\| \leq \|bx\|$ for all $x \in A$,. It can be adapted to Hilbert modules to show that $(*)$ implies $TT^* \geq k^2$ in $\operatorname{Mor}(F)$, and thus TT^* is invertible. Consider then $p = T^*(TT^*)^{-1}T$. It is an idempotent, thus $E = \ker p \oplus \operatorname{Im} p$. Moreover $(TT^*)^{-1}T$ is onto so $\operatorname{Im} p = \operatorname{Im} T^*$ and $T^*(TT^*)^{-1}$ is injective, so $\ker p = \ker T$. □

Proof of the theorem. Let us start with the implication (1) ⇒ (3). By point (3) of the lemma $S := (T : E \to TE) \in \operatorname{Mor}(E, TE)$ and by the point (4) of the lemma SS^* is invertible and 0 is isolated in its spectrum. Since the spectra of SS^* and S^*S coincide outside 0 and since $S^*S = T^*T$, we get (3).
The implication (4) ⇒ (1). Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(0) = g(0) = 0$, $f(t) = 1, g(t) = 1/t$ for $t \neq 0$. Thus f and g are continuous on the spectrum of TT^*. Using the equalities $f(t)t = t$ and $tg(t) = f(t)$, we get $f(TT^*)TT^* = TT^*$ and $TT^*g(TT^*) = f(TT^*)$ from which we deduce $\operatorname{Im} f(TT^*) = \operatorname{Im} TT^*$. But $f(TT^*)$ is a projector (self-adjoint idempotent), hence $\operatorname{Im} TT^*$ is closed and orthocomplemented. Using point (2) of the lemma and the inclusion $\operatorname{Im} TT^* \subset \operatorname{Im} T$, we get (1) (and also the orthocomplementability of $\operatorname{Im} T$).
At this point we have the following equivalences (1) ⇔ (3) ⇔ (4). Replacing T by T^* we get (2) ⇔ (3) ⇔ (4). □

Another result which deserves to be given is:

Proposition 29. *Let H be a Hilbert submodule of E and $T : E \to F$ a A-linear map.*

- *H is orthocomplemented if and only if $i : H \hookrightarrow E \in \operatorname{Mor}(H, E)$.*
- *$T \in \operatorname{Mor}(E, F)$ if and only if the graph of T:*

$$\{(x, y) \in E \times F \ | y = Tx\}$$

  is orthocomplemented.

4.2.3. *Partial isometries.* The following easy result is left as an exercise:

Proposition 30. *(and definition). Let $u \in \operatorname{Mor}(E, F)$. The following assertions are equivalent:*

1. *u^*u is an idempotent,*
2. *uu^* is an idempotent,*

(3) $u^* = u^*uu^*$,
(4) $u = uu^*u$.

u is then called a partial isometry, with initial support $I = \operatorname{Im} u^*$ and final support $J = \operatorname{Im} u$.

Remark 31. If u is a partial isometry, then $\ker u = \ker u^*u$, $\ker u^* = \ker uu^*$, $\operatorname{Im} u = \operatorname{Im} uu^*$ and $\operatorname{Im} u^* = \operatorname{Im} u^*u$. In particular u has closed range and $E = \ker u \oplus \operatorname{Im} u^*$, $F = \ker u^* \oplus \operatorname{Im} u$ where the direct sums are orthogonal.

4.2.4. *Polar decompositions.* All homomorphisms do not admit a polar decomposition. For instance, consider: $T \in \operatorname{Mor}(C[-1,1])$ defined by $Tf = t.f$ (here $C[-1,1]$ is regarded as a Hilbert $C[-1,1]$-module). T is self-adjoint and $|T| : f \mapsto |t|.f$. The equation $T = u|T|$, $u \in \operatorname{Mor}(C[-1,1])$ leads to the constraint $u(1)(t) = \operatorname{sign}(t)$, so $u(1) \notin C[-1,1]$ and u does not exist.

The next result clarifies the conditions to get a polar decomposition:

Theorem 32. *Let $T \in \operatorname{Mor}(E,F)$ such that $\overline{\operatorname{Im} T}$ and $\overline{\operatorname{Im} T^*}$ are orthocomplemented. Then there exists a unique $u \in \operatorname{Mor}(E,F)$, vanishing on $\ker T$, such that*

$$T = u|T|$$

Moreover, u is a partial isometry with initial support $\overline{\operatorname{Im} T^*}$ and final support $\overline{\operatorname{Im} T}$.

Proof. We first assume that T and T^* have dense image. Setting $u_n = T(1/n + T^*T)^{-1/2}$ we get a bounded sequence ($\|u_n\| \leq 1$) such that for all $y \in F$, $u_n(T^*y) = (1/n+T^*T)^{-1/2}TT^*y \to \sqrt{TT^*}(y)$. Thus, by density of $\operatorname{Im} T^*$, $u_n(x)$ converges for all $x \in E$. Let $v(x)$ denotes the limit. Replacing T by T^* above, we also have that $u_n^*(y)$ converges for all $y \in F$, which yields $v \in \operatorname{Mor}(E,F)$. A small computation shows that $u_n|T| - T$ goes to 0 in norm. Thus $v|T| = T$. The homomorphism v is unique by density of $\operatorname{Im}|T|$ and is unitary since $u_n^*u_n(x) \to x$ for all $x \in \operatorname{Im} T^*T$, which proves $v^*v = 1$ and similarly for vv^*.
Now consider the general case and set $E_1 = \overline{\operatorname{Im} T^*}$, $F_1 = \overline{\operatorname{Im} T}$. One applies the first step to the restriction $T_1 \in \operatorname{Mor}(E_1, F_1)$ of T, and we call v_1 the unitary constructed. We set $u(x) = v_1(x)$ if $x \in E_1$ and $u(x) = 0$ if $x \in E_1^\perp = \ker T$. This definition forces the unicity, and it is clear that u is a partial isometry with the claimed initial/final supports. □

Remark 33. u is the strong limit of $T(1/n + T^*T)^{-1/2}$.

4.2.5. *Compact homomorphisms.* Let $x \in E, y \in F$ and define $\theta_{y,x} \in \operatorname{Mor}(E,F)$ by

$$\theta_{y,x}(z) = y.(x,z) \ .$$

The adjoint is given by $\theta_{y,x}^* = \theta_{x,y}$. Then

Definition 34. We define $\mathcal{K}(E,F)$ to be the closure of the linear span of $\{\theta_{y,x};\ x \in E, y \in F\}$ in $\operatorname{Mor}(E,F)$.

One easily check that

- $\|\theta_{y,x}\| \leq \|x\|\|y\|$ and $\|\theta_{x,x}\| = \|x\|^2$,
- $T\theta_{y,x} = \theta_{Ty,x}$ and $\theta_{y,x}S = \theta_{y,S^*x}$,
- $\mathcal{K}(E) := \mathcal{K}(E,E)$ is a closed two-sided ideal of $\operatorname{Mor}(E)$ (and hence a C^*-algebra.

We also prove:

Proposition 35.

$$\mathcal{M}(\mathcal{K}(E)) \simeq \mathrm{Mor}(\mathcal{E})$$

where $\mathcal{M}(A)$ denotes the multiplier algebra of a C^*-algebra A.

Proof. One can show that for all $x \in E$ one can find a unique $y \in E$ such that $x = y. < y, y >$ (a technical exercise: show that the limit $y = \lim x.f_n(\sqrt{(x,x)})$ with $f_n(t) = t^{1/3}.(1/n+t)^{-1}$ exists and satisfies the desired assertion).

It has the consequence that E is a non degenerate $\mathcal{K}(E)$-module (ie, $\mathcal{K}(E).E = E$), indeed $x = y. < y, y >= \theta_{y,y}(y)$. Using an approximate unit $(u_\lambda)_\Lambda$ for $\mathcal{K}(E)$, we can extend the $\mathcal{K}(E)$-module structure of E into a $\mathcal{M}(\mathcal{K}(E))$-module structure:

$$\forall T \in \mathcal{M}(\mathcal{K}(E)), x \in E, \quad T.x = \lim_\Lambda T(u_\lambda).x$$

The existence of the limit is a consequence of $x = \theta_{y,y}(y)$ and $T(u_\lambda).\theta_{y,y} = T(u_\lambda \theta_{y,y}) \to T(\theta_{y,y})$. The limit is just $T(\theta_{y,y}).y$. By the unicity of y, this module structure, extending that of $\mathcal{K}(E)$ is unique.

Hence each $m \in \mathcal{M}(\mathcal{K}(E))$ gives a map $M : E \to E$. For any x, z in E,

$$(z, M.x) = (z, (M \circ \theta_{y,y})(y)) = (z, (m\theta_{y,y}).y) = ((m\theta_{y,y})^* z, (y))$$

thus M has an adjoint: $M \in \mathrm{Mor}(E)$ and M^* corresponds to m^*. The map $\rho : m \to M$ provides a $*$-homomorphism from $\mathcal{M}(\mathcal{K}(E))$ to $\mathrm{Mor}(E)$ which is the identity on $\mathcal{K}(E)$. On the other hand let $\pi : \mathrm{Mor}(E) \to \mathcal{M}(\mathcal{K}(E)$ be the unique $*$-homomorphism, equal to identity on $\mathcal{K}(E)$, associated to the inclusion $\mathcal{K}(E) \subset \mathrm{Mor}(E)$ as a closed ideal. We have $\pi \circ \rho = Id$, and by unicity of the $\mathcal{M}(\mathcal{K}(E))$-module structure of E, $\rho \circ \pi = Id$. □

Let us give some generic examples:

1. Consider A as a Hilbert A-module. We know that for any $a \in A$, there exists $c \in A$ such that $a = cc^*c$. It has the consequence that the map $\gamma_a : A \to A, b \mapsto ab$ is equal to θ_{c,c^*c} and thus is compact. We get a $*$-homomorphism $\gamma : A \to \mathcal{K}(A), a \mapsto \gamma_a$ which has dense image (the linear span of the θ's is dense in $\mathcal{K}(A)$) and clearly injective, because $yb = 0$ for all $b \in A$ implies $y = 0$. Thus γ is an isomorphism;
$$\mathcal{K}(A) \simeq A \ .$$
In particular, $\mathrm{Mor}(A) \simeq \mathcal{M}(A)$, and if $1 \in A$, then $A \simeq \mathrm{Mor}(A) = \mathcal{K}(A)$.
2. For any n, one has in a similar way $\mathcal{K}(A^n) \simeq M_n(A)$ and $\mathrm{Mor}(A^n) \simeq M_n(\mathcal{M}(A))$. If moreover $1 \in A$,
$$(i)\ \ \mathrm{Mor}(A^n) = \mathcal{K}(A^n) \simeq M_n(A) \ .$$
For any Hilbert A-module E, we also have $\mathcal{K}(E^n) \simeq M_n(\mathcal{K}(E))$.

The relations (i) can be extended to arbitrary finitely generated Hilbert A-modules:

Proposition 36. *Let A be a unital C^*-algebra and E a A-Hilbert module. Then the following are equivalent:*

1. *E is finitely generated.*
2. *$\mathcal{K}(E) = \mathrm{Mor}(E)$.*
3. *Id_E is compact.*

In that case, E is also projective (ie, it is a direct summand of A^n for some n).

For the proof we refer to [52].

4.3. **Generalized Fredholm operators.** The Atkinson's theorem claims that for any bounded linear operator on a Hilbert space H, the assertion:

$$\ker F \text{ and } \ker F^* \text{ are finite dimensional,}$$

is equivalent to:

there exists a linear bounded operator G such that $FG-1, GF-1$ are compact . This is a little more subtle on Hilbert A-modules, since first of all the kernel of homomorphisms are A-modules not necessarily free and secondly, replacing the condition "finite dimensional" by "finitely generated", is not enough to recover the previous equivalence. This is why one uses the second assertion as a definition of Fredholm operators in the context of Hilbert modules, and we will see how to adapt the Atkison's classical result to this new notion.

Definition 37. The homomorphism $T \in \mathrm{Mor}(E, F)$ is a *generalized Fredholm operator* if there exists $G \in \mathrm{Mor}(F, E)$ such that:

$$GF - \mathrm{Id} \in \mathcal{K}(E) \quad \text{and} \quad FG - \mathrm{Id} \in \mathcal{K}(F) \ .$$

The following theorem is important to understand the next chapter on KK-theory.

