Rigidity results for $\text{II}_1$ factors and group actions

Lecture 1

Meeting of the GDR Non Commutative Geometry

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Measure preserving actions of countable groups on probability spaces.

\[ \Gamma \sim (X, \mu) \]

Equivalence relation \( R \) on \( X \) given by \( \Gamma \)-orbits:

\[ (x, y) \in R \iff x \text{ and } y \text{ have the same orbit.} \]

Von Neumann algebra \( L^\infty(X) \rtimes \Gamma \).

Central question: how much does the equivalence relation/von Neumann algebra remember of the group action?
The aim is not to give an overview of the many available results. An a priori excuse towards the experts for not mentioning many important developments, earlier results, more recent results,...

But the aim is to concentrate on a small personal selection of theorems and to present their proofs.
1. The actors enter the scene:
   - Examples of actions.
   - Different families of groups.

2. $\mathbb{Z}^2 < \text{SL}(2, \mathbb{Z}) \rtimes \mathbb{Z}^2$ has the relative property (T).

3. Orbit equivalence of actions and relation with 1-cocycles.

Examples of actions

- \( \mathbb{Z} \curvearrowright \mathbb{T} \) by irrational rotation.
- \( \text{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{R}^n/\mathbb{Z}^n \cong \mathbb{T}^n \).

- **Generalized Bernoulli action**:
  Suppose \( \Gamma \curvearrowright I \) and take \( \Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^I \).

- **Profinite actions** of a residually finite group:
  Take a decreasing sequence \( \Gamma_n \) of finite index subgroups of \( \Gamma \) with \( \bigcap_n \Gamma_n = \{e\} \). Set
  \[
  (X, \mu) \to \cdots \to \Gamma/\Gamma_{n+1} \to \Gamma/\Gamma_n \to \cdots \to \Gamma/\Gamma_1.
  \]

**A remark about probability spaces**

‘All’ non-atomic probability spaces are isomorphic with \([0, 1]\) and the Lebesgue measure.

**Convention**: \( \Gamma \curvearrowright (X, \mu) \) always denotes a probability measure preserving action of a countable group.
Basic properties of actions

Let $\Gamma \curvearrowright (X, \mu)$. The action is said to be

- **essentially free** if $\text{Stab}_x = \{e\}$ for almost all $x \in X$,
- **ergodic** if every globally $\Gamma$-invariant $U \subset X$ satisfies $\mu(U) = 0$ or $1$.

### Examples

- $\text{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{R}^n/\mathbb{Z}^n \cong \mathbb{T}^n$ is essentially free and ergodic.
- Let $\Gamma \curvearrowright I$ with infinite orbits and such that $g \in \Gamma \setminus \{e\}$ moves infinitely many $i \in I$.
  Then, $\Gamma \curvearrowright (X_0, \mu_0)^I$ is essentially free and ergodic.

### Exercise 1

Prove the ergodicity of $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ by proving the following stronger property, called (strong) mixing:

$$\lim_{g \to \infty} \mu((U \cdot g) \cap V) = \mu(U) \mu(V) \quad \text{for all } U, V \subset X.$$

**Attention:** the general case $\Gamma \curvearrowright I$ is slightly more subtle.
Families of groups

We shall distinguish several families of groups by looking at their representation theory.

**Definition**

A representation of a group $\Gamma$ is a homomorphism $\pi : \Gamma \to \mathcal{U}(H)$.

(For us, all representations are unitary.)

**Group representations**

- **Example:** the regular representation $\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$:

  $$(\lambda_g \xi)(h) = \xi(g^{-1}h) \quad \text{or equivalently} \quad \lambda_ge_h = e_{gh}.$$  

- **An invariant vector** is a vector $\xi$ with $\pi(g)\xi = \xi$ for all $g \in \Gamma$.

