Rigidity results for II$_1$ factors and group actions

Lecture 2

Meeting of the GDR Non Commutative Geometry

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Formulate and prove Popa’s cocycle superrigidity theorem.

**Ingredients:**

- Von Neumann algebras and $\text{II}_1$ factors: an introduction.
- Weakly mixing actions.
- Popa’s notion of malleability. (Deformation property)
- The relative property $(T)$. (Rigidity property)

Popa’s deformation/rigidity technique.
A von Neumann algebra is a weakly closed unital $\ast$-subalg. of $B(H)$.

Examples:

- $B(H)$ itself, in particular $M_n(\mathbb{C})$.
- $L^\infty(X, \mu)$ (acting on $L^2(X, \mu)$).
- Let $\Gamma$ be a countable group. Consider again the left regular rep. $\lambda : \Gamma \to U(\ell^2(\Gamma)) : (\lambda_g \xi)(h) = \xi(g^{-1}h)$.

$$\mathcal{L}(\Gamma) = \text{weakly closed linear span of } \{\lambda_g \mid g \in \Gamma\}$$

$$= \text{weak closure of } \mathbb{C}\Gamma \text{ represented in } B(\ell^2(\Gamma))$$

is the group von Neumann algebra.
Factors and their types

Definition

A factor is a von Neumann algebra with trivial center.

- $\mathcal{B}(H)$ is a factor.
- $\mathcal{L}(\Gamma)$ is a factor iff $\Gamma$ has infinite conjugacy classes.

Exercise 6 Prove these statements.

Indications for the second statement:

- $\mathcal{L}(\Gamma) \to \ell^2(\Gamma) : x \mapsto x\delta_e$ is injective. Use right translation unitaries, which commute with operators in $\mathcal{L}(\Gamma)$.
- If $z \in \mathcal{Z}(\mathcal{L}(\Gamma))$ and $\xi = z\delta_e$, then $\xi(ghg^{-1}) = \xi(h)$ for all $g, h \in \Gamma$. 
Factors and their types

Definition

A factor is a von Neumann algebra with trivial center.

- $B(H)$ is a factor.
- $L(\Gamma)$ is a factor iff $\Gamma$ has infinite conjugacy classes.

The formula $\tau(x) = \langle \delta_e, x\delta_e \rangle$ defines a tracial state on $L(\Gamma)$.

Murray - von Neumann types of factors

Let $M$ be a factor. Then $M$ is said to be

- type I, if $M \cong M_n(\mathbb{C})$ or $B(H)$,
- type $\text{II}_1$, if $M$ admits a tracial state and $M \not\cong M_n(\mathbb{C})$,
- type $\text{II}_\infty$, if $M$ admits ‘an infinite trace’ and $M \not\cong B(H)$,
- type III, in all the other cases.

For every ICC group $\Gamma$, we get the $\text{II}_1$ factor $L(\Gamma)$. 
The theory of II$_1$ factors is quite hard

Very difficult to decide if two II$_1$ factors are isomorphic.

- Murray-von Neumann, 1943.
  \( \mathcal{L}(S_\infty) \) and \( \mathcal{L}(\mathbb{F}_2) \) are non-isomorphic.

- McDuff, 1969.
  Continuum of non-isomorphic II$_1$ factors.

- Connes, 1976.
  All \( \mathcal{L}(G) \) for \( G \) amenable ICC are isomorphic and yield the so-called hyperfinite II$_1$ factor.

- Open problem. \( \mathcal{L}(\mathbb{F}_n) \cong \mathcal{L}(\mathbb{F}_m) \) for \( n \neq m \) ?

- Conjecture (Connes).
  If \( \Gamma \) is an ICC property (T) group and \( \mathcal{L}(\Gamma) \cong \mathcal{L}(\Lambda) \), then \( \Gamma \cong \Lambda \).
The bimodule $L^2(M)$

Let $(M, \tau)$ be a tracial von Neumann algebra.