Theorem 38. *Let A be a unital C^*-algebra, $\mathcal{E}$ a Hilbert countably generated A-module and F a generalized Fredholm operator on $\mathcal{E}$.*

1. *If $\mathrm{Im}\, F$ is closed, then $\ker F$ and $\ker F^*$ are finitely generated Hilbert modules.*
2. *There exists a compact perturbation G of F such that $\mathrm{Im}\, G$ is closed.*

Proof. (1) Since $\mathrm{Im}\, F$ is closed, so is $\mathrm{Im}\, F^*$ and both are orthocomplemented by, respectively, $\ker F^*$ and $\ker F$. Let $P \in \mathrm{Mor}(\mathcal{E})$ be the orthogonal projection on $\ker F$. Since F is a generalized Fredholm operator, there exists $G \in \mathrm{Mor}(\mathcal{E})$ such that $Q = 1 - GF$ is compact. In particular, Q is equal to Id on $\ker F$ and:

$$QP : \mathcal{E} = \ker F \oplus \mathrm{Im}\, F^* \to \mathcal{E}, \ x \oplus y \mapsto x \oplus 0.$$

Since QP is compact, its restriction: $QP|_{\ker F} : \ker F \to \ker F$ is also compact, but $QP|_{\ker F} = \mathrm{Id}_{\ker F}$ hence proposition 36 implies that $\ker F$ is finitely generated. The same argument works for $\ker F^*$.

(2) Let us denote by π the projection homomorphism:

$$\pi : \mathrm{Mor}(\mathcal{E}) \to C(\mathcal{E}) := \mathrm{Mor}(\mathcal{E})/\mathcal{K}(\mathcal{E}) \ .$$

Since $\pi(F)$ is invertible in $C(\mathcal{E})$ it has a polar decomposition: $\pi(F) = \omega.|\pi(F)|$. Any unitary of $C(\mathcal{E})$ can be lift to a partial isometry [52]. Let U be such a lift of the unitary ω. Using $|\pi(F)| = \pi(|F|)$, it follows that:

$$F = U|F| \mod \mathcal{K}(\mathcal{E}) \ .$$

Since $\pi(|F|)$ is also invertible, and positive, we can form $\log(\pi(|F|))$ and choose a self-adjoint $H \in \mathrm{Mor}(\mathcal{E})$ with $\pi(H) = \log(\pi(|F|))$. Then:

$$\pi(Ue^H) = \omega\pi(|F|) = \pi(F)$$

that is, Ue^H is a compact perturbation of F (and thus is a generalized Fredholm operator). U is a partial isometry, hence has a closed image, and e^H is invertible in $\mathrm{Mor}(\mathcal{E})$, hence Ue^H has closed image and the theorem is proved. □

4.4. **Tensor products.**

4.4.1. *Inner tensor products.* Let E be a Hilbert A-module, F a Hilbert B-module and $\pi : A \to \mathrm{Mor}(F)$ a $*$-homomorphism. We define a sesquilinear form on $E \otimes_A F$ by setting:

$$\forall x, x' \in E, y, y' \in F, \quad (x \otimes y, x' \otimes y')_{E\otimes F} := (y, (x, x')_E \cdot y')_F$$

where we have set $a \cdot y = \pi(a)(y)$ to lighten the formula. This sesquilinear form is a B-valued scalar product: only the positivity axiom needs some explanation. Set:

$$b = (\sum_i x_i \otimes y_i, \sum_i x_i \otimes y_i) = \sum_{i,j} (y_i, (x_i, x_j).y_j)$$

where π has been omitted. Let us set $P = ((x_i, x_j))_{i,j} \in M_n(A)$. The matrix P provides a (self-adjoint) compact homomorphism of A^n, which is positive since:

$$\forall a \in A^n, (a, Pa)_{A^n} = \sum_{i,j} a_i^*(x_i, x_j)a_j = (\sum_i x_i a_i, \sum_j x_j a_j) \geq 0\ .$$

This means that $P = Q^*Q$ for some $Q \in M_n(A)$. On the other hand, one can consider P as a homomorphism on F^n and setting $y = (y_1, \dots, y_n) \in F^n$:

$$b = (y, Py) = (Qy, Qy) \geq 0\ .$$

Thus $E \otimes_A F$ is a pre-Hilbert module in the generalized sense (i.e. we do not require that the inner product is definite) and the Hausdorff completion of $E \otimes_A F$ is a Hilbert B-module denoted in the same way.

Proposition 39. *Let $T \in \mathrm{Mor}(E)$ and $S \in \mathrm{Mor}(F)$.*

- *$T \otimes 1 : x \otimes y \mapsto Tx \otimes y$ defines a homomorphism of $E \otimes_A F$.*
- *If S commutes with π then $1 \otimes S : x \otimes y \mapsto x \otimes Sy$ is a homomorphism which commutes with any $T \otimes 1$.*

Remark 40. 1. Even if T is compact, $T \otimes 1$ is not compact in general. Same thing for S.
2. In general $1 \otimes S$ is not defined.

4.4.2. *Outer tensor products.* Now forget the homomorphism π and consider the tensor product over $\mathbb{C}$ of E and F. We set:

$$\forall x, x' \in E, y, y' \in F, \quad (x \otimes y, x' \otimes y')_{E\otimes F} := (x, x')_E \otimes (y, y')_F \in A \otimes B\ .$$

By default, we use the spatial tensor product of A and B. This defines a pre-Hilbert $A \otimes B$-module in the generalized sense (the proof of positivity uses similar arguments) and the Hausdorff completion will be denoted $E \otimes_{\mathbb{C}} F$.

Examples 41. Let H be a separable Hilbert space. Then:

$$H \otimes_{\mathbb{C}} A \simeq H_A$$

4.4.3. *Connections.* We turn back to internal tensor products. We keep notations of the corresponding subsection. A. Connes and G. Skandalis [14] introduced the notion of connection to bypass the non existence, in general, of $1 \otimes S$.

Definition 42. Consider two C^*-algebras A and B. Let E be a Hilbert A-module and F be a Hilbert B-module. Assume there is a $*$-morphism

$$g : A \to \mathcal{L}(F)$$

and take the inner tensor product $E \otimes_A F$. Given $x \in E$ we define a homomorphism

$$\begin{aligned} T_x : E &\rightarrow E \otimes_A F \\ y &\mapsto x \otimes y \end{aligned}$$

whose adjoint is given by

$$\begin{aligned} T_x^* : E \otimes_A F &\rightarrow F \\ z \otimes y &\mapsto g((x, z))y \end{aligned}$$

If $S \in \mathcal{L}(F)$, an *S-connection* on $E \otimes_A F$ is given by an element

$$G \in \mathcal{L}(E \otimes_A F)$$

such that for all $x \in E$:

$$\begin{aligned} T_x S - G T_x &\in \mathcal{K}(F, E \otimes_A F) \\ S T_x^* - T_x^* G &\in \mathcal{K}(E \otimes_A F, F) \ . \end{aligned}$$

Proposition 43. (1) *If $[\pi, S] \subset \mathcal{K}(F)$ then S-connections exists.*

(2) *If G_i, $i = 1, 2$ are S_i-connections, then $G_1 + G_2$ is a $S_1 + S_2$-connection and $G_1 G_2$ is a $S_1 S_2$-connection.*

(3) *For any S-connection G, $[G, \mathcal{K}(E) \otimes 1] \subset \mathcal{K}(E \otimes_A F)$.*

(4) *The space of 0-connections is exactly:*

$\{G \in \mathrm{Mor}(F, E \otimes_A F) \mid (\mathcal{K}(E) \otimes 1)G$ *and* $G(\mathcal{K}(E) \otimes 1)$ *are subsets of* $\mathcal{K}(E \otimes_A F)\}$

All these assertions are important during the construction of the Kasparov product. For the proof, see [14]

5. KK-Theory

5.1. **Kasparov modules and Homotopies.** Given two C^*-algebras A and B a *Kasparov A-B-module* (shortly Kasparov module) is given by a triple

$$x = (\mathcal{E}, \pi, F)$$

where $\mathcal{E} = \mathcal{E}^0 \bigoplus \mathcal{E}^1$ is a $(\mathbb{Z}/2\mathbb{Z})$-graded Hilbert countably generated B-module, $\pi : A \rightarrow \mathcal{L}(\mathcal{E})$ is a $*$-morphism of degree 0 with respect to the graduation, and $F \in \mathcal{L}(\mathcal{E})$. These data are required to satisfy the following properties:

$$\begin{aligned} \pi(a)(F^2 - 1) &\in \mathcal{K}(\mathcal{E}) \qquad \text{for all } a \in A \\ [\pi(a), F] &\in \mathcal{K}(\mathcal{E}) \qquad \text{for all } a \in A. \end{aligned}$$

We denote the set of Kasparov A-B-modules by $E(A, B)$.

Let us immediately define the equivalence relation leading to the definition of KK-groups. We denote $B([0,1]) := C([0,1], B)$.

Definition 44. A *homotopy* between two Kasparov A-B-modules $x = (\mathcal{E}, \pi, F)$ and $x' = (\mathcal{E}', \pi', F')$ is a Kasparov A-$B([0,1])$-module $\tilde{x}$ such that:

$$\begin{aligned} (ev_{t=0})_*(\tilde{x}) &= x, \\ (ev_{t=1})_*(\tilde{x}) &= x'. \end{aligned} \tag{5.1}$$

Here $ev_{t=\cdot}$ is the evaluation map at $t = \cdot$. Homotopy between Kasparov A-B-modules is an equivalence relation. If there exists a homotopy between x and x' we write $x \sim_h x'$.

The set of homotopy classes of Kasparov A-B-modules is denoted $KK(A, B)$.

There is a natural *addition* on $E(A,B)$: if $x=(\mathcal{E},\pi,F)$ and $x'=(\mathcal{E}',\pi',F')$ belong to $E(A,B)$, their sum $x+x'\in E(A,B)$ is defined by

$$x+x' \quad = \quad (\mathcal{E}\oplus\mathcal{E}',\pi\oplus\pi',F\oplus F').$$

A Kasparov $A-B$-module $x=(\mathcal{E},\pi,F)$ is called *degenerate* if for all $a\in A$, $\pi(a)(F^2-1)=0$ and $[\pi(a),F]=0$. It follows:

Proposition 45. *Degenerate elements of* $E(A,B)$ *are homotopic to* $(0,0,0)$.
The addition of Kasparov $A-B$*-modules provides a structure of abelian group to* $KK(A,B)$.

Proof. Let $x=(\mathcal{E},\pi,F)\in E(A,B)$ be a degenerate element. Set $\tilde{x}=(\tilde{\mathcal{E}},\tilde{\pi},\tilde{F})\in E(A,B([0,1]))$ with

$$\begin{aligned}\tilde{\mathcal{E}} &= C_0([0,1[\,,\mathcal{E})\\ \tilde{\pi}(a)\xi(t) &= \pi(a)\xi(t),\\ \tilde{F}\xi(t) &= F\xi(t).\end{aligned}$$

Then $\tilde{x}$ is a homotopy between x and $(0,0,0)$.

One can easily show that addition of Kasparov modules makes sense at the level of their homotopy classes. Thus $KK(A,B)$ admits a commutative semi-group structure with $(0,0,0)$ as a neutral element. Eventually, the opposite in $KK(A,B)$ of $x=(\mathcal{E},\pi,F)\in E(A,B)$ is represented by:

$$(\mathcal{E}^{op},\pi,-F).$$

where $\mathcal{E}^{op}$ is $\mathcal{E}$ with the opposite graduation: $(\mathcal{E}^{op})^i=\mathcal{E}^{1-i}$. Indeed, the module $(\mathcal{E},\pi,F)\oplus(\mathcal{E}^{op},\pi,-F)$ is homotopically equivalent to the degenerate module

$$(\mathcal{E}\oplus\mathcal{E}^{op},\pi\oplus\pi,\begin{pmatrix}0&1\\1&0\end{pmatrix})$$

This can be realized with the homotopy

$$G_t \quad = \quad \cos(\frac{\pi t}{2})\begin{pmatrix}0&-F\\F&0\end{pmatrix}+\sin(\frac{\pi t}{2})\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

□

5.2. **Operations on Kasparov modules.** Let us explain the functoriality of the KK-groups with respect to its variables. The following both operations on Kasparov modules make sense on KK-groups:

- **Pushforward along $*$-morphisms: covariance in the second variable.**

  Let $x=(\mathcal{E},\pi,F)\in E(A,B)$ and let $g:B\to C$ be a $*$-morphism. We define an element $g_*(x)\in E(A,C)$ by

  $$g_*(x) \quad = \quad (\mathcal{E}\otimes_g C,\pi\otimes 1,F\otimes id),$$

  where $\mathcal{E}\otimes_g C$ is the inner tensor product of the Hilbert B-module $\mathcal{E}$ with the Hilbert C-module C endowed with the left action of B given by g.
- **Pullback along $*$-morphisms: contravariance in the first variable.**

  Let $x=(\mathcal{E},\pi,F)\in E(A,B)$ and let $f:C\to A$ be a $*$-morphism. We define an element $f^*(x)\in E(C,B)$ by

  $$f^*(x) \quad = \quad (\mathcal{E},\pi\circ f,F).$$

Provided with this operations, KK-theory is a bifunctor from the category (of pairs) of C^*-algebras to the category of abelian groups.

