5. Pseudodifferential calculus and further topics (work in progress)

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Summary

1. Lie groupoids
2. Foliations
3. Holonomy groupoid of a singular foliation
4. The $C^*$-algebra of a singular foliation
5. Pseudodifferential calculus and further topics
   - 1. Pseudodifferential calculus
   - 2. Tangent groupoid
   - 3. Quotient Lie groupoids
   - 4. Lie algebroids
   - 5. Continuous family groupoids
Why did we spend so much energy to construct a $C^*$-algebra?

Take a compact manifold $M$. Let $X_1, \ldots X_n$ be vector fields on $M$ such that $[X_i, X_j] = \sum_k f_{i,j,k} X_k$.

- Show that $\Delta = \sum X_i^* X_i$ is essentially selfadjoint on $L^2(M)$; on $L^2(\text{Leaf})$.
- Where does the resolvant of $\Delta$ live?
- Where does the index of such an operator live?
1. Pseudodifferential calculus
The ‘cotangent bundle’; symbols

We fix a foliated manifold \((M, \mathcal{F})\). We also fix an atlas \(\mathcal{U}\).

The symbols should be (polyhomogeneous) functions defined on the cotangent bundle. What is this bundle?

Denote by \(F^*\) the union of \((\mathcal{F}_x)^*\): nice locally compact space.

Locally, \(\mathcal{F}\) (being finitely generated) is a quotient of a trivial bundle: \(\mathbb{R}^n/N_x\). Then \(F^*\) is a closed subspace of \((\mathbb{R}^n)^* \times M\).

A homogeneous symbol is then just the restriction to \(F^*\) of a symbol on \((\mathbb{R}^n)^* \times M\).
Pseudodifferential operators

The pseudodifferential operators should be smoothing outside the ‘units’.

So, we start with a bi-submersion \((U, t, s)\) in our atlas together with an identity bisection \(i.e.\) a submanifold \(V \subset U\) such that \(s, t\) coincide on \(V\) (and are local diffeomorphisms).

We further give:

- A local isomorphism \(\varphi : U \sim N\) where \(N\) is the normal bundle of the inclusion \(V \subset U\) - only defined near \(V\);
- A cut-off function \(\chi: \) smooth function on \(U\); \(\chi(u) = 1\) on \(V\) and \(\chi(u) = 0\) for \(u\) far from \(V\) (when \(\varphi\) is not defined);
- A symbol \(a\) on \(N^*\) (in the usual sense: polyhomogeneous).
Pseudodifferential operators (2)

We can then make sense of an expression like

\[ k_a(u) = \int a(s(u), \xi) \exp(i\phi(u)\xi) \chi(u) \, d\xi \]

**Theorem**

*In this way we get a multiplier of our \( \ast \)-algebra \( \mathcal{A}_U \).*

**Proposition**

*Modulo lower order, the operator \( k_a \) only depends on the restriction of \( a \) to \( F^* \).*

This means that if \( a \) is of order \( m \) and vanishes on \( F^* \), there exists \( b \) of order \( m - 1 \) such that \( k_b \) and \( k_a \) define the same multiplier.
Order zero pseudodifferential operators

We can then form the extension of order 0 pseudodifferential operators. We then prove the following facts

- If $a$ is of negative order, then $k_a \in C^*(M, \mathcal{F})$;
- if $a$ is of order 0, then $k_a$ is bounded (in every representation $\pi$); it is a multiplier of $C^*(M, \mathcal{F})$.
- If $a$ is of order 0, and its symbol vanishes outside $\mathcal{F}^*$, then $k_a \in C^*(M, \mathcal{F})$.

Take then the closure of the set of such operators: we will find an exact sequence of $C^*$-algebras:

$$0 \to C^*(G) \to \Psi^*(G) \to C_0(S\mathcal{F}^*) \to 0.$$
Elliptic operators

Proposition

Positive order elliptic operators define unbounded multipliers of the $C^*$-algebra.

Same proof as Vassout in the Lie groupoid case.

We may then deduce:

- In particular formally selfadjoint elliptic (pseudo)differential operators are essentially self adjoint in every representation: in particular in $L^2(M)$, in $L^2(Leaf)$.
- They have the same spectrum in every faithful representation.
- We thus have an analytic index map $K^*(F^*) \to K^*(C^*(M, \mathcal{F}))$.

Conclusion

Our $C^*$-algebra is suited for index problems of differential operators.
2. The ‘tangent groupoid’

The analogue of the ‘tangent groupoid’ of Alain Connes: foliation on \( \tilde{M} = M \times \mathbb{R} \) with leaves

- \((x, 0)\) for \(x \in M\);
- \(L \times \{\lambda\}\) for \(\lambda \in \mathbb{R}^*\), \(L\) leaf in \(M\).

