Singular foliations

III. Holonomy groupoid of a singular foliation

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Summary

1. Lie groupoids
2. Foliations
3. Holonomy groupoid of a singular foliation
   1. Bi-submersions
   2. Bisections
   3. Equivalence of bi-submersions
   4. The holonomy groupoid
4. The $C^*$-algebra of a singular foliation
5. Pseudodifferential calculus and further topics
Introduction

How to define the holonomy groupoid? No transversals...

Germs of (local) diffeomorphisms preserving the foliation. Two problems.

1. Have to take a quotient: in the regular case, only the transverse action matters.
2. Topology?
1. Bi-submersions

Transverse maps and pull back foliations

Recall:

**Definition**

A *foliation* on $M$ is a locally finitely generated submodule of $\mathcal{C}_c^\infty(M; TM)$ stable under Lie brackets.

If $(M, \mathcal{F})$ is a foliation and $f : N \to M$ is a smooth map, the induced module $f^*(\mathcal{F}) = \mathcal{C}_c^\infty(N) \otimes \mathcal{C}_c^\infty(M) \mathcal{F}$ is a submodule of $\mathcal{C}_c^\infty(N; f^*(TM))$.

**Definition**

A smooth map $f : N \to M$ is said to be transverse to $\mathcal{F}$ if

$$f^*(\mathcal{F}) + (df)(\mathcal{C}_c^\infty(N; TN)) = \mathcal{C}_c^\infty(N; f^*(TM)).$$

A submersion is transverse!

If $f : N \to M$ is transverse, then pull back foliation on $N$:

$$f^{-1}(\mathcal{F}) = \{ X \in \mathcal{C}_c^\infty(N; TN); \ df(X) \in f^*(\mathcal{F}) \}$$
**Bi-submersions**

“Pieces of the holonomy groupoid.”

**Definition**

Let \((M, \mathcal{F})\) be a foliation. A *bi-submersion* of \((M, \mathcal{F})\) is a manifold \(U\) with two submersions \(s, t : U \to M\) such that each fiber of \(s\) maps through \(t\) into the same leaf in a submersive way.

Precise definition depends on the foliation (not only the partition into leaves).

**Definition**

Let \((M, \mathcal{F})\) be a foliation. A *bi-submersion* of \((M, \mathcal{F})\) is a manifold \(U\) with two submersions \(s, t : U \to M\) such that

\[
s^{-1}(\mathcal{F}) = t^{-1}(\mathcal{F}) = \ker dt + \ker ds.
\]
Examples

- Assume \( \mathcal{F} \) is the foliation associated with a Lie groupoid \( G \) (\( M = G^{(0)} \) the leaves are the orbits) and \( \mathcal{F} = \#(C^\infty_c(A)) \), where \( A \) is the algebroid.
  Then \( (G, t, s) \) is a bi-submersion of \( (M, \mathcal{F}) \).

1. If \((U, t, s)\) is a bi-submersion, \((U, s, t)\) is a bi-submersion the inverse bi-submersion.

2. If \((U_1, t_1, s_1)\) and \((U_2, t_2, s_2)\) are bi-submersions the fibered product \( U = U_1 \times_{s_1, t_2} U_2 \) with \( t_1 \) and \( s_2 \) gives rise to a bi-submersion the composition bi-submersion.

3. Bi-submersion representing the identity.
   Given a finite family \((\xi_1, \ldots, \xi_k)\) of vector fields generating the foliation near a point \( x_0 \), we form a bi-submersion \((U, t, s)\) where
   - \( U \) is an open neighborhood of \((0, x_0)\) in \( \mathbb{R}^k \times M \);
   - \( s(\lambda, x) = x \);
   - \( t(\lambda, x) = \exp_x \sum \lambda_i \xi_i \).
Remarks on the dimension

- The dimension of $U$ is not the same for all $U$.
- If $t(u) = x$, $\dim U \geq \dim M + \dim \mathcal{F}_x$.
- If we start with a minimal set of sections near $x_0$, we obtain a bi-submersion with this minimal dimension.
2. Bisections

How to compare bi-submersions?

Definition

Let \((U, t, s)\) be a bi-submersion. A bisection of \(U\) is a submanifold \(V \subset U\) such that the restrictions of both \(s\) and \(t\) on \(V\) are local diffeomorphisms.

Proposition

Let \((U, t, s)\) be a bi-submersion and \(u \in U\). There is a bisection \(V\) of \(U\) such that \(u \in V\).