We record another useful operation in KK-theory:

- **Suspension**:
  Let $x = (\mathcal{E}, \pi, F) \in E(A, B)$ and let D be a C^*-algebra. We define an element $\tau_D(x) \in E(A \otimes D, B \otimes D)$ by
  $$\tau_D(x) \quad = \quad (\mathcal{E} \otimes_{\mathbb{C}} D, \pi \otimes 1, F \otimes id).$$
  Here we take the external tensor product $\mathcal{E} \otimes_{\mathbb{C}} D$, which is a $B \otimes D$-Hilbert module.

5.3. **Examples of Kasparov modules and of homotopies between them.**

5.3.1. *Kasparov Modules coming from homomorphisms between C^*-algebras.* Let A, B be two C^*-algebras and $f : A \to B$ a $*$-homomorphism. Since $\mathcal{K}(B) \simeq B$, the following:

$$[f] := (B, f, 0)$$

defines a Kasparov $A - B$-module. If A and B are $\mathbb{Z}_2$-graded, f has to be a homomorphism of degree 0 (ie, respecting the grading).

5.3.2. *Atiyah's Ell.* Let X be a compact Hausdorff topological space. Take $A = C(X)$ be the algebra of continuous functions on X and let $B = \mathbb{C}$. Then

$$E(A, B) = Ell(X)$$

the ring of generalized elliptic operators on X as defined by M. Atiyah. Below we give two concrete examples of such Kasparov modules:

- Assume X is a compact smooth manifold, let $A = C(X)$ as above and let $B = \mathbb{C}$. Let E and E' be two smooth vector bundles over X and denote by π the action of $A = C(X)$ by multiplication on $L^2(X, E) \oplus L^2(X, E')$. Given a zero order pseudo-differential operator
  $$P : C^\infty(E) \to C^\infty(E')$$
  with parametrix $Q : C^\infty(E') \to C^\infty(E)$ the triple
  $$x_P \quad = \quad \left(L^2(X, E) \oplus L^2(X, E'),\ \pi,\ \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix} \right)$$
  defines an element in $E(A, B) = E(C(X), \mathbb{C})$.
- Let X be a compact spin^c manifold of dimension $2n$, let $A = C(X)$ be as above and let $B = \mathbb{C}$. Denote by $S = S^+ \otimes S^-$ the complex spin bundle over X and let
  $$\not{D} : L^2(X, S) \to L^2(X, S)$$
  be the corresponding Dirac operator. Let π be the action of $A = C(X)$ by multiplication on $L^2(X, S)$. Then, the triple
  $$x_{\not{D}} \quad = \quad \left(L^2(X, S),\ \pi,\ \frac{\not{D}}{\sqrt{1 + \not{D}^2}} \right)$$
  defines an element in $E(A, B) = E(C(X), \mathbb{C})$.

5.3.3. *Compact perturbations.* Let $x = (\mathcal{E}, \pi, F) \in E(A, B)$. Let $P \in \mathrm{Mor}(\mathcal{E})$ which satisfy:

$$\forall a \in A,\ \pi(a).P \in \mathcal{K}(\mathcal{E}) \text{ and } P.\pi(a) \in \mathcal{K}(\mathcal{E}) \tag{5.2}$$

Then:

$$x \quad \sim_h \quad (\mathcal{E}, \pi, F + P).$$

The homotopy is the obvious one: $(\mathcal{E} \otimes C([0,1]), \pi \otimes \mathrm{Id}, F + tP)$. In particular, when B is unital, we can always choose a representant $(\mathcal{E}, \pi, G)$ with $\mathrm{Im}\, G$ closed (cf. theorem 38).

5.3.4. *(Quasi) Self-adjoint representants.* There exists a representant $(\mathcal{E}, \pi, G)$ of $x = (\mathcal{E}, \pi, F) \in E(A, B)$ satisfying:

$$\pi(a)(G - G^*) \in \mathcal{K}(\mathcal{E}) \ . \tag{5.3}$$

Just take $(\mathcal{E} \otimes C([0,1]), \pi \otimes \mathrm{Id}, F_t)$ as a homotopy where

$$F_t = (tF^*F + 1)^{1/2} F (tF^*F + 1)^{-1/2}$$

Then $G = F_1$ satisfies (5.3). Now, $H = (G + G^*)/2$ is self-adjoint and $P = (G - G^*)/2$ satisfies (5.2) thus $(\mathcal{E}, \pi, H)$ is another representant of x.

Note that (5.3) is often useful in practice and is added as an axiom in many definitions of KK-theory, like the original one of Kasparov. It was observed in [47] that it could be omitted.

5.3.5. *Unitarly equivalent modules.* Let (E_i, π_i, F_i), $i = 1, 2$, be two Kasparov modules such that there exists a unitary $u : E_1 \to E_2$ with:

$$uF_1u^* = F_2 \text{ and } \forall a \in A,\ u\pi_1(a)u^* - \pi_2(a) \in \mathcal{K}(\mathcal{E}_2)$$

We then say that (E_1, F_1) and (E_2, F_2) are unitarly equivalent. Unitarly equivalent Kasparov modules are, up to the addition of degenerate modules, homotopic. Consider for instance:

$$\left(E_1 + E_2, \pi_1 \oplus \pi_2, \begin{pmatrix} \cos t & -u^* \sin t \\ u \sin t & \cos t \end{pmatrix} \begin{pmatrix} F_1 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \cos t & u^* \sin t \\ -u \sin t & \cos t \end{pmatrix} \right).$$

5.3.6. *Relationship with ordinary K-theory.* Let B be a unital C^*-algebra. A finitely generated ($\mathbb{Z}/2\mathbb{Z}$-graded) projective B-module $\mathcal{E}$ is a submodule of some $B^N \oplus B^N$ and can then be endowed with a structure of Hilbert B-module. Afterwards, $\mathrm{Id}_{\mathcal{E}}$ is a compact morphism (prop. 36), thus:

$$(\mathcal{E}, \iota, 0) \in E(\mathbb{C}, B)$$

where ι is just multiplication by complex numbers. This provides a group homomorphism $K_0(B) \to KK(\mathbb{C}, B)$.

Conversely, let $(\mathcal{E}, 1, F) \in E(\mathbb{C}, B)$ be any Kasparov module where we have chosen F with closed range (see above): $\ker F$ is then a finitely generated $\mathbb{Z}/2\mathbb{Z}$-graded projective B-module. Consider $\widetilde{\mathcal{E}} = \{\xi \in C([0,1], \mathcal{E}) \mid \xi(1) \in \ker F\}$ and $\widetilde{F}(\xi) : t \mapsto F(\xi(t))$. Then $(\widetilde{\mathcal{E}}, 1, \widetilde{F})$ provides a homotopy between $(\mathcal{E}, 1, F)$ and $(\ker F, 1, 0)$. This also gives an inverse of the previous group homomorphism.

5.3.7. *A non trivial generator of $KK(\mathbb{C},\mathbb{C})$.* In the special case $B=\mathbb{C}$, we get $KK(\mathbb{C},\mathbb{C})\simeq K_0(\mathbb{C})\simeq\mathbb{Z}$ and under this isomorphisms, the following triple:

$$\left(L^2(\mathbb{R})^2, 1, \frac{1}{\sqrt{1+H}}\begin{pmatrix} 0 & -\partial_x+x \\ \partial_x+x & 0\end{pmatrix}\right) \quad \text{where} \quad H=-\partial_x^2+x^2$$

corresponds to $+1$. It is an easy exercise to check that ∂_x+x and H are essentially self-adjoint as unbounded operators on $L^2(\mathbb{R})$, that H has a compact resolvant, ∂_x+x has a Fredholm index equal to $+1$, and thus that the previous element is unambiguously defined and satisfies the required claim.

5.4. **Ungraded Kasparov modules and KK_1.** Sometimes, triple $(\mathcal{E},\pi,F)$ satisfying axioms (5.1) arise with no natural grading for $\mathcal{E}$, and consequently with no diagonal/antidiagonal decompositions for π, F. We then speak about ungraded Kasparov A-B-modules and the corresponding set is denoted by $E^1(A,B)$. The direct sum is defined in the same way, as well as the homotopy, which is this times an element of $E^1(A,B[0,1])$. The homotopy is still an equivalence relation denoted $\sim_h$ and the quotient $E^1(A,B)/\sim_h$ inherits a structure of abelian group as before. Let C_1 be the Clifford Algebra over $\mathbb{C}$. It is the graded C^*-algebra generated by an element ε of order 1 satisfying $\varepsilon^*=\varepsilon$ and $\varepsilon^2=1$. We have:

Proposition 46. *The following map:*

$$\begin{array}{rccc} (5.4) \qquad & E^1(A,B) & \longrightarrow & E(A,B\otimes C_1) \\ & (\mathcal{E},\pi,F) & \longmapsto & (\mathcal{E}\otimes C_1, \pi\otimes \mathrm{Id}, F\otimes\varepsilon) \end{array}$$

induces an isomorphism between $E^1(A,B)/\sim_h$ and $KK_1(A,B)=KK(A,B\otimes C_1)$.

Proof. The grading of C_1 gives the one of $\mathcal{E}\otimes C_1$ and the map (5.4) gives easily a homomorphism c from $KK_1(A,B)$ to $KK(A,B\otimes C_1)$.

Now let $y=(\mathcal{E},\pi,F)\in E(A,B\otimes C_1)$. The multiplication by ε on the right of $\mathcal{E}$ makes sense, even if B is not unital, and one has $\mathcal{E}_1=\mathcal{E}_0\varepsilon$. It follows that $\mathcal{E}=\mathcal{E}_0\oplus\mathcal{E}_1\simeq\mathcal{E}_0\oplus\mathcal{E}_0$ and any $T\in \mathrm{Mor}(\mathcal{E})$, thanks to the $B\otimes C_1$-linearity, has the following expression:

$$T=\begin{pmatrix} Q & P \\ P & Q\end{pmatrix} \qquad P,Q\in \mathrm{Mor}_B(\mathcal{E}_0)$$

Thus $F=\begin{pmatrix} 0 & P \\ P & 0\end{pmatrix}$, $\pi=\begin{pmatrix} \pi_0 & 0 \\ 0 & \pi_0\end{pmatrix}$ and $c^{-1}[y]=[\mathcal{E}_0,\pi_0,P]$. □

Remark 47. The opposite of $(\mathcal{E},\pi,F)$ in $KK_1(A,B)$ is represented by $(\mathcal{E},\pi,-F)$. One may wonder why we have to decide if a Kasparov module is graded or not. Actually, If we forget the $\mathbb{Z}/2\mathbb{Z}$ graduation of a graded Kasparov $A-B$-module $x=(\mathcal{E},\pi,F)$ and consider it as an ungraded module, then we get the trivial class in $KK_1(A,B)$. Indeed, the graduation of x implies that F has a matricial decomposition $\begin{pmatrix} 0 & Q \\ P & 0\end{pmatrix}$ and

$$\begin{pmatrix} 0 & -P \\ -Q & 0\end{pmatrix}=\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & Q \\ P & 0\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}=\rho_{-\pi/2}F\rho_{+\pi/2}$$

Thus, as operators on the Hilbert module $\mathcal{E}$ where the graduation is forgotten, we get

$$-F=\rho_{-\pi/2}F\rho_{+\pi/2}$$

It follows that $F_t = \rho_{-t\pi/2} F \rho_{+t\pi/2}$ provides a ungraded homotopy between x and $-x$ and thus $x = 0 \in KK_1(A, B)$.

Examples 48. Take again the example of the Dirac operator D on a spin^c manifold X whose dimension is odd. There is no natural $\mathbb{Z}/2\mathbb{Z}$ graduation for the spinor bundle. The previous triple $x_{\not{D}}$ belongs this time to $E^1(C(X), \mathbb{C})$.

5.5. **The Kasparov product.** In this section we construct the product

$$KK(A, B) \otimes KK(B, C) \to KK(A, C)\ .$$

It will satisfies the properties explained in 3. Actually:

Theorem 49. *Let* $x = (\mathcal{E}, \pi, F) \in E(A, B)$ *and* $x = (\mathcal{E}', \pi', F') \in E(B, C)$ *be two Kasparov modules. Set*

$$\mathcal{E}'' \quad = \quad \mathcal{E} \otimes_B \mathcal{E}'$$

and

$$\pi'' \quad = \quad \pi \otimes 1$$

Then there exists a unique, up to homotopy, F'*-connection on* $\mathcal{E}''$ *denoted by* F'' *such that*

- $(\mathcal{E}'', \pi'', F'') \in E(A, C)$
- $\pi''(a)\,[F'', F \otimes 1]\,\pi''(a)$ *is nonnegative modulo* $\mathcal{K}(\mathcal{E}'')$ *for all* $a \in A$.

