Rigidity results for $\text{II}_1$ factors and group actions

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**Introduction**

In recent years, Sorin Popa has obtained far reaching rigidity results for Bernoulli actions of groups with Kazhdan’s property (T). The Bernoulli action of $\Gamma$ is defined as the shift action on the infinite product probability space $(X_0, \mu_0)^\Gamma$.

Popa proved that the equivalence relation given by the orbits of a Bernoulli action of a Kazhdan group, entirely remembers the group and the action. In fact, these actions are even essentially remembered by their group measure space $\text{II}_1$ factor. This was the first occurrence in the literature of a theorem deducing conjugacy of actions from mere isomorphism of the associated von Neumann algebras.

**Summary**

The major topics of the lecture course will be the following.

- Introduce enough background material from the theory of $\text{II}_1$ factors, including the definition of a $\text{II}_1$ factor and the group measure space construction of Murray and von Neumann (also called crossed product construction). There will be a special emphasis on bimodules (Connes’ correspondences), providing a crucial technique in the study of $\text{II}_1$ factors.

- Introduce enough background material about groups and their actions on probability spaces, including (relative) property (T) and the notion of a (weakly) mixing action.

- Prove Popa’s cocycle superrigidity theorem for Bernoulli actions of Kazhdan groups. As a corollary, we will see that their orbit equivalence relation remembers the group action.

- Prove a sample von Neumann algebra rigidity theorem. We will not prove the strongest available theorems. Nevertheless, the presented proof will be sufficiently general to classify, up to isomorphism of $\text{II}_1$ factors, a large family of (generalized) Bernoulli action crossed products. It also provides a fairly simple 1-parameter family of $\text{II}_1$ factors without outer automorphisms.

**Prerequisites**

I hope that the following list of prerequisites gives a rough impression of the level at which the course will start.

- The very basics of von Neumann algebras, including von Neumann’s bicommutant theorem.
- Weak and strong topology on a von Neumann algebra.
- The notion of a (tracial) state and the corresponding GNS-construction $L^2(M, \tau)$.
- The infinite product $\prod_f (X_0, \mu_0)$ of probability spaces.