More precisely, let \(\lambda : M \times \mathbb{R} \to \mathbb{R}\) be the second projection. Let \(\tilde{\mathcal{F}} \subset T(M \times \mathbb{R})\) be generated by \((\lambda X, 0), X \in \mathcal{F}\). It is a foliation.

- Its holonomy groupoid is a field \((\tilde{G}_\lambda)_{\lambda \in \mathbb{R}}\) of groupoids.
- For \(\lambda \neq 0\), \(\tilde{G}_\lambda = G\) is the holonomy groupoid of \((M, \mathcal{F})\).
- \(\tilde{G}_0\) is the field \((\mathcal{F}_x)_{x \in M}\).

Its \(C^*\)-algebra is isomorphic (via Fourier) to \(C_0(F^*)\).
Tangent groupoid (2)

As in the Lie groupoid case, restricting \( \tilde{G} \) to \([0, 1]\), we get an exact sequence:

\[
0 \to C^*(M, \mathcal{F}) \otimes C_0((0, 1]) \to C^*(\tilde{G}_{[0,1]}) \xrightarrow{ev_0} C_0(F^*) \to 0
\]

whence a morphism \([ev_1] \circ [ev_0]^{-1} : K_0(C_0(F^*)) \to K_0(C^*(M, \mathcal{F}))\).

Proposition (cf. Monthubert-Pierrot)

This morphism coincides analytic index of elliptic pseudodifferential operators.
3. Quotient Lie groupoids

One could (should...) systematize our construction of the holonomy groupoid - and its $C^*$-algebra.

A quotient Lie groupoid is a groupoid which is a ‘suitable quotient’ of a manifold. The composition is defined and smooth ‘upstairs’; algebraic properties: associativity, units, inverse, ‘downstairs’.

More precisely, we have:

- a manifold $\mathcal{W}$ (not necessarily connected) with a pair of submersions $s, t : \mathcal{W} \to M$.
- a ‘nice’ equivalence relation $\sim$ on $\mathcal{W}$ which respects both $s$ and $t$: (if $x \sim y$, then $s(x) = s(y)$ and $t(x) = t(y)$).
Quotient groupoids (2)

- A composition $W^{(2)} = \{(x, y) \in W \times W; \ s(x) = t(y)\} \rightarrow W$ (which may as well be defined only locally) satisfying
  - $s(xy) = s(y)$ and $t(xy) = t(x)$;
  - compatible with $\sim$: if $x \sim x'$ and $y \sim y'$, then $xy \sim x'y'$.
  - Associative modulo $\sim$: $(xy)z \sim x(yz)$.
  - Units and inverses modulo $\sim$

We can then form as above:

- The $\ast$-algebra - as a quotient of $C^\infty_c(W, \Omega^{1/2})$ using half densities on $\ker s \times \ker t$.
- The representations of $G + \text{an } L^1$-estimate, whence $C^*$-algebra.
- A pseudodifferential calculus, an index map, a tangent groupoid...
- Reduced $C^*$-algebra ?
Quotient groupoids (3)

Question
What do we mean by ‘suitable’ equivalence relations?

- They should be given by families of pairs of submersions $U \xrightarrow{p,q} W$ - with $s \circ p = s \circ q$ and $t \circ p = t \circ q$. We then have $p(u) \sim q(u)$.

- All this works if $\sim$ is given by partial diffeomorphisms.

Proposition

*Every algebroid is integrable!*
4. Lie algebroids

One should be able to do something better...

Let us examine the example of non integrable Lie groupoid we met.

- The groupoid we get is $(S^2 \times S^2) \rtimes (SU(2) \times SU(2))/\mathbb{R}$.
- It is Morita equivalent to the group $\mathbb{R}/(\mathbb{Z} + \alpha\mathbb{Z})$ where $\alpha$ is an irrational number.
- The $C^*$-algebra we obtain is just $\mathcal{K}$...

**Question**

Is there a better way to define the $C^*$-algebra of such a quotient group?

It should be $K$-equivalent to $C^*(\mathbb{T}^2)$, with infinite dimensional $K$-groups.
5. Continuous family groupoids

A. Paterson:
Locally compact groupoids which are smooth only in the ‘group direction’: the fibers of $s$ and $t$ are manifolds.

One then forms:
- pseudodifferential calculus;
- analytic index;
- tangent groupoid...

Examples of such groupoids
- Lie group actions on locally compact spaces;
- Closed saturated subsets in Lie groupoids.
Continuous family groupoids (2)

Question
Find a framework which generalizes both Paterson’s setting and ours.

1. $C_{\infty,0}$-foliations on locally compact spaces. One can define them in terms of bi-submersions. One can then again construct:
   - their groupoid;
   - their $C^*$-algebra;
   - a pseudo differential calculus;
   - an analytic index;
   - a tangent groupoid...

2. Quotient continuous family groupoids
And, this is the right place to stop...

Thank you for your patience.

Thank you Claire and sorry for all the trouble