$(\mathcal{E}'', \pi'', F'')$ *is the Kasparov product of* x *and* x'. *It enjoys all the properties described in section 3.*

Idea of the proof. We just explain the construction of the operator F''. For a complete proof, see for instance [29, 14]. A very naive idea for F'' could be $F \otimes 1 + 1 \otimes F'$ but the trouble is that the operator $1 \otimes F'$ is not well defined in general. We can overcome this first difficulty by replacing the not well defined $1 \otimes F'$ by any F'-connection G on $\mathcal{E}''$, and try $F \otimes 1 + G$. We get into a second problem which is that the axioms of Kasparov module are not satisfied in general with this candidate for F'': for instance $(F^2 - 1) \otimes 1 \in \mathcal{K}(E) \otimes 1 \not\subset \mathcal{K}(E'')$ as soon as E'' is not finitely generated.

The case of tensor product of elliptic self-adjoint differential operators on a closed manifold M, indicates us the good way. If D_1 and D_2 are two such operators and H_1,H_2 the natural L^2 spaces on which they act, then the bounded operator on $H_1 \otimes H_2$:

$$\frac{D_1}{\sqrt{1 + D_1^2}} \otimes 1 + 1 \otimes \frac{D_2}{\sqrt{1 + D_2^2}} \tag{5.5}$$

inherits the same problem than $F \otimes 1 + G$ but:

$$D'' := \frac{1}{\sqrt{2 + D_1^2 \otimes 1 + 1 \otimes D_2^2}} (D_1 \otimes 1 + 1 \otimes D_2)$$

has better properties: $D''^2 - 1$ and $[C(M), D'']$ belong to $\mathcal{K}(H_1 \otimes H_2)$. Note that

$$D'' = \sqrt{M}.\frac{D_1}{\sqrt{1 + D_1^2}} \otimes 1 + \sqrt{N}.1 \otimes \frac{D_2}{\sqrt{1 + D_2^2}}$$

with

$$M = \frac{1 + D_1^2 \otimes 1}{2 + D_1^2 \otimes 1 + 1 \otimes D_2^2} \text{ and } N = \frac{1 + 1 \otimes D_2^2}{2 + D_1^2 \otimes 1 + 1 \otimes D_2^2}\ .$$

The operators M, N are bounded on $H_1 \otimes H_2$, positive, and satisfy $M + N = 1$. We thus see that in that case, the naive idea (5.5) can be corrected by combining the involved operators with some adequate "partition of unity".

Turning back to our problem, this calculation leads us to look for the good operator F'' in the following form:

$$F'' = \sqrt{M}.F \otimes 1 + \sqrt{N}G \ .$$

We need to have that F'' is a F'-connection, and satisfies $a.(F''^2 - 1) \in \mathcal{K}(E'')$ and $[a, F''] \in \mathcal{K}(E'')$ for all $a \in A$ (by a we mean $\pi''(a)$). Using the previous form for F'', a small computation shows that these assertions become true if all the following conditions hold:

($\imath$) M is a 0-connection (equivalently, N is a 1-connection),
($\imath\imath$) $[M, F \otimes 1]$, $N.[F \otimes 1, G]$, $[G, M]$, $N(G^2 - 1)$ belong to $\mathcal{K}(E'')$,
($\imath\imath\imath$) $[a, M]$, $N.[G, a]$ belong to $\mathcal{K}(E'')$.

At this point there is a miracle:

Theorem 50 (Kasparov's technical theorem)**.** *Let J be a C^*-algebra and denote by $\mathcal{M}(J)$ its multipliers algebra. Assume there are two subalgebras A_1, A_2 of $\mathcal{M}(J)$ and a linear subspace $\triangle \subset \mathcal{M}(J)$ such that*

$$\begin{array}{rcl} A_1\, A_2 & \subset & J \\ [\triangle, A_1] & \subset & J \end{array}$$

Then there exist two nonnegative elements $M, N \in \mathcal{M}(J)$ with $M + N = 1$ such that

$$\begin{array}{rcl} M\, A_1 & \subset & J \\ N\, A_2 & \subset & J \\ [M, \triangle] & \subset & J \end{array}$$

For a proof, see [25].
Now, to get ($\imath$), ($\imath\imath$), ($\imath\imath\imath$), we apply this theorem with:

$$\begin{array}{rcl} A_1 & = & C^*\langle \mathcal{K}(\mathcal{E}) \otimes 1,\, \mathcal{K}(\mathcal{E}'')\rangle \\ A_2 & = & C^*\langle G^2 - 1,\, [G, F \otimes 1]\,,\, [G, \pi'']\rangle \\ \triangle & = & Vect\langle \pi''(A),\, G,\, F \otimes 1\rangle \end{array}$$

This gives us the correct F''. □

5.6. **Equivalence and duality in KK-theory.** With the Kasparov product come the following notions:

Definition 51. Let A, B be two C^*-algebras.

- One says that A and B are *KK-equivalent* if there exist $\alpha \in KK(A, B)$ and $\beta \in KK(B, A)$ such that:

  $$\alpha \otimes \beta = 1 \in KK(A, A) \text{ and } \beta \otimes \alpha = 1 \in KK(B, B).$$

  In that case, the pair (α, β) is called a *KK-equivalence* and it gives rise to isomorphisms

  $$KK(A \otimes C, D) \simeq KK(B \otimes C, D) \text{ and } KK(C, A \otimes D) \simeq KK(C, B \otimes D)$$

  given by Kasparov products for all C^*-algebras C, D.

- One says that A and B are *KK-dual (or Poincaré dual)* if there exist $\delta \in KK(A \otimes B, \mathbb{C})$ and $\lambda \in KK(\mathbb{C}, A \otimes B)$ such that:
$$\lambda \underset{B}{\otimes} \delta = 1 \in KK(A, A) \text{ and } \lambda \underset{A}{\otimes} \delta = 1 \in KK(B, B) \ .$$
In that case, the pair (λ, δ) is called a KK-duality and it gives rise to isomorphisms
$$KK(A \otimes C, D) \simeq KK(C, B \otimes D) \text{ and } KK(C, A \otimes D) \simeq KK(B \otimes C, B \otimes D)$$
given by Kasparov products for all C^*-algebras C, D.

We continue this paragraph with classical computations illustrating these notions.

5.6.1. *Bott periodicity.* Let $\beta \in KK(\mathbb{C}, C_0(\mathbb{R}^2))$ be represented by the Kasparov module:

$$(\mathcal{E}, \pi, C) = \left(C_0(\mathbb{R}^2) \oplus C_0(\mathbb{R}^2),\ 1,\ \frac{1}{\sqrt{1+c^2}} \begin{pmatrix} 0 & c_- \\ c_+ & 0 \end{pmatrix} \right).$$

where c_+, c_- are the operators given by pointwise multiplication by $x - \imath y$ and $x + \imath y$ respectively and $c = \begin{pmatrix} 0 & c_- \\ c_+ & 0 \end{pmatrix}$.

Let $\alpha \in KK(C_0(\mathbb{R}^2), \mathbb{C})$ be represented by the Kasparov module:

$$(\mathcal{H}, \pi, F) = \left(L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2),\ \pi,\ \frac{1}{\sqrt{1+D^2}} \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \right)$$

where $\pi : C_0(\mathbb{R}^2) \to \mathcal{L}(L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2))$ is the action given by multiplication of functions and the operators D_+ and D_- are given by

$$\begin{aligned} D_+ &= \partial_x + \imath \partial y \\ D_- &= -\partial_x + \imath \partial y. \end{aligned}$$

and $D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$.

Theorem 52. *α and β provide a KK-equivalence between $C_0(\mathbb{R}^2)$ and $\mathbb{C}$*

This is the Bott periodicity theorem in the bivariant K-theory framework.

Proof. Let us begin with the computation of $\beta \otimes \alpha \in KK(\mathbb{C}, \mathbb{C})$. We have an identification:

$$\mathcal{E} \underset{C_0(\mathbb{R}^2)}{\otimes} \mathcal{H} \simeq \mathcal{H} \oplus \mathcal{H} \tag{5.6}$$

where on the right the first copy of $\mathcal{H}$ stands for $\mathcal{E}_0 \underset{C_0(\mathbb{R}^2)}{\otimes} \mathcal{H}_0 \oplus \mathcal{E}_1 \underset{C_0(\mathbb{R}^2)}{\otimes} \mathcal{H}_1$ and the second for $\mathcal{E}_0 \underset{C_0(\mathbb{R}^2)}{\otimes} \mathcal{H}_1 \oplus \mathcal{E}_1 \underset{C_0(\mathbb{R}^2)}{\otimes} \mathcal{H}_0$. One checks directly that under this identification the following operator

$$G = \frac{1}{\sqrt{1+D^2}} \begin{pmatrix} 0 & 0 & D_- & 0 \\ 0 & 0 & 0 & -D_+ \\ D_+ & 0 & 0 & 0 \\ 0 & -D_- & 0 & 0 \end{pmatrix} \tag{5.7}$$

is an F-connection. On the other hand, under the identification (5.7), the operator $C \otimes 1$ gives:

$$\frac{1}{\sqrt{1+c^2}}\begin{pmatrix} 0 & 0 & 0 & c_- \\ 0 & 0 & c_+ & 0 \\ 0 & c_- & 0 & 0 \\ c_+ & 0 & 0 & 0 \end{pmatrix} \tag{5.8}$$

It follows immediately that $\beta \otimes \alpha$ is represented by:

$$\delta = \left(\mathcal{H} \oplus \mathcal{H}, 1, \frac{1}{\sqrt{1+c^2+D^2}}\mathcal{D}\right) \tag{5.9}$$

where $\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}$; $\mathcal{D}_+ = \begin{pmatrix} D_+ & c_- \\ c_+ & -D_- \end{pmatrix}$ and $\mathcal{D}_- = \mathcal{D}_+^*$. Observe that, denoting by ρ the rotation in $\mathbb{R}^2$ of angle $\pi/4$:

$$\begin{aligned}
\begin{pmatrix} \rho^{-1} & 0 \\ 0 & \rho \end{pmatrix}\begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}\begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} &= \begin{pmatrix} 0 & \rho^{-1}\mathcal{D}_-\rho^{-1} \\ \rho\mathcal{D}_+\rho & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & \imath(\partial_y - y) & -\partial_x + x \\ 0 & 0 & \partial_x + x & -\imath(\partial_y + y) \\ \imath(\partial_y + y) & -\partial_x + x & 0 & 0 \\ \partial_x + x & \imath(-\partial_y + y) & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & x - \partial_x \\ x + \partial_x & 0 \end{pmatrix} \otimes 1 + 1 \otimes \begin{pmatrix} 0 & \imath(\partial_y - y) \\ \imath(\partial_y + y) & 0 \end{pmatrix}
\end{aligned}$$

Of course

$$\delta \sim_h \left(\mathcal{H} \oplus \mathcal{H}, 1, \frac{1}{\sqrt{1+c^2+D^2}}\begin{pmatrix} 0 & \rho^{-1}\mathcal{D}_-\rho^{-1} \\ \rho\mathcal{D}_+\rho & 0 \end{pmatrix}\right)$$

and the computation above shows that δ coincides with the Kasparov product $u \otimes u$ with $u \in KK(\mathbb{C}, \mathbb{C})$ given by:

$$u = \left(L^2(\mathbb{R})^2, 1, \frac{1}{\sqrt{1+x^2+\partial_x^2}}\begin{pmatrix} 0 & x - \partial_x \\ x + \partial_x & 0 \end{pmatrix}\right)$$

A simple exercise shows that $\partial_x + x : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is essentially self-adjoint with one dimensional kernel and zero dimensional cokernel, thus $1 = u = u \otimes u \in KK(\mathbb{C}, \mathbb{C})$.

Let us turn to the computation of $\alpha \otimes \beta \in KK(C_0(\mathbb{R}^2), C_0(\mathbb{R}^2))$: it is a Kasparov product over $\mathbb{C}$, thus it commutes:

$$\alpha \otimes \beta = \tau_{C_0(\mathbb{R}^2)}(\beta) \otimes \tau_{C_0(\mathbb{R}^2)}(\alpha) \tag{5.10}$$

but the two copies of $C_0(\mathbb{R}^2)$ above are not the same (the first one comes from α, the second from β: think about it as functions of the variable x for the first and of the variable y for the second) and we can not factorize directly $\tau_{C_0(\mathbb{R}^2)}$ in the right hand side of (5.10) in order to use the value of $\beta \otimes \alpha$. This is where a classical argument, known as the rotation trick of Atiyah, is necessary:

Lemma 53. *Let* $\phi : C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}^2) \to C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}^2)$ *be the flip automorphism:* $\phi(f)(x, y) = f(y, x)$. *Then:*

$$[\phi] = 1 \in KK(C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}^2), C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}^2))$$

Proof of the lemma. Let us denote by I_2 the identity matrix of $M_2(\mathbb{R})$. Use a continuous path of isometries of $\mathbb{R}^4$ connecting $\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$ to $\begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}$. This gives a homotopy $\phi \sim_h \mathrm{Id}$. □

Now

$$\begin{aligned} \alpha \otimes \beta &= \tau_{C_0(\mathbb{R}^2)}(\beta) \otimes \tau_{C_0(\mathbb{R}^2)}(\alpha) = \tau_{C_0(\mathbb{R}^2)}(\beta) \otimes [\phi] \otimes \tau_{C_0(\mathbb{R}^2)}(\alpha) \\ &= \tau_{C_0(\mathbb{R}^2)}(\beta \otimes \alpha) = \tau_{C_0(\mathbb{R}^2)}(1) = 1 \in KK(C_0(\mathbb{R}^2), C_0(\mathbb{R}^2)) \end{aligned} \tag{5.11}$$

□

5.6.2. *Self duality of $C_0(\mathbb{R})$.* With the same elements as before, we get:

Corollary 54. *The algebra $C_0(\mathbb{R})$ is Poincaré dual to itself.*

Other examples of Poincaré dual algebras will be given later.