Just take \(V\) to have a tangent space \(T_uV\) at \(u\) such that \(\ker ds \oplus T_uV = T_uU\) and \(\ker dt \oplus T_uV = T_uU\).
Bisections and local diffeomorphisms

If $V$ is a bisection, $t|_V \circ s|_V^{-1}$ is a local diffeomorphism which preserves the foliation.

**Definition**

Let $(U, t, s)$ be a bi-submersion $u \in U$ and $\varphi$ a local diffeomorphism of $M$. We say that $U$ carries $\varphi$ at $u$ if there is a bisection through $u$ whose associated diffeomorphism near $u$ is (the germ of) $\varphi$.

- The identity bi-submersion carries the identity local diffeomorphism.
- If $U$ carries $\varphi$, the inverse bi-submersion of $U$ carries $\varphi^{-1}$.
- If $U$ carries $\varphi$ and $V$ carries $\psi$, the composition $U \circ V$ carries $\varphi \circ \psi$.
- Let $\varphi$ be a local diffeomorphism preserving the foliation, $x_0 \in \text{dom}\varphi$ and $(U, t, s)$ an identity bi-submersion at $x$. Then $(U, \varphi \circ t, s)$ is a bi-submersion which carries $\varphi$. 

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3. Equivalence of bi-submersions

**Definition**

Let \((U, t_U, s_U)\) and \((V, t_V, s_V)\) be bi-submersions. Let \(u \in U\) and \(v \in V\). We say that the germs of \(U\) at \(u\) and of \(V\) at \(v\) are equivalent (write \(u \sim v\)) if there is a neighborhood \(U'\) of \(u\) in \(U\) and a smooth map \(f : U' \to V\) over \(s\) and \(t\) such that \(f(u) = v\).

**Theorem**

Let \((U, t_U, s_U)\) and \((V, t_V, s_V)\) be bi-submersions, \(u \in U\) and \(v \in V\).

- If the germs of \(U\) at \(u\) and of \(V\) at \(v\) are equivalent, then \(U\) carries at \(u\) the same bisections as \(V\) at \(v\).
- Conversely, if there is a bisection carried by \(U\) at \(u\) and by \(V\) at \(v\), then the germs of \(U\) at \(u\) and of \(V\) at \(v\) are equivalent.
4. The holonomy groupoid

Atlases

It follows from the above theorem that ‘equivalence’ is an equivalence relation.

The holonomy groupoid is the quotient of a ‘suitable’ family of bisubmersions by this equivalence relation.

Which bi-submersions to take?

**Definition**

An *atlas* is a family \((U_i, t_i, s_i)_{i \in I}\) of bi-submersions, ‘stable’ by composition and by inverse and such that \(\bigcup_i t_i(U_i) = M\).

More precisely, a family \((U_i, t_i, s_i)_{i \in I}\) is an atlas if the set of germs of local diffeomorphisms carried by all the \(U_i\)’s is a (sub) groupoid (of the groupoid of all germs of diffeomorphisms).
How to choose an atlas?

- If the foliation is given by a Lie groupoid $G$, $G$ itself is an atlas. And every open subgroupoid of $G$ is an atlas.

- We can compare the atlases: write $\mathcal{U} \sim \mathcal{V}$ if they carry the same (germs of) local diffeomorphisms. Define similarly $\mathcal{U} \prec \mathcal{V}$.

- There is a minimal choice of atlas: the path holonomy atlas, the one generated by small exponentiation of vector fields.

- Various other choices of atlases:
  - all bi-submersions,
  - bi-submersions which preserve the leaves...
Holonomy groupoid

**Definition**

- The holonomy groupoid $G_\mathcal{U}$ associated to an atlas $\mathcal{U} = (U_i, t_i, s_i)$ is the quotient of $\bigcup_i U_i$ by the relation $\sim$.
- The *holonomy groupoid* of the foliation is the one associated to the minimal atlas.

**Proposition**

- For any atlas $\mathcal{U}$, the set $G_\mathcal{U}$ is a groupoid with $G^{(0)} = M$.
- If $\mathcal{U}$ is the minimal atlas, the orbits are the leaves.
The holonomy groupoid is a Lie groupoid exactly in the almost regular case (Debord).

Topology: quotient topology.
Usually bad: the different fibers have different dimensions.

‘Often’ nice manifold structure on the fibers $G^x = t^{-1}(x)$. Dimension $\dim \mathcal{F}_x$.
Is $G^x$ always a manifold?