- **A sequence of almost invariant vectors** is a sequence $(\xi_n)$ in $H$ satisfying

  $$\|\xi_n\| = 1 \quad \text{and} \quad \|\pi(g)\xi_n - \xi_n\| \to 0 \quad \text{for all} \quad g \in \Gamma.$$
Families of groups

- Finite groups, abelian groups, solvable groups.
- **Γ is amenable** if the regular representation admits a sequence of almost invariant vectors.
  - All abelian groups are amenable.
  - Stable under extensions.
- **Γ has Kazhdan’s property (T)** if every representation \( \pi \) that admits a sequence of almost invariant vectors, actually has a non-zero invariant vector.
- The pair \( \Lambda < \Gamma \) has the **relative property (T)** if every representation \( \pi \) of \( \Gamma \) that admits a sequence of almost invariant vectors, actually has a non-zero \( \Lambda \)-invariant vector.

**Exercise 2** If \( \Lambda < \Gamma \) has the relative property (T) with \( \Lambda \) being infinite, then \( \Gamma \) is non-amenable.

**Exercise 3** If \( \Gamma \) has property (T), so does every quotient of \( \Gamma \).
Examples

- The pair $\mathbb{Z}^2 < SL(2, \mathbb{Z}) \rtimes \mathbb{Z}^2$ has the relative property (T).
- The groups $SL(n, \mathbb{Z})$ have property (T) for $n \geq 3$.
- The groups $SL(n, \mathbb{Z}) \rtimes \mathbb{Z}^n$ have property (T) for $n \geq 3$.
- The groups $Sp(2n, \mathbb{Z})$ have property (T) for $n \geq 2$.
- Lattices in ‘higher rank simple Lie groups’ have property (T).
- (Shalom, 2006) The groups $SL(n, \mathbb{Z}[X_1, \ldots, X_k])$ have property (T) for $n \geq \max\{3, k + 2\}$ (but this is not an optimal bound in general).
Relative property (T) for $\mathbb{Z}^2 < \text{SL}(2, \mathbb{Z}) \rtimes \mathbb{Z}^2$

Let $\pi : \text{SL}(2, \mathbb{Z}) \rtimes \mathbb{Z}^2 \to \mathcal{U}(H)$ be a unitary representation.

**Tool**: spectral decomposition of $\pi$ restricted to $\mathbb{Z}^2$.

- Duality of $\mathbb{T}^2$ and $\mathbb{Z}^2$ by $\langle (\alpha \beta), (\chi \gamma) \rangle = \alpha^x \beta^y \in \mathbb{T}$.
- For every unit vector $\xi \in H$, a probability measure $\mu_\xi$ on $\mathbb{T}^2$.
- $\langle \pi(\overrightarrow{x})\xi, \xi \rangle = \int_{\mathbb{T}^2} \langle \overrightarrow{\alpha}, \overrightarrow{x} \rangle \, d\mu_\xi(\overrightarrow{\alpha})$
- If $P$ is the orthogonal projection of $H$ onto the $\mathbb{Z}^2$-invariant vectors, then $\langle P\xi, \xi \rangle = \mu_\xi\{(1 \, 1)\}$
- For $A \in \text{SL}(2, \mathbb{Z}) \sim \mathbb{T}^2$, one has $\mu_\xi \circ A = \mu_{\pi(A^t)\xi}$. 
Relative property (T) for $\mathbb{Z}^2 < \text{SL}(2, \mathbb{Z}) \rtimes \mathbb{Z}^2$

Let now $(\xi_n)$ be a sequence of almost invariant unit vectors.

Denote $K = [-\pi, \pi) \times [-\pi, \pi)$ and identify $K = \mathbb{T}^2$.

Probability measures $\mu_\xi$ on $K$ such that

- $\langle \pi\left(\begin{smallmatrix} \hat{x} \\ \hat{y} \end{smallmatrix}\right) \xi, \xi \rangle = \int_K \exp(i(x\alpha + y\beta)) \, d\mu_\xi(\alpha, \beta)$

- $\mu_\xi \circ A = \mu_{\pi(A^t)}\xi$ where $A$ acts on $K$ under the ident. $K = \mathbb{T}^2$, implying that $|\mu_{\xi_n}(A(V)) - \mu_{\xi_n}(V)| \to 0$ for all $V \subset K$.

