Singular foliations
1. Lie groupoids

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1. Locally compact groupoids

**Groupoids**

Definition

A *groupoid* is a small category in which each arrow is invertible.

A groupoid has therefore two sets:

- the set $G^{(0)}$ of *objects*;
- the set $G = G^{(1)}$ of arrows.
Groupoids (2)

We have several maps:

- Source and target maps $s, t : G \rightarrow G^{(0)}$;
- Unit map $u : G^{(0)} \rightarrow G$;
- Inverse $G \rightarrow G$ - denoted by $x \mapsto x^{-1}$;
- Composition from $G^{(2)} = \{(x, y) \in G \times G; s(x) = t(y)\}$ to $G$.

These maps satisfy:

- $s(xy) = s(y)$; $t(xy) = t(x)$; associativity $(xy)z = x(yz)$;
- $s(u(a)) = t(u(a)) = a$; $u(t(x))x = x = xu(s(x))$;
- $s(x^{-1}) = t(x)$; $t(x^{-1}) = s(x)$; $x^{-1}x = u(s(x))$ and $xx^{-1} = u(t(x))$. 
Locally compact groupoids (J. Renault)

- $G$, $G^{(0)}$ are locally compact spaces;
- all maps are continuous;
- $s, t$ are open.

Examples

1. A space is a groupoid: every arrow is a unit.
2. A group is a groupoid: only one unit.
A locally compact group $\Gamma$ acting continuously on a locally compact space. $G = \{(x, g, y) \in X \times \Gamma \times X; \ gy = x\}$ denoted by $G = X \rtimes \Gamma$.

Note that

$(x, g, y) \mapsto (x, g)$ homeomorphism $G \simeq X \times \Gamma$ and

$(x, g, y) \mapsto (g, y)$ homeomorphism $G \simeq \Gamma \times X$.

- $G^{(0)} = X$.
- $s(x, g, y) = y$ and $t(x, g, y) = x$.
- $u(x) = (x, 1, x)$;
- $(x, g, y)(y, h, z) = (x, gh, z)$ – note that $(gh)z = g(hz) = gy = x$. 
2. The $C^*$-algebra of a locally compact groupoid
(J. Renault)

Convolution

Let $G$ be a locally compact groupoid.

If $G = X \rtimes \Gamma$, we may define the (full or reduced) crossed product $C_0(X) \rtimes \Gamma$ which is a completion of $C_c(\Gamma; C_0(X))$. Note that it is also the completion of $C_c(\Gamma \times X) = C_c(G)$.

In general, for $f, g \in C_c(G)$,

- We put $f^*(x) = f(x^{-1})$.
- We want to form $f \star g$ by a formula

$$f \star g(x) = \int_{yz=x} f(y) \, g(z).$$

In other words, we want to have an integration along the fibers of the composition $G(2) = \{ (x, y); \ s(x) = t(y) \} \to G$. 

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Haar systems

- **Continuity:** $C_c(G^{(2)}) \to C_c(G)$.
- **Associativity:** Fubini + invariance:
  - $\{(y, z); \ yz = x\} \simeq G^{t(x)} \simeq G_{s(x)}$ through the maps $(y, z) \mapsto y$ and $(y, z) \mapsto z$.
  - Measures on these sets which correspond to each other under these identifications.
  - Measures $\lambda^u$ on $G^u$ such that $y \mapsto xy$ maps $\lambda^{s(x)}$ to $\lambda^{t(x)}$.

**Definition**

(J. Renault) A Haar system is a family $(\lambda^u)_{u \in G(0)}$ of measures on $G$ such that the support of $\lambda^u$ is in $G^u$ and satisfying the above invariance and continuity properties.

**Proposition**

*Endowed with the above involution and product,* $C_c(G)$ is a $\ast$-algebra.
Representations

Let $G$ be a locally compact groupoid. Put $G^{(0)} = X$. Let $\lambda = (\lambda^u)_{u \in X}$ be a Haar system on $G$. If $\mu$ is a measure on $X$, we define the measure $\mu \circ \lambda$ on $G$ by setting $\mu \circ \lambda(f) = \int_X (\lambda^u(f)) \, d\mu(u)$.

We define $\widetilde{\mu \circ \lambda}$ to be the image of $\mu \circ \lambda$ under the map $x \mapsto x^{-1}$. The measure $\mu$ is said to be quasi-invariant if $\mu \circ \lambda$ is equivalent to $\widetilde{\mu \circ \lambda}$.

A representation of $G$ is a triple $(\mu, \mathcal{H}, \theta)$ where:

- $\mu$ is a (class of) quasi-invariant measure on $X$;
- $\mathcal{H} = (H_u)_{u \in X}$ is a measurable field of Hilbert spaces;
- $\theta$ is a unitary representation of $G$ on $\mathcal{H}$:
  - For $x \in G$, $\theta(x)$ is a unitary operator from $H_{s(x)}$ to $H_{t(x)}$;
  - for (almost) all composable $(x, y)$ we have $\theta(xy) = \theta(x)\theta(y)$;
  - the field $\theta(x)_{x \in G}$ is measurable.
The $C^*$-algebra

Associated to a representation of $G$ is a representation of $C_c(G)$ on $\mathbb{H} = \int_X H_u \, d\mu(u)$ through a formula:

$$(\pi_\theta(f)(\xi))(u) = \int f(x)\delta(x)^{1/2}\theta(x)\xi(s(x)) \, d\lambda^u(x)$$

for $f \in C_c(G)$, $\xi \in \mathbb{H}$ and $u \in X$. Here $\delta$ is a Radon-Nykodym derivative.