- Hilbert space $L^2(M)$ by completing $M$ for the scalar product $\langle x, y \rangle = \tau(x^* y)$, which defines the $L^2$-norm $\|x\|_2 = \sqrt{\tau(x^* x)}$.
- We sometimes write $\hat{x}$ to distinguish $x \in M$ and $\hat{x} \in L^2(M)$.
- $L^2(M)$ is an $M$-$M$-bimodule: $x \cdot \hat{y} \cdot z = \hat{xyz}$

Concrete interpretations:

- $M = L^\infty(X)$ yields $L^2(X)$.
- $M = L(\Gamma)$ yields $l^2(\Gamma)$ and bimodule $\lambda_g \cdot e_h \cdot \lambda_k = e_{ghk}$.

Polar decomposition:

- Define $L^2(M)^+$ as the closure of $M^+$ inside $L^2(M)$.
- Every $\xi \in L^2(M)$ has a canonical polar decomposition $\xi = v|\xi|$ where $v \in M$ is a partial isometry and $|\xi| \in L^2(M)^+$.
- If $u_1, u_2 \in U(M)$ and $u_1 \xi = \xi u_2$, then $u_1 v = v u_2$. 
Weakly mixing actions

**Definition**

The action $\Gamma \curvearrowright (X, \mu)$ is called 

- **strongly mixing**, if $\lim_{g \to \infty} \mu(U \cdot g \cap V) = \mu(U) \mu(V)$ for all $U, V \subset X$.

- **weakly mixing**, if there exists a sequence $(g_n)$ in $\Gamma$ such that $\lim_{n \to \infty} \mu(U \cdot g_n \cap V) = \mu(U) \mu(V)$ for all $U, V$.

Consider $\Gamma \curvearrowright (X_0, \mu_0)^I$. Then, 

- strongly mixing iff Stab $i$ is finite for all $i \in I$,
- weakly mixing iff every $i \in I$ has an infinite orbit.

Profinite actions are the typical actions that are not weakly mixing.
Properties of weakly mixing actions

**Exercise 7** The action $\Gamma \acts (X, \mu)$ is ergodic iff for all non-negligible $U, V \subset X$, there exists $g \in \Gamma$ such that $U \cdot g \cap V$ is non-negligible.

Let $\Gamma \acts (X, \mu)$ be a probability measure preserving action. Then the following statements are equivalent.

1. $\Gamma \acts (X, \mu)$ is weakly mixing: $\exists g_n, \mu(U \cdot g_n \cap V) \to \mu(U) \mu(V)$.

2. For all $U_1, \ldots, U_n \subset X$ non-negligible, there exists $g \in \Gamma$ such that $U_i \cdot g \cap U_j$ is non-negligible for all $i, j$.

3. If $\Gamma \acts (Y, \eta)$ is ergodic, the diagonal action $\Gamma \acts X \times Y$ is ergodic.

4. The diagonal action $\Gamma \acts X \times X$ is ergodic.

The non-trivial implications are $2 \Rightarrow 3$ and $4 \Rightarrow 1$. 
Proof of 2 ⇒ 3

**Given**: an action $\Gamma \triangleleft (X, \mu)$ such that for all $U_1, \ldots, U_n \subset X$ non-negligible, $\exists g$ with $U_i \cdot g \cap U_j$ non-negligible for all $i, j$.

**To prove**: $\Gamma \triangleleft X \times Y$ is ergodic whenever $\Gamma \triangleleft Y$ is ergodic.

Let $U \subset X \times Y$ be $\Gamma$-invariant.