To prove: $\mu_\xi\{(0,0)\} \neq 0$ for some $\xi$.

In fact, we prove: $\mu_{\xi_n}\{(0,0)\} \to 1$.

$$\int_K \cos(\alpha) \, d\mu_{\xi_n}(\alpha, \beta) = \text{Re}\langle \pi\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \xi_n, \xi_n \rangle \to 1 \quad \text{and same with } \cos(\beta)$$

Conclusion: $\mu_{\xi_n}(U) \to 1$ for every neighborhood $U$ of $0 \in K$.

But then: $|\mu_{\xi_n}(A(V)) - \mu_{\xi_n}(V)| \to 0$ for all $V \subset \mathbb{R}^2$ and the linear action of $A \sim \mathbb{R}^2$. 
Relative property (T) for $\mathbb{Z}^2 < \text{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$

Consider the partition $\mathbb{R}^2 \setminus \{(0, 0)\} = \mathcal{V}_1 \sqcup \mathcal{V}_2 \sqcup \mathcal{V}_3 \sqcup \mathcal{V}_4$.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$A(\mathcal{V}_1 \sqcup \mathcal{V}_2) = \mathcal{V}_1$$

But, 

$$\mu_{\xi_n}(A(\mathcal{V})) \approx \mu_{\xi_n}(\mathcal{V})$$

Hence, $\mu_{\xi_n}(\mathcal{V}_2) \to 0$.

Similarly, $\mu_{\xi_n}(\mathcal{V}_i) \to 0$ for all $i = 1, 2, 3, 4$.

Hence, $\mu_{\xi_n}\{(0, 0)\} \to 1$. QED
Orbit equivalence

**Definition**

The actions $\Gamma \sim (X, \mu)$ and $\Lambda \sim (Y, \eta)$ are called

- **orbit equivalent** if there exists a (measure preserving) isomorphism $\Delta : X \to Y$ such that $\Delta(x \cdot \Gamma) = \Delta(x) \cdot \Lambda$ for almost all $x \in X$.

- **conjugate** if there exists an iso. $\Delta : X \to Y$ and an iso. $\delta : \Gamma \to \Lambda$ such that $\Delta(x \cdot g) = \Delta(x) \cdot \delta(g)$ for almost all $x \in X$ and $g \in \Gamma$.

**Exercise 4**  Essentially free actions of finite groups $\Gamma, \Lambda$ are orbit equivalent iff $|\Gamma| = |\Lambda|$. (Hint : fundamental domain.)

**Theorem (Ornstein, Weiss, 1980)**

All essentially free and ergodic actions of all infinite amenable groups are orbit equivalent.

Radically different results in the non-amenable case.
Popa’s orbit equivalence superrigidity theorem

**Theorem (Popa, 2005)**

Let $\Gamma$ be a group with infinite normal subgroup $H \triangleleft \Gamma$ having the relative property (T).

Let $\Gamma \curvearrowright I$ such that $H$ acts with infinite orbits. Consider $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^I$.

Every essentially free ergodic action that is orbit equivalent with $\Gamma \curvearrowright (X, \mu)$, is actually conjugate to $\Gamma \curvearrowright (X, \mu)$.

In other words : the orbit equivalence relation of $\Gamma \curvearrowright (X, \mu)$ entirely remembers the group and the action.

**w-rigid group** : having an infinite normal subgroup with relative (T).

Many groups are w-rigid : $\text{SL}(2, \mathbb{Z}) \rtimes \mathbb{Z}^2$, property (T) groups, direct products with arbitrary groups.
Orbit equivalence and 1-cocycles

Let $\Gamma \sim (X, \mu)$ be a probability measure preserving action.