Proof. The automorphism ψ of $C_0(\mathbb{R})^{\otimes^3}$ given by $\psi(f)(x,y,z) = f(z,x,y)$ is homotopic to the identity thus:

$$\begin{aligned} \beta \underset{C_0(\mathbb{R})}{\otimes} \alpha &= \tau_{C_0(\mathbb{R})}(\beta) \otimes \tau_{C_0(\mathbb{R})}(\alpha) = \tau_{C_0(\mathbb{R})}(\beta) \otimes [\psi] \otimes \tau_{C_0(\mathbb{R})}(\alpha) \\ &= \tau_{C_0(\mathbb{R})}(\beta \otimes \alpha) = \tau_{C_0(\mathbb{R})}(1) = 1 \in KK(C_0(\mathbb{R}), C_0(\mathbb{R})) \end{aligned} \tag{5.12}$$

□

Exercise 55. Let

$$\beta_c = \left(C_0(\mathbb{R}) \otimes C_1, 1, \frac{x}{\sqrt{x^2+1}} \otimes \varepsilon \right) \in KK(\mathbb{C}, C_0(\mathbb{R}) \otimes C_1)$$

$$\alpha_c = \left(L^2(\mathbb{R}, \Lambda^*\mathbb{R}), \pi, \frac{1}{\sqrt{1+\Delta}}(d+\delta) \right) \in KK(C_0(\mathbb{R}) \otimes C_1, \mathbb{C})$$

where $(d+\delta)(a+bdx) = -b' + a'dx$, $\Delta = (d+\delta)^2$ and $\pi(f \otimes \varepsilon)$ sends $a + bdx$ to $f(b+adx)$.
Show that β_c, α_c provide a KK-equivalence between $\mathbb{C}$ and $C_0(\mathbb{R}) \otimes C_1$ (Hints: compute directly $\beta_c \otimes \alpha_c$, then use the commutativity of the Kasparov product over $\mathbb{C}$ and check that the flip of $(C_0(\mathbb{R}) \otimes C_1)^{\otimes^2}$ is 1 to conclude about the computation of $\alpha_c \otimes \beta_c$).

5.6.3. *A simple Morita equivalence.* Let $\imath_n = (M_{1,n}(\mathbb{C}), 1, 0) \in E(\mathbb{C}, M_n(\mathbb{C}))$ where the $M_n(\mathbb{C})$-module structure is given by multiplication by matrices on the right. Note that $[\imath_n]$ is also the class of the homomorphism $\mathbb{C} \to M_n(\mathbb{C})$ given by the left up corner inclusion. Let also $\jmath_n = (M_{n,1}(\mathbb{C}), m, 0) \in E(M_n(\mathbb{C}), \mathbb{C})$ where m is multiplication by matrices on the left. Then immediately:

$$\imath_n \otimes \jmath_n \sim_h (\mathbb{C}, 1, 0) \text{ and } \jmath_n \otimes \imath_n \sim_h (M_n(\mathbb{C}), 1, 0)$$

thus $\mathbb{C}$ and $M_n(\mathbb{C})$ are KK-equivalent and this is an example of a Morita equivalence. The map in K-theory associated with $\jmath$: $\cdot \otimes \jmath_n : K_0(M_n(\mathbb{C})) \to \mathbb{Z}$ is just the trace homomorphism. Consider similarly the Kasparov elements $\imath \in E(\mathbb{C}, \mathcal{K}(\mathcal{H}))$ associated to the homomorphism $\imath : \mathbb{C} \to \mathcal{K}(\mathcal{H})$ given by the choice of a rank one projection and $\jmath = (\mathcal{H}, m, 0) \in E(\mathcal{K}(\mathcal{H}), \mathbb{C})$ where m is just the action of compact operators on $\mathcal{H}$: they provide a KK-equivalence between $\mathcal{K}$ and $\mathbb{C}$.

5.6.4. *$C_0(\mathbb{R})$ and C_1.* We leave the proof of the following result as an exercise:

Proposition 56. *The algebras $C_0(\mathbb{R})$ and C_1 are KK-equivalent.*

Hint for the proof: Consider

$$\widetilde{\alpha} = \left(L^2(\mathbb{R}, \Lambda^*\mathbb{R}), m, \frac{1}{\sqrt{1+\Delta}}(d+\delta)\right) \in KK(C_0(\mathbb{R}), C_1)$$

where d, δ, Δ are defined in the previous exercise, $m(f)(\xi) = f\xi$, and the C_1-right module structure of $L^2(\mathbb{R}, \Lambda^*\mathbb{R})$ is given by $(a + bdx) \cdot \varepsilon = -ib + iadx$. Consider also:

$$\widetilde{\beta} = \left(C_0(\mathbb{R})^2, \varphi, \frac{x}{\sqrt{1+x^2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \in KK(C_1, C_0(\mathbb{R}))$$

where $\varphi(\varepsilon)(f, g) = (-ig, if)$. Prove that they provide the desired KK-equivalence.

$\square$

Exercise 57.
(1) Check that $\tau_{C_1} : KK(A, B) \to KK(A \otimes C_1, B \otimes C_1)$ is an isomorphism.
(2) Check that under τ_{C_1} and the Morita equivalence $M_2(\mathbb{C}) \sim \mathbb{C}$, the elements α_c, β_c of the previous exercise coincide with $\widetilde{\alpha}, \widetilde{\beta}$ and recover the KK-equivalence between C_1 and $C_0(\mathbb{R})$.

Remark 58. At this point, one sees that $KK_1(A, B) = KK(A, B(\mathbb{R}))$, $(B(\mathbb{R}) := C_0(\mathbb{R}) \otimes B)$ can also be presented in the following different ways:

$$E_1(A, B)/\sim_h \simeq KK(A, B \otimes C_1) \simeq KK(A \otimes C_1, B) \simeq KK(A(\mathbb{R}), B)$$

5.7. **Computing the Kasparov product without its definition.** Computing the product of two Kasparov modules is in general quite hard, but we are very often in one of the following situations.

5.7.1. *Use of the functorial properties.* Thanks to the functorial properties listed in 3, a lot of products can be deduced from known, already computed, ones. For instance, in the proof of the Bott periodicity (the KK-equivalence between $\mathbb{C}$ and $C_0(\mathbb{R}^2)$) one had to compute two products: the first one was directly computed, the second one was deduced from the first using the properties of the Kasparov product and a simple geometric fact. Examples of this kind are numerous.

5.7.2. *Maps between K-theory groups.* Let A, B be two unital (if not, add a unit) C^*-algebras, $x \in KK(A, B)$ be given by a Kasparov module $(\mathcal{E}, \pi, F)$ where F has a closed range and assume that we are interested by the map $\phi_x : K_0(A) \to K_0(B)$ associated with x in the following way:

$$y \in K_0(A) \simeq KK(\mathbb{C}, A); \quad \phi_x(y) = y \otimes x$$

This product takes a particular simple form when y is represented by $(\mathcal{P}, 1, 0)$ with $\mathcal{P}$ a finitely generated projective A-module (see 5.3.6):

$$y \otimes x = \left(\mathcal{P} \underset{A}{\otimes} \mathcal{E}, \pi \otimes 1, F \otimes \mathrm{Id}\right) = (\ker(F \otimes \mathrm{Id}), 1, 0)\,.$$

5.7.3. *Kasparov elements constructed from homomorphisms.* Sometimes, Kasparov classes $y \in KK(B,C)$ can be explicitly represented as Kasparov products of classes of homomorphisms with inverses of such classes. Assume for instance that $y = [e_0]^{-1} \otimes [e_1]$ where $e_0 : \mathcal{C} \to B$, $e_1 : \mathcal{C} \to C$ are homomorphisms of C^*-algebras and e_0 produces an invertible element in KK-theory (for instance: $\ker e_0$ is K-contractible and: B is nuclear or $\mathcal{C}, B$ K-nuclear, see [48, 16]). Then computing a Kasparov product $x \otimes y$ where $x \in KK(A,B)$ amounts to lift x to $KK(A,\mathcal{C})$, that is find $x' \in KK(A,\mathcal{C})$ such that $(e_0)_*(x') = x$ and restrict this lift to $KK(A,C)$, that is evaluate $x" = (e_1)_*(x')$. It follows from the properties of the product that $x" = x \otimes y$.

Examples 59. Consider the tangent groupoid $\mathcal{G}_{\mathbb{R}}$ of $\mathbb{R}$ and let $\delta = [e_0]^{-1} \otimes [e_1] \otimes \mu$ be the associated deformation element: $e_0 : C^*(\mathcal{G}_{\mathbb{R}}) \to C^*(T\mathbb{R}) \simeq C_0(\mathbb{R}^2)$ is evaluation at $t = 0$, $e_1 : C^*(\mathcal{G}_{\mathbb{R}}) \to C^*(\mathbb{R} \times \mathbb{R}) \simeq \mathcal{K}(L^2(\mathbb{R})) \simeq \mathcal{K}$ is evaluation at $t = 1$ and $\mu = (L^2(\mathbb{R}), m, 0) \in KK(\mathcal{K}, \mathbb{C})$ gives the Morita equivalence $\mathcal{K} \sim \mathbb{C}$.

Let $\beta \in KK(\mathbb{C}, C_0(\mathbb{R}^2))$ be the element used in paragraph 5.6.1. Then $\beta \otimes \delta$ is easy to compute. The lift $\beta' \in KK(\mathbb{C}, C^*(\mathcal{G}_{\mathbb{R}}))$ is produced using the pseudodifferential calculus for groupoids (see below) and can be presented as a family $\beta' = (\beta_t)$ with:

$$\beta_0 = \beta;\ t > 0, \beta_t = \left(C^*(\mathbb{R} \times \mathbb{R}, \frac{dx}{t}), 1, \frac{1}{\sqrt{1 + x^2 + t^2\partial_x^2}} \begin{pmatrix} 0 & x - t\partial_x \\ x + t\partial_x & 0 \end{pmatrix} \right)$$

Then after restricting at $t = 1$ and applying the Morita equivalence; it just remains the index of the Fredholm operator appearing in β_1, that is $+1$, and this proves $\beta \otimes \delta = 1$.

Observe that by unicity of the inverse, we conclude that $\delta = \alpha$.

Examples 60. (Boundaries homomorphisms in long exact sequences) Let

$$0 \to I \underset{i}{\to} A \underset{p}{\to} B \to 0$$

be a short exact sequence of C^*-algebras either admitting a completely positive, norm decreasing linear section or assume that I, A, B are K-nuclear ([48]). Let $C_p = \{(a, \varphi) \in A \oplus C_0([0,1[, B) \ |p(a) = \varphi(0)\}$ be the cone of the homomorphism $p : A \to B$ and denote by d the homomorphism: $C_0(]0,1[, B) \hookrightarrow C_p$ given by $d(\varphi) = (0, \varphi)$ and by e the homomorphism: $I \to C_p$ given by $e(a) = (a, 0)$. Thanks to the hypotheses, $[e]$ is invertible in KK-theory. One can set $\delta = [d] \otimes [e]^{-1} \in KK(C_0(\mathbb{R}) \otimes B, I)$ and using the Bott periodicity $C_0(\mathbb{R}^2) \underset{\text{KK}}{\sim} \mathbb{C}$ in order to identify:

$$KK_2(C, D) = KK(C_0(\mathbb{R}^2) \otimes C, D) \simeq KK(C, D),$$

the connecting maps in the long exact sequences:

$$\cdots \to KK_1(I, D) \to KK(B, D) \xrightarrow{i^*} KK(A, D) \xrightarrow{p^*} KK(I, D) \to KK_1(B, D) \to \cdots,$$

$$\cdots \to KK_1(C, B) \to KK(C, I) \xrightarrow{i_*} KK(C, A) \xrightarrow{p_*} KK(C, B) \to KK_1(C, I) \to \cdots$$

are given by the appropriate Kasparov products with δ.

Index theorems

6. Introduction to Pseudodifferential operators on groupoids

The historical motivation for developing the pseudodifferential calculus on groupoids comes from A. Connes, who introduced implicitly this notion for foliations. Later on, this calculus was axiomatized and studied on general groupoids by several authors [36, 37, 50].