- Let $\Sigma$ be the Polish space of measurable subsets $\mathcal{W} \subset Y$ with $d(\mathcal{V}, \mathcal{W}) = \|X_\mathcal{V} - X_\mathcal{W}\|_2$.
- Define $F : X \to \Sigma : F(x) = U_x \ (= \{y \in Y \mid (x, y) \in U\})$. Then, $F(x \cdot g) = F(x) \cdot g$.
- Let $\mathcal{V} \in \Sigma$ be an essential value of $F$.
- **Claim.** $\mathcal{V}$ is $\Gamma$-invariant. Indeed, $\forall \varepsilon > 0$, we find a non-negligible $\mathcal{O} \subset X$ such that $d(F(x), \mathcal{V}) < \varepsilon$ for all $x \in \mathcal{O}$. Then, $d(\mathcal{V} \cdot g, \mathcal{V}) < 2\varepsilon$ whenever $\mathcal{O} \cap \mathcal{O} \cdot g$ non-negligible. So, $d(\mathcal{V} \cdot h, \mathcal{V}) < 4\varepsilon$ if $\exists g$ with $(\mathcal{O} \cdot h) \cdot g \cap \mathcal{O}$ and $\mathcal{O} \cdot g \cap \mathcal{O}$ n-n.
- Thus, $F$ only takes values $\emptyset$ and $Y$. By ergodicity, we are done.
Proof of $4 \Rightarrow 1$

**Given:** an action $\Gamma \acts (X, \mu)$ with $\Gamma \acts X \times X$ ergodic.

**To prove:** $\exists g_n$ such that $\mu(U \cdot g_n \cap V) \to \mu(U) \mu(V), \ \forall U, V \subset X$.

Consider trace $\tau$ on $L^\infty(X)$ given by $\mu$ and action $\sigma_g$ on $L^\infty(X)$.

**Suppose the contrary.** Take $\varepsilon > 0$ and $a_1, \ldots, a_n \in L^\infty(X)$ such that $\tau(a_i) = 0$ and $\sum_{i,j=1}^{n} |\tau(a_i \sigma_g(a_j)^*)|^2 \geq \varepsilon$ for all $g \in \Gamma$.

- Write $\xi := \sum_{i=1}^{n} a_i \otimes a_i^*$ where we view $\xi \in L^2(X \times X)$. Then,
  \[ \langle \xi, (\sigma_g \otimes \sigma_g)(\xi) \rangle \geq \varepsilon \quad \text{for all } g \in \Gamma, \quad \langle 1 \otimes 1, \xi \rangle = 0 \]
- Let $\eta \in L^2(X \times X)$ be the unique element of minimal norm in
  \[ \overline{\text{conv}}\{(\sigma_g \otimes \sigma_g)(\xi) \mid g \in \Gamma\} \subset L^2(X \times X) \]
- But then, $\eta \neq 0$, $\eta$ is invariant under the diagonal action and $\langle 1 \otimes 1, \eta \rangle = 0$. Contradiction with the ergodicity of $\Gamma \acts X \times X$. 

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Popa’s malleability of Bernoulli-like actions

Definition (Popa, 2005)

The action $\Gamma \curvearrowright (X, \mu)$ is called $s$-malleable if there exist a flow $(\alpha_t)_{t \in \mathbb{R}}$ and an involutive automorphism $\beta$ of $X \times X$ satisfying

- $\alpha_t$ and $\beta$ commute with the diagonal $\Gamma$-action,
- $\alpha_1(x, y) = (y, \ldots)$
- $\beta \alpha_t \beta = \alpha_{-t}$ and $\beta(x, y) = (x, \ldots)$

The actions $\Gamma \curvearrowright [0, 1]^I$ are $s$-malleable.

- It suffices to construct such a flow $\alpha_t^0$ and $\beta^0$ on $[0, 1] \times [0, 1]$, and then take the diagonal product to obtain $\alpha_t$ and $\beta$ on $[0, 1]^I \times [0, 1]^I$, which automatically commutes with the $\Gamma$-action.
Popa’s cocycle superrigidity theorem

**Polish group of finite type**: closed subgr. of $\mathcal{U}(M)$ for tracial $(M, \tau)$.

**Theorem (Popa, 2005)**

Let $\Gamma \curvearrowright I$ and let $G$ be a Polish group of finite type.