**Polish group**: topological group with separable, complete metric inducing the topology.

Countable groups, compact second countable groups, $\mathcal{U}(H)$.

**Definition**

A **1-cocycle** for $\Gamma \sim X$ with values in a Polish group $G$, is a measurable map

$$\omega : X \times \Gamma \to G$$

satisfying

$$\omega(x, gh) = \omega(x, g) \omega(x \cdot g, h)$$

‘Homomorphism’ from the transformation groupoid to $G$.

**Cohomologous cocycles**:

$\omega_1, \omega_2$ are called cohomologous if $\exists \varphi : X \to G$ such that

$$\omega_1(x, g) = \varphi(x) \omega_2(x, g) \varphi(x \cdot g)^{-1}.$$ 

Homomorphisms $\Gamma \to G$: cocycles not depending on $x \in X$.  

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Orbit equivalence and 1-cocycles

For the rest of this lecture: all actions essentially free and ergodic.

From orbit equivalence to a 1-cocycle (Zimmer)

Let $\Delta : X \to Y$ be an orbit equivalence of $\Gamma \bowtie (X, \mu)$ and $\Lambda \bowtie (Y, \eta)$.

Define $\omega : X \times \Gamma \to \Lambda$ by $\Delta(x \cdot g) = \Delta(x) \cdot \omega(x, g)$.

Then, $\omega$ is a 1-cocycle. **Exercise 5** Check this.

Proposition

Suppose that $\omega$ is cohomologous to a homomorphism $\delta : \Gamma \to \Lambda$.

Suppose moreover that $\Gamma$ has no finite normal subgroups.

Then, $\delta$ is an isomorphism and the actions $\Gamma \bowtie (X, \mu)$ and $\Lambda \bowtie (Y, \eta)$ are conjugate.

So: solving the cocycle = turning the orbit equiv. into a conjugacy.

We now prove this proposition.
Solving cocycle implies conjugacy of the actions

**Data:** Actions $\Gamma \curvearrowleft (X, \mu)$ and $\Lambda \curvearrowleft (Y, \eta)$, orbit equivalence $\Delta : X \to Y$, 1-cocycle $\omega : X \times \Gamma \to \Lambda : \Delta(x \cdot g) = \Delta(x) \cdot \omega(x, g)$.

- Suppose that $\delta : \Gamma \to \Lambda$ homomorphism and $\varphi : X \to \Lambda$ measurable such that $\delta(g) = \varphi(x)^{-1} \omega(x, g) \varphi(x \cdot g)$.
- Write $\tilde{\Delta} : X \to Y : \tilde{\Delta}(x) = \Delta(x) \cdot \varphi(x)$. So, $\tilde{\Delta}(x \cdot g) = \tilde{\Delta}(x) \cdot \delta(g)$.

**Problem:** is $\tilde{\Delta}$ essentially bijective?

**But:** $X = \bigcup_n X_n$ with $\tilde{\Delta}|_{X_n}$ injective, measure preserving.

- **Claim:** Ker $\delta$ is finite and hence trivial.
  Indeed, take $U \subset X$ non-negligible and $\tilde{\Delta}|_U$ injective. Then, the $U \cdot g$, with $g \in \text{Ker} \delta$, are essentially disjoint.

- Suppose that $\tilde{\Delta}$ is not essentially injective. Find $g \neq e$ and $U \subset X$ with $\tilde{\Delta}(x \cdot g) = \tilde{\Delta}(x)$ for all $x \in U$. Derive contradiction.

QED
Prove Popa’s cocycle superrigidity theorem:

For good generalized Bernoulli actions $\Gamma \curvearrowright (X_0, \mu_0)^I$ of a $\omega$-rigid group $\Gamma$, every 1-cocycle with values in a countable group is cohomologous to a homomorphism.

Implies the orbit equivalence superrigidity theorem stated before: the orbit equivalence relation entirely remembers the group and the action.

Cocycle superrigidity theorem holds true for more general ‘target groups’.