The following example illustrate how the pseudodifferential calculus on groupoids arise in our approach of index theory. If P is a partial differential operator on $\mathbb{R}^n$:

$$P(x, D) = \sum_{|\alpha|\leq d} c_\alpha(x) D_x^\alpha$$

we may associate to it the following asymptotic operator:

$$P(x, tD) = \sum_{|\alpha|\leq d} c_\alpha(x) (tD_x)^\alpha$$

by introducing a parameter $t \in]0, 1]$ in front of each ∂_{x_j}. We use above the ordinary convention : $D_x^\alpha = (-i\partial_{x_1})^{\alpha_1} \dots (-i\partial_{x_n})^{\alpha_n}$. We would like to give a (interesting) sense to the limit $t \to 0$. Of course we would not be very happy with $tD \to 0$.

To investigate this question, let us look at $P(x, tD)$ as a left multiplier on $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ rather than a linear operator on $C^\infty(\mathbb{R}^n)$:

$$\begin{aligned} P(x, tD_x)u(x, y) &= \int e^{(x-z).\xi} p(x, t\xi) u(z, y) dz d\xi \\ &= \int e^{\frac{x-z}{t}.\xi} p(x, \xi) u(z, y) \frac{dz d\xi}{t^n} \\ &= \int e^{(X-Z).\xi} p(x, \xi) u(x - t(X - Z), x - tX) dZ d\xi \end{aligned}$$

In the last line we have introduced the notation $X = \frac{x-y}{t}$ and performed the change of variables $Z = \frac{z-y}{t}$.

At this point, assume that u depends also on t in the following way:

$$u(x, y, t) = \widetilde{u}(y, \frac{x-y}{t}, t), \quad \widetilde{u} \in C^\infty(\mathbb{R}^{2n} \times [0, 1])$$

It follows:

$$\begin{aligned} P(x, tD_x)u(x, y) &= \int e^{(X-Z).\xi} p(x, \xi) \widetilde{u}(x - tX, Z, t) dZ d\xi \\ &\xrightarrow{t\to 0} \int e^{(X-Z).\xi} p(x, \xi) \widetilde{u}(x, Z, 0) dZ d\xi \\ &= P(x, D_X)\widetilde{u}(x, X, 0) \end{aligned}$$

Observations

- $P(x, D_X)$ is a partial differential operator in the variable X with constant coefficients, depending smoothly on a parameter x and with symbol coinciding with the one of $P(x, D_x)$ in the sense that: $\sigma(P(x, D_X)(x, X, \xi) = P(x, \xi)$. In particular, $P(x, D_X)$ is invariant by the translation $X \mapsto X + X_0$. Of course, $P(x, D_X)$ is nothing else, up to a Fourier transform in X, than the symbol $P(x, \xi)$ of $P(x, D_x)$. In other words, denoting by $S_X(T\mathbb{R}^n)$ the space of smooth functions $f(x, X)$ rapidly decreasing in X

and by $\mathcal{F}_X$ the Fourier transform with respect to the variable X, we have a commutative diagram:

$$\begin{tikzcd}
S_X(T\mathbb{R}^n) \arrow[r, "P(x,D_X)"] \arrow[d, "\mathcal{F}_X"'] & S_X(T\mathbb{R}^n) \arrow[d, "\mathcal{F}_X"'] \\
S_\xi(T^*\mathbb{R}^n) \arrow[r, "P(x,\xi)"] & S_\xi(T^*\mathbb{R}^n)
\end{tikzcd}$$

where $P(x, D_X)$ acts as a left multiplier on the convolution algebra $S_X(T\mathbb{R}^n)$ and $P(x, \xi)$ acts as a left multiplier on the functions algebra $S_\xi(T^*\mathbb{R}^n)$ (equipped with the pointwise multiplication of functions).

- u and $\widetilde{u}$ are related by the bijection:

$$\begin{array}{rccc} \phi : & \mathbb{R}^{2n} \times [0,1] & \longrightarrow & \mathcal{G}_{\mathbb{R}^{2n}} \\ & (x, X, t) & \longmapsto & (x + tX, x, t) \text{ if } t > 0 \\ & (x, X, 0) & \longmapsto & (x, X, 0) \end{array}$$

  ($\phi^{-1}(x, y, t) = (y, (x - y)/t, t)$, $\phi^{-1}(x, X, 0) = (x, X, 0)$). In fact, the smooth structure of the tangent groupoid $\mathcal{G}_{\mathbb{R}^{2n}}$ is defined by requiring that ϕ is a diffeomorphism. Thus $\widetilde{u} \in C^\infty(\mathbb{R}^{2n} \times [0, 1])$ means $u \in C^\infty(\mathcal{G}_{\mathbb{R}^n})$.

Thus $P(x, D_X)$ is another way to look at, and even, another way to **define**, the symbol of $P(x, D_x)$. What is important for us is that it appears as a "limit" of a family P_t constructed with P, and the pseudodifferential calculus on the tangent groupoid of $\mathbb{R}^n$ will make us able to give a rigorous meaning to this limit and perform interesting computations.

The material below is taken from [36, 37, 50]. Let G be a Lie groupoid, with units space $G^{(0)} = V$ and with a smooth (right) Haar system $d\lambda$. We assume that V is a compact manifold and that the s-fibers G_x, $x \in V$, have no boundary. We denote by U_γ the map induced on functions by right multiplication by γ, that is:

$$U_\gamma : C^\infty(G_{s(\gamma)}) \longrightarrow C^\infty(G_{r(\gamma)});\ U_\gamma f(\gamma') = f(\gamma'\gamma)$$

Definition 61. A G-operator is a continuous linear map $P : C_c^\infty(G) \longrightarrow C^\infty(G)$ such that:

(i) P is given by a family $(P_x)_{x\in V}$ of linear operators $P_x : C_c^\infty(G_x) \to C^\infty(G_x)$ and:

$$\forall f \in C_c^\infty(G), \quad P(f)(\gamma) = P_{s(\gamma)} f_{s(\gamma)}(\gamma)$$

where f_x stands for the restriction $f|_{G_x}$.

(ii) The following invariance property holds:

$$U_\gamma P_{s(\gamma)} = P_{r(\gamma)} U_\gamma$$

Let P be a G-operator and denote by $k_x \in C^{-\infty}(G_x \times G_x)$ the Schwartz kernel of P_x, for each $x \in V$, as obtained from the Schwartz kernel theorem applied to the manifold G_x provided with the measure $d\lambda_x$.

Thus, using the axiom [i]:

$$\forall \gamma \in G, f \in C^\infty(G), \quad Pf(\gamma) = \int_{G_x} k_x(\gamma, \gamma') f(\gamma') d\lambda_x(\gamma'), \qquad (x = s(\gamma))$$

Next:

$$U_\gamma Pf(\gamma') = Pf(\gamma'\gamma) = \int_{G_x} k_x(\gamma'\gamma, \gamma'') f(\gamma'') d\lambda_x(\gamma''), \qquad (x = s(\gamma))$$

and

$$\begin{aligned} P(U_\gamma f)(\gamma') &= \int_{G_y} k_y(\gamma',\gamma'')f(\gamma''\gamma)d\lambda_y(\gamma''), \qquad (y=r(\gamma)) \\ &\overset{\eta=\gamma''\gamma}{=} \int_{G_x} k_y(\gamma',\eta\gamma^{-1})f(\eta)d\lambda_x(\eta), \qquad (x=s(\gamma)) \end{aligned}$$

where the last line uses the invariance property of Haar systems. Thus axiom [ii] is equivalent to the following equalities of distributions on $G_x \times G_x$, for all $x \in V$:

$$\forall \gamma \in G, \quad k_x(\gamma'\gamma,\gamma'') = k_y(\gamma',\gamma''\gamma^{-1}) \quad (x=s(\gamma),\ y=r(\gamma))$$

Setting $k_P(\gamma) := k_{s(\gamma)}(\gamma, s(\gamma))$, we get $k_x(\gamma,\gamma') = k_P(\gamma\gamma'^{-1})$, and the linear operator $P : C_c^\infty(G) \to C^\infty(G)$ is given by:

$$P(f)(\gamma) = \int_{G_x} k_P(\gamma\gamma'^{-1})d\lambda_x(\gamma') \quad (x=s(\gamma))$$

and we may consider k_P as a single distribution on G acting on smooth functions on G by convolution. With a slight abuse of terminology, we will refer to k_P as the Schwartz (or convolution) kernel of P.

We will say that P is *smoothing* if $k_P \in C^\infty(G)$ and is *compactly supported* or *uniformly supported* if k_P is compactly supported (which implies that each P_x is properly supported).

Let us develop some examples of G operators.

Examples 62. (1) if $G = G^{(0)} = V$ is just a set, then $G_x = \{x\}$ for all $x \in V$. The axiom [ii] is empty and the axiom [ii] implies that a G-operator is given by pointwise multiplication by a smooth function $P \in C^\infty(V)$: $Pf(x) = P(x).f(x)$.

(2) $G = V \times V$ the pair groupoid, and the Haar system $d\lambda$ is given in the obvious way by a single measure dy on V:

$$d\lambda_x(y) = dy \text{ under the identification } G_x = V \times \{x\} \simeq V$$

It follows that for any G-operator P:

$$Pg(z,x) = \int_{V\times\{x\}} k_P(z,y)g(y,x)d\lambda_x(y,x) = \int_V k_P(z,y)g(y,x)dy$$

which proves immediately that $P_x = P_y$ are equal as linear operators on $C^\infty(V)$ under the obvious identifications $V \simeq V \times \{x\} \simeq V \times \{y\}$.

(3) Let $p : X \to Z$ a submersion, and $G = X \underset{Z}{\times} X = \{(x,y) \in X \times X\ |p(x) = p(y)\}$ the associated subgroupoid of the pair groupoid $X \times X$. The manifold G_x can be identified with the fiber $p^{-1}(p(x))$. The axiom [ii] implies that for any G-operator P, we have $P_x = P_y$ as linear operators on $p^{-1}(p(x))$ as soon as $y \in p^{-1}(p(x))$. Thus, P is actually given by a family $\tilde{P}_z$, $z \in Z$ of operators on $p^{-1}(z)$, with the relation $P_x = \tilde{P}_{p(x)}$.

(4) Let $G = E$ be the total space of a (euclidean, hermitian) vector bundle $p : E \to V$, with $r = s = p$. The Haar system $d_x w$, $x \in V$, is given by the metric structure on the fibers of E. We have here:

$$Pf(v) = \int_{E_x} k_P(v-w)f(w)d_x w \qquad (x = p(v))$$

Thus, for all $x \in V$, P_x is a convolution operator on the linear space E_x.

(5) Let $G = \mathcal{G}_V = TV \times \{0\} \sqcup V \times V \times]0,1]$ be the tangent groupoid of V. It can be viewed as a family parametrized by $[0,1]$ of groupoids G_t, where $G_0 = TV$ and $G_t = V \times V$ for $t > 0$. A $\mathcal{G}_V$-operator is given by a family P_t of G_t-operators, and $(P_t)_{t>0}$ is a family parametrized by t of operators on $C_c^\infty(V)$ while P_0 is a family parametrized by $x \in V$ of translation invariant operators on T_xV. The $\mathcal{G}_V$-operators are thus a blend of examples 2 and 4.

We turn now to the definition of pseudodifferential operators on a Lie groupoid G.

Definition 63. A G-operator P is a G-pseudodifferential operator of order m if:

(1) The Schwartz kernel k_P is smooth outside $G^{(0)}$.
(2) For every distinguished chart $\psi : U \subset G \to \Omega \times s(U) \subset \mathbb{R}^{n-p} \times \mathbb{R}^p$ of G:

$$\begin{tikzcd} U \arrow[rr, "\psi"] \arrow[dr, "s"'] & & \Omega \times s(U) \arrow[dl, "p_2"] \\ & s(U) & \end{tikzcd}$$

the operator $(\psi^{-1})^* P \psi^* : C_c^\infty(\Omega \times s(U)) \to C_c^\infty(\Omega \times s(U))$ is a smooth family parametrized by $s(U)$ of pseudodifferential operators of order m on Ω.

We will use very few properties of this calculus. We content ourselves with some examples and a list of properties. The reader can find a complete presentation in [50, 49, 37, 36, 35].

Examples 64. In the previous five examples, a G-pseudodifferential operator is:

(1) an operator of pointwise multiplication by a smooth function on V;
(2) a single pseudodifferential operator on V;
(3) a smooth family parametrized by Z of pseudodifferential operators in the fibers: it is exactly the notion of [7];
(4) a family parametrized by $x \in V$ of convolution operators in E_x such that the underlying distribution k_P identifies with the Fourier transform of a symbol on E (that is, a smooth function on E satisfying the standard decay conditions with respect to its variable in the fibers);
(5) the data provided by an asymptotic pseudodifferential operator on V together with one of its complete symbol, the choice of it depending on the gluing in $\mathcal{G}_V$: it is quite close from the notions studied in [23, 8, 22]

A G-pseudodifferential operator P has a *compact support* if there exists a compact set $K \subset G$ such that $\mathrm{supp}(Pf) \subset K.\mathrm{supp}(f)$ for all $f \in C_c^\infty(G)$. It turns out that the space $\Psi_c^*(G)$ of compactly supported G-pseudodifferential operators is an involutive algebra.