If $\Lambda \lhd \Gamma$ has the relative property (T) and acts with infinite orbits on $I$, then every 1-cocycle for $\Gamma \curvearrowright (X_0, \mu_0)^I$ with values in $G$ is cohomologous to a homomorphism $\Gamma \to V$.

\[ \omega : X \times \Gamma \to G : \omega(x, gh) = \omega(x, g) \omega(x \cdot g, h) \]

$\omega_1 \sim \omega_2$ if $\exists \varphi : X \to G$, $\omega_1(x, g) = \varphi(x) \omega_2(x, g) \varphi(x \cdot g)^{-1}$.

\[ \sim \]

Implies Popa’s orbit equivalence superrigidity theorem.

We prove this theorem. We shall make the extra assumptions that

- $\Gamma$ has the property (T),
- $\mu_0$ is non-atomic, so that $\Gamma \curvearrowright (X_0, \mu_0)^I$ is really s-malleable.
Proof of cocycle superrigidity

We have $\Gamma \ltimes (X, \mu)$ s-malleable and weakly mixing.

Let $\mathcal{G} \subset \mathcal{U}(M)$ a closed subgroup and $\omega : X \times \Gamma \to \mathcal{G}$ a 1-cocycle.

Flow $(\alpha_t)_{t \in \mathbb{R}}$ on $X \times X$, commuting with $\Gamma$ and $\alpha_1(x, y) = (y, \cdots)$.

- Consider the diagonal action $\Gamma \ltimes Z = X \times X$ and the 1-cocycles for $\Gamma \ltimes Z$,

  \[ \omega_0 : Z \times \Gamma \to \mathcal{G} : \omega_0(x, y, g) = \omega(x, g) , \]
  \[ \omega_t : Z \times \Gamma \to \mathcal{G} : \omega_t(z, g) = \omega_0(\alpha_t(z), g) . \]

- **Part I**: by property (T), the 1-cocycles $\omega_t$ are ‘locally constant modulo cohomology’. But then, $\omega_0 \sim \omega_1$.

- **Part II**: once $\omega_0 \sim \omega_1$ as cocycles with values in $\mathcal{U}(M)$, a very general weak mixing argument will conclude the proof.
Proof of cocycle superrigidity, Part I

We have set $Z = X \times X$ with the diagonal $\Gamma$-action.

Also
\[
\omega_0 : Z \times \Gamma \to G : \omega_0(x, y, g) = \omega(x, g),
\]
\[
\omega_t : Z \times \Gamma \to G : \omega_t(z, g) = \omega_0(\alpha_t(z), g).
\]

Define unitary representations
\[
\pi_t : \Gamma \to \mathcal{U}(L^2(Z) \otimes L^2(M)) : (\pi_t(g)\xi)(z) = \omega_t(z, g)\xi(z \cdot g)\omega_0(z, g)^*.
\]

By property (T), $\pi_t$ has a non-zero invariant vector for $t$ small.

So, for small $t = \frac{1}{n}$, we get $\varphi : Z \to L^2(M)$ satisfying
\[
\omega_t(z, g) \varphi(z \cdot g) = \varphi(z) \omega_0(z, g).
\]

By polar decomposition, we may assume that $\varphi$ takes values in the partial isometries of $M$.

We cheat and assume that $\varphi$ takes values in $\mathcal{U}(M)$.

So, $\omega_0 \sim \omega_t$ as cocycles with values in $\mathcal{U}(M)$.

Applying $\alpha_t$, also $\omega_{2t} \sim \omega_t, \ldots$, and finally $\omega_0 \sim \omega_1$. 

Proof of cocycle superrigidity, Part II

The second part follows from two general principles.

From now on, \( \Gamma \sim (X, \mu) \) weakly mixing.
- \( G \) a Polish group with bi-invariant metric.
- \( \omega : X \times \Gamma \to G \) a 1-cocycle.

Consider \( \Gamma \sim X \times X \) and the cocycles \( \omega_0, \omega_1 \).

Proposition 1

If \( \omega_0 \sim \omega_1 \), then \( \omega \) is cohomologous to a homomorphism.