**Definition**

- The full $C^*$-algebra $C^*(G)$ of $G$ is the completion with respect to the norm $f \mapsto \sup_\theta \|\pi_\theta(f)\|$.

- Natural representation $\rho_u$ on $L^2(G_u)$. The reduced $C^*$-algebra $C_r^*(G)$ of $G$ is the completion with respect to the norm $f \mapsto \sup_u \|\rho_u(f)\|$.
The $L^1$ estimate

In order to show that this supremum ($\sup_\theta \|\pi_\theta(f)\|$) is finite, we need an estimate.

For $f \in C_c(G)$ put

$$
\|f\|_1 = \sup_u \max \left\{ \int_{G^u} |f(x)| \, d\lambda_u(x), \int_{G^u} |f(x)| \, d\lambda_u(x) \right\}.
$$

Proposition

For every representation $\theta$ of the groupoid, $\|\pi_\theta(f)\| \leq \|f\|_1$.

Theorem (Renault)

Every representation of $C^*(G)$ is the integrated form of the representation of the groupoid.
3. Lie groupoids

**Definition**

We will say that $G$ is a Lie groupoid if $G$ and $M = G^{(0)}$ are manifolds, all the maps are smooth and $s, t$ are submersions.

Examples are provided by Lie groups acting smoothly on smooth manifolds.

Associated to a Lie groupoid is its *Lie algebroid*. It is given by a vector bundle - most naturally defined as the normal bundle to the inclusion $u : M \to G$, *i.e.* $u^* TG/du(TM)$.

There is a Lie algebra structure on the sections of this bundle:

- we replace the bundle $u^* TG/du(TM)$ by $(\ker ds)|_M$;
- we consider these sections as $G$ invariant sections of the bundle $\ker ds$.

Finally, there is a bundle map called *anchor* $\sharp : A \to TM$ defined by $dt : (\ker ds)|_M \to TM$ such that:

- $\sharp$ is a Lie algebra morphism $\sharp[a, b] = [\sharp a, \sharp b]$ if $a, b$ are sections of $A$
- $[a, fb] = f[a, b] + \sharp a(f)b$ if $a, b$ are sections of $A$ and $f \in C^\infty(M)$. 
A Lie algebroid is a bundle $A$ on a manifold $M$, with a bundle map $\# : A \to TM$ and a Lie bracket on the sections satisfying the above properties.

We say that the algebroid $A$ is *integrable* if it is the Lie algebroid associated with a Lie groupoid $G$.

The problem of integrability of Lie algebroids has a long history.

**Example**

Of a non integrable algebroid (Almeida, McKenzie).

Consider $SU(2) \times SU(2)$ acting on $S^2 \times S^2$. Transitive groupoid $G$ algebroid $A$ with isotropy $T^2$. Let $\mathbb{R}$ embed irrationally on $T^2$. Then $A/\mathbb{R}$ is a non integrable Lie algebroid.
Integrability of Lie algebroids

Crainic and Fernandes characterized integrable algebroids.

Only a few facts:

1. The obstruction is given a countable subgroup of the center of the Lie algebra \( \ker \sharp_x \) at each point. It is the image of map \( \pi_2(\text{Leaf}_x) \to Z(\ker \sharp_x) \).

   The algebroid is integrable if
   - this group is discrete.
   - It is uniformly discrete: Image of the nearby leaves.

2. If \( \sharp: A \to TM \) is almost injective, \( i.e. \) injective on a dense set, then \( A \) is integrable (Debord).
4. Pseudodifferential operators

Let $G$ be a Lie groupoid. The (Lie algebra of sections of the) Lie algebroid acts on $C_c^\infty(G)$ by unbounded multipliers. The algebra generated is the algebra of differential operators.

Using Fourier transform, one can write a differential operator $P$ (acting by left multiplication on $f$):

$$(Pf)(x) = \int_{x=yz} \exp(i\langle \phi(y), \xi \rangle) a(t(x), \xi) \chi(y)f(z) \, d\xi \, dy$$

Where

- $\varphi : U \simeq A$ is the phase: a local isomorphism defined on a neighborhood $U$ of $M$ in $G$.
- $\chi$ is the cut-off function: $\chi$ smooth; $\chi(y) = 1$ on $M$; $\chi(y) = 0$ for $u \notin U$.
- $a \in C^\infty(A^*)$ is a polynomial on $\xi$ where $A^*$ is the total space of the vector bundle dual to $A$. It is called the symbol of $P$. 

Order 0 pseudodifferential operators

We can then make sense of an expression like that for much more general symbols: the poly-homogeneous ones: \( a(u, \xi) \sim \sum_{k \in \mathbb{N}} a_{m-k}(u, \xi) \) where \( a_j \) is homogeneous of order \( j \) (outside a neighborhood of \( M \subset A^\ast \)).

\( m \) is called the order of \( a \) and the associated operator and \( a_m \) the principal symbol.

Proposition (Connes; Monthubert-Pierrot; Nistor-Weinstein-Xu)

- Negative order pseudodifferential operators define elements of \( C^\ast(G) \).
- Zero order pseudodifferential operators define multipliers of \( C^\ast(G) \).

Together with multiplicativity of the principal symbol, this gives an exact sequence of \( C^\ast \)-algebras:

\[
0 \rightarrow C^\ast(G) \rightarrow \Psi^\ast(G) \rightarrow C_0(SA^\ast) \rightarrow 0
\]

(the same with reduced).