The principal symbol of a G-pseudodifferential operator P of order m is defined as a function $\sigma_m(P)$ on $A^*(G) \setminus G^{(0)}$ by:

$$\sigma_m(P)(x,\xi) = \sigma_{pr}(P_x)(x,\xi)$$

where $\sigma_{pr}(P_x)$ is the principal symbol of the pseudodifferential operator P_x on the manifold G_x. Conversely, given a symbol f of order m on $A^*(G)$ together with the following data:

(1) A smooth embedding $\theta : \mathcal{U} \to AG$, where $\mathcal{U}$ is a open set in G containing $G^{(0)}$, such that $\theta(G^{(0)}) = G^{(0)}$ and $\theta(\gamma) \in A_{s(\gamma)}G$ for all $\gamma \in \mathcal{U}$;

(2) A smooth compactly supported map $\phi : G \to \mathbb{R}_+$ such that $\phi^{-1}(1) = G^{(0)}$;

we get a G-pseudodifferential operator $P_{f,\theta,\phi}$ by the formula:

$$u \in C_c^\infty(G),\ P_{f,\theta,\phi}u(\gamma) = \int_{\substack{\gamma' \in G_{s(\gamma)},\\ \xi \in A^*_{r(\gamma)}(G)}} e^{-i\theta(\gamma'\gamma^{-1}).\xi} f(r(\gamma), \xi)\phi(\gamma'\gamma^{-1})u(\gamma')d\lambda_{s(\gamma)}(\gamma')$$

The principal symbol of $P_{f,\theta,\phi}$ is just the leading part of f.

The principal symbol map respects products. An operator is *elliptic* when its principal symbol never vanishes and in that case it has a parametrix inverting it modulo $\Psi_c^{-\infty}(G) = C_c^\infty(G)$.

Operators of negative order in $\Psi_c^*(G)$ are actually in $C^*(G)$, while zero order operators are in the multiplier algebra $\mathcal{M}(C^*(G))$.

All these definitions and properties extend immediately to the case of operators acting between sections of bundles on $G^{(0)}$ pulled back to G with the range map r. The space of compactly supported pseudodifferential operators on G acting on sections of r^*E and taking values in sections of r^*F will be noted $\Psi_c^*(G, E, F)$. If $F = E$ we get an algebra denoted by $\Psi_c^*(G, E)$.

Examples 65. (1) The family given by $P_t = P(x, tD_x)$ for $t > 0$ and $P_0 = P(x, D_X)$ described in the introduction of this section is a G-pseudodifferential operator with G the tangent groupoid of $\mathbb{R}^n$.

(2) More generally, let V be a closed manifold. Let f be a symbol on V. We get a $\mathcal{G}_V$-pseudodifferential operator P by setting:

$$(t > 0) \quad P_t u(x, y, t) = \int_{z \in V, \xi \in T_x^*V} e^{\frac{\exp_x^{-1}(z)}{t}.\xi} f(x, \xi)u(z, y)\frac{dzd\xi}{t^n}$$

$$P_0 u(x, X, 0) = \int_{Z \in T_xV, \xi \in T_x^*V} e^{(X-Z).\xi} f(x, \xi)u(x, Z)dZd\xi$$

Moreover, P_1 is a pseudodifferential operator on the manifold V which admits f as a complete symbol.

7. INDEX THEOREM FOR SMOOTH MANIFOLDS

The purpose of this last lecture is to present a proof of the Atiyah-Singer index theorem using deformation groupoids and show how it generalizes to conical pseudomanifolds. The results presented here come from recent works of the authors together with a joint work with V. Nistor [19, 20, 18], we refer to [19, 20] for the proofs.

The KK-element associated to a deformation groupoid

Before going to the description of the index maps, let us describe a useful and classical construction [13, 27].

A smooth groupoid G is called a *deformation groupoid* if:

$$G = G_1 \times \{0\} \cup G_2 \times]0, 1] \rightrightarrows G^{(0)} = M \times [0, 1],$$

where G_1 and G_2 are smooth groupoids with unit space M. That is, G is obtained by gluing $G_2 \times]0, 1] \rightrightarrows M \times]0, 1]$ which is the groupoid G_2 over M parameterized by $]0, 1]$ with the groupoid $G_1 \times \{0\} \rightrightarrows M \times \{0\}$.

In this situation one can consider the saturated open subset $M\times]0,1]$ of $G^{(0)}$. Using the isomorphisms $C^*(G|_{M\times]0,1]})\simeq C^*(G_2)\otimes C_0(]0,1])$ and $C^*(G|_{M\times\{0\}})\simeq C^*(G_1)$, we obtain the following exact sequence of C^*-algebras:

$$0 \longrightarrow C^*(G_2)\otimes C_0(]0,1]) \xrightarrow{i_{M\times]0,1]}} C^*(G) \xrightarrow{ev_0} C^*(G_1) \longrightarrow 0$$

where $i_{M\times]0,1]}$ is the inclusion map and ev_0 is the *evaluation map* at 0, that is ev_0 is the map coming from the restriction of functions to $G|_{M\times\{0\}}$.

We assume now that $C^*(G_1)$ is nuclear. Since the C^*-algebra $C^*(G_2)\otimes C_0(]0,1])$ is contractible, the long exact sequence in KK-theory shows that the group homomorphism $(ev_0)_* = \cdot\otimes[ev_0] : KK(A,C^*(G))\to KK(A,C^*(G_1))$ is an isomorphism for each C^*-algebra A.

In particular with $A=C^*(G)$ we get that $[ev_0]$ is invertible in KK-theory: there is an element $[ev_0]^{-1}$ in $KK(C^*(G_1),C^*(G))$ such that $[ev_0]\otimes[ev_0]^{-1}=1_{C^*(G)}$ and $[ev_0]^{-1}\otimes[ev_0]=1_{C^*(G_1)}$.

Let $ev_1 : C^*(G)\to C^*(G_2)$ be the evaluation map at 1 and $[ev_1]$ the corresponding element of $KK(C^*(G),C^*(G_2))$.

The *KK-element associated to the deformation groupoid* G is defined by:

$$\delta = [ev_0]^{-1}\otimes[ev_1]\in KK(C^*(G_1),C^*(G_2))\ .$$

We will see several examples of this construction in the sequel.

The analytical index

Let M be a closed manifold and consider its tangent groupoid:

$$\mathcal{G}^t_M := TM\times\{0\}\cup M\times M\times]0,1] \rightrightarrows M\times[0,1]$$

Let us construct the associated KK-element.

Using the partition $M\times[0,1]=M\times\{0\}\cup M\times]0,1]$ into saturated open and closed subsets of the units space of the tangent groupoid, we get the following short exact sequence of C^*-algebras:

$$0\to C^*(\mathcal{G}^t_M|_{M\times]0,1]})\overset{i}{\to} C^*(\mathcal{G}^t_M)\overset{e_0^M}{\to} C^*(G^t_M|_{M\times\{0\}})\to 0$$

where i comes from the inclusion of functions and e_0^M is the evaluation map at 0. Moreover $C^*(\mathcal{G}^t_M|_{M\times]0,1]}) = C^*(M\times M\times]0,1])\simeq\mathcal{K}(L^2(M))\otimes C_0(]0,1])$ and $C^*(G^t_M|_{M\times\{0\}})=C^*(TM)\simeq C_0(T^*M)$. Thus we have

$$0\to\mathcal{K}(L^2(M))\otimes C_0(]0,1])\to C^*(\mathcal{G}^t_M)\overset{e_0^M}{\to} C_0(T^*M)\to 0$$

Since the C^*-algebras involved here are nuclear, we can apply the six terms exact sequence associated to this exact sequence of C^*-algebras. We get that $(e_0^M)_*$ is invertible or in other words that $[e_0^M]\in KK(C^*(\mathcal{G}^t_M),C_0(T^*M))$ is invertible and admits an inverse (with respect to the Kasparov product) $[e_0^M]^{-1}\in KK(C_0(T^*M),C^*(\mathcal{G}^t_M))$.

Let $e_1^M : C^*(\mathcal{G}^t_M)\to C^*(\mathcal{G}^t_M|_{M\times\{1\}})\simeq\mathcal{K}(L^2(M))$ be the evaluation map at 1. We define:

$$\partial_M := [e_0^M]^{-1}\otimes[e_1^M]\in KK(C_0(T^*M),\mathcal{K})\simeq KK(C_0(T^*M),\mathbb{C})\ .$$

The analytical index is then [13]

$$\begin{array}{rccc} Inda_M := (e_1^M)_*\circ(e_0^M)_*^{-1} : & KK(\mathbb{C},C_0(T^*M)) & \to & KK(\mathbb{C},\mathcal{K}(L^2(M)) \\ & \simeq K_0(C_0(T^*M)) & & \simeq\mathbb{Z} \end{array}$$

or in terms of Kasparov product

$$Inda_M = \cdot \otimes \partial_M \ .$$

Using the notion of pseudodifferential calculus for $\mathcal{G}_M^t$, it is easy to justify that this map is the usual analytical index map. Indeed, let $f(x,\xi)$ be an elliptic zero order symbol and consider the $\mathcal{G}_M^t$-pseudodifferential operator, $P_f = (P_t)_{0\leq t\leq 1}$, defined as in example 65. Then f provides a K-theory class $[f] \in K_0(C^*(TM)) \simeq K_0(C_0(T^*M))$ while P provides a K-theory class $[P] \in K_0(C^*(\mathcal{G}_M^t))$ and:

$$(e_0^M)_*([P]) = [f] \in K_0(C^*(TM))$$

Thus:

$$[f] \otimes [e_0^M]^{-1} \otimes [e_1^M] = [P_1] \in K_0(\mathcal{K})$$

and $[P_1]$ coincides with $\mathrm{Ind}(P_1)$ under $K_0(\mathcal{K}) \simeq \mathbb{Z}$.
Since P_1 has principal symbol equal to the leading part of f, and since every class in $K_0(C_0(T^*M))$ can be obtained with a zero order elliptic symbol, the claim is justified.

To be complete, let us explain that the analytical index map is the Poincaré dual of the homomorphism in K-homology associated with the obvious map: $M \to \{\cdot\}$. Indeed, thanks to the obvious homomorphism $\Psi : C^*(TM) \otimes C(M) \to C^*(TM)$ given by multiplication, ∂_M can be lifted into an element $D_M = \Psi_*(\partial_M) \in KK(C^*(TM)\otimes C(M), \mathbb{C}) = K^0(C^*(TM)\otimes C(M))$, called the *Dirac element*. This Dirac element yields the well known Poincaré duality between $C_0(T^*M)$ and $C(M)$ ([14, 30, 19]), and in particular it gives an isomorphism:

$$\cdot \underset{C^*(TM)}{\otimes} D_M : \ K_0(C^*(TM)) \xrightarrow{\simeq} K^0(C(M))$$

whose inverse is induced by the principal symbol map.
One can then easily show the following proposition:

Proposition 66. *Let $q : M \to \cdot$ be the projection onto a point. The following diagram commutes:*

$$\begin{tikzcd}
K^0(T^*M) \arrow[r, "\mathrm{PD}"] \arrow[d, "Ind_a"'] & K_0(M) \arrow[d, "q_*"] \\
\mathbb{Z} \arrow[r, "="] & \mathbb{Z}
\end{tikzcd}$$

The topological index
Take an embedding $M \to \mathbb{R}^n$, and let $p : N \to M$ be the normal bundle of this embedding. The vector bundle $TN \to TM$ admits a complex structure, thus we have a Thom isomorphism:

$$T : K_0(C^*(TM)) \xrightarrow{\simeq} K_0(C^*(TN))$$

given by a KK-equivalence:

$$T \in KK(C^*(TM), C^*(TN)) \ .$$

T is called the *Thom element* [29].

The bundle N identifies with an open neighborhood of M into $\mathbb{R}^n$, so we have the excision map:

$$j : C^*(TN) \to C^*(T\mathbb{R}^n).$$

Consider also: $B : K_0(C^*(T\mathbb{R}^n)) \to \mathbb{Z}$ given by the isomorphism $C^*(T\mathbb{R}^n) \simeq C_0(\mathbb{R}^{2n})$ together with Bott periodicity.

The *topological index map* Ind_t is the composition:

$$K(C^*(TM)) \xrightarrow{T} K(C^*(TN)) \xrightarrow{j_*} K(C^*(T\mathbb{R}^n)) \xrightarrow[\simeq]{B} \mathbb{Z}$$

This classical construction can be reformulated with groupoids.