Proposition 2

Let \( G \subset G_1 \) where \( G_1 \) is still a Polish group with bi-invariant metric.

If \( \omega \) is cohomologous to a homomorphism when viewed as a \( G_1 \)-valued cocycle, then the same holds true for \( \omega \) viewed as a \( G \)-valued cocycle.
Proof of Proposition 1

ω₀ ∼ ω₁ meaning

ω(x, g) = φ(x, y) ω(y, g) φ(x · g, y · g)⁻¹  (⋆)

Tool: Polish group \( \mathcal{U}(X, \mathcal{G}) = \{ F : X → \mathcal{G} \} / \)equality a.e.

\[ d(F_1, F_2) = \int_X d(F_1(x), F_2(x)) \, d\mu(x). \]

▶ View \( \varphi : X → \mathcal{U}(X, \mathcal{G}) \) and take essential value \( \varphi_0 ∈ \mathcal{U}(X, \mathcal{G}) \).

▶ Define \( \tilde{ω}(y, g) = \varphi_0(y) \omega(y, g) \varphi_0(y · g)⁻¹ \).

Claim: \( \tilde{ω} \) is independent of \( y ∈ X \).

▶ Let \( ε > 0 \) and take \( U ⊂ X \) with \( d(φ(x, ·), φ_0(·)) < ε \) for \( x ∈ U \).

▶ View \( G ⊂ \mathcal{U}(X, \mathcal{G}) \) as constant functions. By (⋆), \( d(G, \tilde{ω}(·, g)) < 2ε \) when \( U ∩ U · g \) non-negligible.

▶ Since \( \tilde{ω}(·, h) = \tilde{ω}(·, g) \tilde{ω}(· · g, g⁻¹ h) \), we have \( d(G, \tilde{ω}(·, h)) < 4ε \) when \( \exists g, U ∩ U · g \) and \( U ∩ (U · h⁻¹) · g \) non-negligible.

▶ But \( d(G, \tilde{ω}(·, h)) < 4ε \) holds for all \( h ∈ Γ \) and \( ε > 0 \) QED
Proof of Proposition 2

We have \( \omega : X \times \Gamma \to G \subset G_1 \).

We know \( \delta(g) = \varphi(x) \omega(x, g) \varphi(x \cdot g)^{-1} \) for \( \varphi : X \to G_1, \delta : \Gamma \to G_1 \).

- We may assume that 1 is essential value of \( \varphi \).
- We prove that \( \delta \) and \( \varphi \) take values in \( G \).
  So, \( \omega \sim \delta \) as cocycles with values in \( G \).
- Let \( \varepsilon > 0 \) and \( d(\varphi(x), 1) < \varepsilon \) for \( x \in U \).
- **Claim**: \( \delta \) takes values in \( G \).
  If \( U \cap U \cdot g \) non-negligible, \( d(\delta(g), G) < 2\varepsilon \).
  Since \( h = gg^{-1} h \), we have \( d(\delta(h), G) < 4\varepsilon \)
  if \( \exists g, U \cap U \cdot g \) and \( U \cap (U \cdot h^{-1}) \cdot g \) non-negligible.
- For every \( \varepsilon > 0 \), the set \( \{ x \in X \mid d(\varphi(x), G) < \varepsilon \} \) is non-negligible and \( \Gamma \)-invariant. Hence, it is the whole of \( X \). QED
Challenging exercise

Exercise 8

Prove the cocycle superrigidity theorem when $\Gamma$ no longer has property (T), but only an infinite normal subgroup $\Lambda$ with the relative property (T) and acting weakly mixingly.

- In Part I, you prove that $\omega_0 \sim \omega_1$ as cocycles on $\Lambda$ with values in $U(M)$.
- Proposition 1 applies as such and yields $\omega$ cohomologous to a homomorphism on $\Lambda$ and with values in $U(M)$.
- Extend the technique of Proposition 2 to get that $\omega$ is cohomologous to a homomorphism on the whole of $\Gamma$ and with values in $G$. 