First, let us give a description of T, or rather of its inverse, in terms of groupoids. Recall the construction of the Thom groupoid. We begin by pulling back TM over N in the groupoid sense:

$$\text{Let :} \qquad {}^*p^*(TM) = N \underset{M}{\times} TM \underset{M}{\times} N \rightrightarrows N.$$

$$\text{Let :} \qquad \mathcal{T}_N = TN \times \{0\} \sqcup {}^*p^*(TM) \times]0,1] \rightrightarrows N \times [0,1]$$

This *Thom groupoid* and the Morita equivalence between ${}^*p^*(TM)$ and TM provides the KK-element:

$$\tau_N \in KK(C^*(TN), C^*(TM)) \ .$$

This element is defined exactly as ∂_M is. Precisely, the evaluation map at 0, $\tilde{e}_0 : C^*(\mathcal{T}_N) \to C^*(TN)$ defines an invertible KK-element. We let $\tilde{e}_1 : C^*(\mathcal{T}_N) \to C^*({}^*p^*(TM))$ be the evaluation map at 1. The Morita equivalence between the groupoids TM and ${}^*p^*(TM)$ leads to a Morita equivalence between the corresponding C^*-algebra and thus to a KK-equivalence $\mathcal{M} \in KK(C^*({}^*p^*(TM)), C^*(TM))$. Then

$$\tau_N := [\tilde{e}_0]^{-1} \otimes [\tilde{e}_1] \otimes \mathcal{M} \ .$$

We have the following:

Proposition 67. [20] *If T is the KK-equivalence giving the Thom isomorphism then:*

$$\tau_N = T^{-1}.$$

This proposition also applies to interpret the isomorphism $B : K_0(C^*(T\mathbb{R}^n)) \to \mathbb{Z}$.

Indeed, consider the embedding $\cdot \hookrightarrow \mathbb{R}^n$. The normal bundle is just $\mathbb{R}^n \to \cdot$ and we get as before:

$$\tau_{\mathbb{R}^n} \in KK(C^*(T\mathbb{R}^n), \mathbb{C})$$

Using the previous proposition we get: $B = \cdot \otimes \tau_{\mathbb{R}^n}$.

Remark also that $\mathcal{T}_{\mathbb{R}^n} = \mathcal{G}_{\mathbb{R}^n}$.

Finally the topological index:

$$\mathrm{Ind}_t = \tau_{\mathbb{R}^n} \circ j_* \circ \tau_N^{-1}$$

is entirely described with (deformation) groupoids.

The equality of the indices

Everything in our presentation of index maps is given by Kasparov products with:

1. classes of homomorphisms coming from restrictions/inclusions between groupoids,
2. inverses of such classes,
3. explicit Morita equivalences.

This implies the commutativity of:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "="] & \mathbb{Z} \arrow[r, "="] & \mathbb{Z} \\
K_0(C^*(\mathcal{G}_M)) \arrow[u] \arrow[r] \arrow[d, "\simeq"'] & K_0(C^*(\mathcal{G}_N)) \arrow[u] \arrow[r] \arrow[d, "\simeq"'] & K_0(C^*(\mathcal{G}_{\mathbb{R}^n})) \arrow[u] \arrow[d, "\simeq"'] \\
K_0(C^*(TM)) \arrow[r] & K_0(C^*(TN)) \arrow[r] & K_0(C^*(T\mathbb{R}^n))
\end{tikzcd}$$

Thus we recover:

$$\mathrm{Ind}_a = \mathrm{Ind}_t$$

8. The case of pseudomanifolds with isolated singularities

As we explained earlier, the proof of the K-theoretical form of the Atiyah-Singer presented in these lectures extends very easily to the case of pseudomanifolds with isolated singularities. This is achieved as soon as one uses the correct notion of *tangent space* of the pseudomanifold and for a pseudomanifold X with one conical point (the case of several isolated singularities is similar), this is the noncommutative tangent space defined in section 1.5:

$$T^{\mathsf{S}}X = X^- \times X^- \cup T\overline{X^+} \rightrightarrows X^\circ$$

It will replace in the sequel the ordinary tangent space of a smooth manifold. Moreover, it gives rise to another deformation groupoid which will replace the ordinary tangent groupoid of a smooth manifold:

$$\mathcal{G}^t_X = T^{\mathsf{S}}X \times \{0\} \cup X^\circ \times X^\circ \times]0,1] \rightrightarrows X^\circ \times [0,1]$$

We call $\mathcal{G}^t_X$ the *tangent groupoid* of X. It can be provided with a smooth structure such that $T^{\mathsf{S}}X$ is a smooth subgroupoid. Moreover both are amenable so their reduced and maximal C^*-algebras coincide and are nuclear.

With these choices of $T^{\mathsf{S}}X$ as a tangent space for X and of $\mathcal{G}^t_X$ as a tangent groupoid, one can follow step by step all the constructions made in the previous section.

8.1. The analytical index.

Using the partition $X^\circ \times [0,1] = X^\circ \times \{0\} \cup X^\circ \times]0,1]$ into saturated open and closed subsets of the units space of the tangent groupoid, we define the KK-element associated to the tangent groupoid of X:

$$\partial_X := [e_0]^{-1} \otimes [e_1] \in KK(C^*(T^{\mathsf{S}}X), \mathcal{K}) \simeq KK(C^*(T^{\mathsf{S}}X), \mathbb{C}) \ ,$$

where $e_0 : C^*(\mathcal{G}^t_X) \to C^*(\mathcal{G}^t_X|_{X^\circ\times\{0\}}) \simeq C^*(T^{\mathsf{S}}X)$ is the evaluation at 0 and $e_1 : C^*(\mathcal{G}^t_X) \to C^*(\mathcal{G}^t_X|_{X^\circ\times\{1\}}) \simeq \mathcal{K}(L^2(X))$ is the evaluation at 1.

Now we can define the analytical index exactly as we did for closed smooth manifolds. Precisely the *analytical index* for X is set to be the map:

$$\mathrm{Ind}^X_a = \cdot \otimes \partial_X : KK(\mathbb{C}, C^*(T^{\mathsf{S}}X)) \to KK(\mathbb{C}, \mathcal{K}(L^2(X^\circ))) \simeq \mathbb{Z} \ .$$

The interpretation of this map as the Fredholm index of an appropriate class of elliptic operators is possible and done in [32].

8.2. **The Poincaré duality.** Continuing the analogy with smooth manifolds, we explain in this paragraph that the analytical index map for X is Poincaré dual to the index map in K-homology associated to the obvious map : $X \to \{.\}$.

The algebras $C(X)$ and $C^{\bullet}(X) := \{f \in C(X) \mid f \text{ is constant on } cL\}$ are isomorphic. If g belongs to $C^{\bullet}(X)$ and f to $C_c(T^{\mathsf{S}}X)$, let $g \cdot f$ be the element of $C_c(T^{\mathsf{S}}X)$ defined by $g \cdot f(\gamma) = g(r(\gamma))f(\gamma)$. This induces a *-morphism

$$\Psi : C(X) \otimes C^*(T^{\mathsf{S}}X) \to C^*(T^{\mathsf{S}}X) \ .$$

The *Dirac element* is defined to be

$$D_X := [\Psi] \otimes \partial_X \in KK(C(X) \otimes C^*(T^{\mathsf{S}}X), \mathbb{C}).$$

We recall

Theorem 68. [19] *There exists a (dual-Dirac) element* $\lambda_X \in KK(\mathbb{C}, C(X) \otimes C^*(T^{\mathsf{S}}X))$ *such that*

$$\lambda_X \underset{C(X)}{\otimes} D_X = 1_{C^*(T^{\mathsf{S}}X)} \in KK(C^*(T^{\mathsf{S}}X), C^*(T^{\mathsf{S}}X)) \ ,$$

$$\lambda_X \underset{C^*(T^{\mathsf{S}}X)}{\otimes} D_X = 1_{C(X)} \in KK(C(X), C(X)) \ .$$

This means that $C(X)$ *and* $C^*(T^{\mathsf{S}}X)$ *are Poincaré dual.*

Remark 69. The explicit construction of λ_X, which is heavy and technical, can be avoided. In fact, the definitions of $T^{\mathsf{S}}X$, $\mathcal{G}_X^t$ and thus that of D_X, can be extended in a very natural way to the case of an arbitrary pseudomanifold and the proof of the Poincaré duality can be done using a recursive argument on the depth of the stratification, starting with the case depth$=0$, that is with the case of smooth closed manifolds. This is the subject of [18].

The theorem implies that:

$$\begin{array}{ccc} KK(\mathbb{C}, C^*(T^{\mathsf{S}}X)) \simeq K_0(C^*(T^{\mathsf{S}}X)) & \to & K(C(X), \mathbb{C}) \simeq K^0(C(X)) \\ x & \mapsto & x \underset{C^*(T^{\mathsf{S}}X)}{\otimes} D_X \end{array}$$

is an isomorphism. It is explained in [32] how to in interpret its inverse as a principal symbol map, and one also get the analogue of proposition 66:

Proposition 70. *Let* $q : X \to \cdot$ *be the projection onto a point. The following diagram commutes:*

$$\begin{tikzcd} K_0(C^*(T^{\mathsf{S}}X)) \arrow[r, "\mathrm{PD}"] \arrow[d, "Ind_a^X"'] & K_0(X) \arrow[d, "q_*"] \\ \mathbb{Z} \arrow[r, "="] & \mathbb{Z} \end{tikzcd}$$

8.3. **The topological index.**

Thom isomorphism Take an *embedding* $X \hookrightarrow c\mathbb{R}^n = \mathbb{R}^n \times [0, +\infty[/\mathbb{R}^n \times \{0\}$. This means that we have a map which restricts to an embedding in the usual sense $X^\circ \to \mathbb{R}^n \times]0, +\infty[$ and which sends c to the image of $\mathbb{R}^n \times \{0\}$ in $c\mathbb{R}^n$. Moreover we ask the embedding on $X^- = L \times]0, 1[$ to be of the form $j \times \mathrm{Id}$ where j is an embedding of L in $\mathbb{R}^n$.

Such an embedding provides a *conical normal bundle*. Precisely, let $p : N^\circ \to X^\circ$ be the normal bundle associated with $X^\circ \hookrightarrow \mathbb{R}^n \times]0,+\infty[$. We can identify $N^\circ|_{X^-} \simeq N^\circ|_L \times]0,1[$, and set :

$$N = \bar{c}N^\circ|_L \cup N^\circ|_{X^+} \ .$$

Thus N is the pseudomanifold with an isolated singularity obtained by gluing the closed cone $\bar{c}N^\circ|_L := N^\circ|_L \times [0,1]/N^\circ|_L \times \{0\}$ with $N^\circ|_{X^+}$ along their common boundary $N^\circ|_L \times \{1\} = N^\circ|_{\partial X^+}$. Moreover $p : N \to X$ is a conical vector bundle.

The *Thom groupoid* is then:

$$\mathcal{T}_N = T^{\mathsf{S}}N \times \{0\} \sqcup {}^*p^*(T^{\mathsf{S}}X) \times]0,1] \ .$$

It is a deformation groupoid. The corresponding KK-element gives the *inverse Thom element*:

$$\tau_N \in KK(C^*(T^{\mathsf{S}}N), C^*(T^{\mathsf{S}}X)) \ .$$

Proposition 71. [20] *The following map is an isomorphism.*

$$K(C^*(T^{\mathsf{S}}N)) \xrightarrow{\cdot \otimes \tau_N} K(C^*(T^{\mathsf{S}}X))$$

Roughly speaking, the inverse of $\cdot \otimes \tau_N$ is the *Thom isomorphism* for the "vector bundle" $T^{\mathsf{S}}N$ "over" $T^{\mathsf{S}}X$. One can show that it really restricts to usual Thom homomorphism on regular parts.

Excision The groupoid $T^{\mathsf{S}}N$ is identified with an open subgroupoid of $T^{\mathsf{S}}c\mathbb{R}^n$ and we have an excision map:

$$j : C^*(T^{\mathsf{S}}N) \to C^*(T^{\mathsf{S}}\mathbb{R}^n) \ .$$

Bott element Consider $c \hookrightarrow c\mathbb{R}^n$.
The (conical) normal bundle is $c\mathbb{R}^n$ itself. Remark that $\mathcal{G}^t_{c\mathbb{R}^n} = \mathcal{T}_{c\mathbb{R}^n}$. Then

$$\tau_{c\mathbb{R}^n} \in KK(C^*(T^{\mathsf{S}}c\mathbb{R}^n), \mathbb{C})$$

gives an isomorphism:

$$B = (\cdot \otimes \tau_{c\mathbb{R}^n}) : K_0(C^*(T^{\mathsf{S}}c\mathbb{R}^n)) \to \mathbb{Z}$$

Definition 72. The *topological index* is the morphism

$$\mathrm{Ind}_t^X = B \circ j_* \circ \tau_N^{-1} : \ K_0(C^*(T^{\mathsf{S}}X)) \to \mathbb{Z}$$

One can make exactly the same proof as in the smooth case to get

Theorem 73.

$$Ind_a^X = Ind_t^X$$

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