# $K$-DUALITY FOR STRATIFIED PSEUDOMANIFOLDS

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## Introduction

This paper takes place in a longstanding project aiming to study index theory and related questions on stratified pseudomanifolds using tools and concepts from noncommutative geometry.

The key observation at the beginning of this project is that, in is $K$-theoritic form and before the use of the Chern character or heat kernels methods to produce explicit formulas, the Atiyah-Singer index theorem [2] involves ingredients that should survive to the singularities allowed in a stratified pseudomanifold. This is possible, from our opinion, as soon as one accepts reasonable generalizations and new presentation of certain classical objects on smooth manifolds, making sense on stratified pseudomanifolds.

The first instance of these classical objects that need to be adapted to singularities is the notion of *tangent space*. Since index maps in [2] are defined on the $K$-theory of the tangent spaces of smooth manifolds, one must have a similar space adapted to stratified pseudomanifold. Moreover, such a space should satisfy natural attempts. It should coincide with the usual notion on the regular part of the pseudomanifold and incorporate in some way copies of usual tangent spaces of strata, while keeping enough smoothness to allow interesting computations. Moreover, it should be Poincaré dual in $K$-theory (shortly, $K$-dual) to the pseudomanifold itself. This $K$-theoritic property involves bivariant $K$-theory and was proved between smooth manifolds and their tangent spaces by G. Kasparov [15] and A. Connes-G. Skandalis [9].

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In [11], we introduced a candidate to be the tangent space of a pseudomanifold with isolated conical singularities. It appeared to be a smooth groupoid, leading to a noncommutative $C^*$-algebra, and we prooved that it fulfills the expected $K$-duality.

In [17], the second author gave a notion of noncommutative symbols, living on the noncommutative tangent space of a pseudomanifold with isolated conical singularities, as well as a notion of ellipticity, such that the classical results on manifolds are brought to the conical case: elliptic symbols describe the $K$-theory of the tangent space and the isomorphism between the $K$-theory of the noncommutative tangent space and the $K$-homology of the conical pseudomanifold itself resulting from [11] can be concretely interpreted as a symbol map sending an appropriate class of elliptic operators (namely, fully elliptic $b$-pseudodifferential operators), to there elliptic noncommutative symbols.

In [4], the noncommutative tangent space together with other deformation groupoids was used to construct analytical and topological index maps, and their equality is prooved. As expected, these index maps are straight generalizations of those of [2] for manifolds.

The present paper is devoted to the construction of the noncommutative tangent space for general stratified pseudomanifold and the proof of the $K$-duality. It is thus a sequel of [11], but can be read independently. At first glance, one should have expected that the technics of [11] iterate easely to give the general result. In fact, if the definition of the groupoid giving the noncommutative tangent space itself is natural and intuitive in the general case, its smoothness is quite intricate and brings issues that did not exist in the conical case. We have given here a detailed treatment of this point, since we believe that this material will be usefull in further studies about the geometry of stratified spaces. Another difference with [11] is that we have given up the explicit construction of a dual Dirac element. Instead, we use an easely defined Dirac element and then proove the Poincaré duality by a recursive argument, based on the stability of removing minimal strata in a pseudomanifold and then "doubling" it to get a new pseudomanifold, less singular. The difficulty in this approach is moved to the proof of the commutativity of certain diagrams in $K$-theory, necessary to apply the five lemma and to continue the recursion.

The interpretation of this $K$-duality in terms of noncommutative symbols and pseudodifferential operators, as well as the construction of index maps together with the statement of an index theorem, is postponed to forthcoming papers.

This approach of index theory on singular spaces in the framework of noncommutative geometry takes place in a long history of past and present resarch works. But the specific questions about Poincaré duality, topological index map and statement of an Atiyah-Singer like theorem are quite recent and attract an increasing interest [24, 21, 28, 25].

## 1. Basic definitions

### 1.1. **Around Lie groupoids.**

We refer to [27, 6, 18] for the classical definitions and construction related to groupoids, their Lie algebroids and groupoids $C^*$-algebras. In this section, we fix the notations and recall the less classical definitions and results needed in the sequel. Some material presented here is already in [11, 4].

### 1.1.1. *Pull back groupoids.*

Let $G \rightrightarrows M$ be a locally compact Hausdorff groupoid with source $s$ and range $r$. If $f : N \to M$ is a surjective map, the *pull back* groupoid ${}^*f^*(G) \rightrightarrows N$ of $G$ by $f$ is by definition the set

$$ {}^*f^*(G) := \{(x,\gamma,y) \in N \times G \times N \mid r(\gamma) = f(x),\ s(\gamma) = f(y)\} $$

with the structural morphisms given by

1. the unit map $x \mapsto (x, f(x), x)$,
2. the source map $(x,\gamma,y) \mapsto y$ and range map $(x,\gamma,y) \mapsto x$,
3. the product $(x,\gamma,y)(y,\eta,z) = (x,\gamma\eta,z)$ and inverse $(x,\gamma,y)^{-1} = (y,\gamma^{-1},x)$.

The results of [26] apply to show that the groupoids $G$ and ${}^*f^*(G)$ are Morita equivalent when $f$ is surjective, open and proper.

Let us assume for the rest of this subsection that $G$ is a smooth groupoid and that $f$ is a surjective proper submersion, then ${}^*f^*(G)$ is also a Lie groupoid. Let $(\mathcal{A}(G), q, [\ ,\ ])$ be the Lie algebroid of $G$ (which is defined since $G$ is smooth). Recall that $q : \mathcal{A}(G) \to TM$ is the anchor map. Let $(\mathcal{A}({}^*f^*(G)), p, [\ ,\ ])$ be the Lie algebroid of ${}^*f^*(G)$ and $Tf : TN \to TM$ be the differential of $f$. Then we claim that there exists an isomorphism

$$ \mathcal{A}({}^*f^*(G)) \simeq \{(V,U) \in TN \times \mathcal{A}(G) \mid Tf(V) = q(U) \in TM\} $$

under which the anchor map $p : \mathcal{A}({}^*f^*(G)) \to TN$ identifies with the projection $TN \times \mathcal{A}(G) \to TN$. In particular, if $(U,V) \in \mathcal{A}({}^*f^*(G))$ with $U \in T_xN$ and $V \in \mathcal{A}_y(G)$, then $y = f(x)$.

### 1.1.2. *Subalgebras and exact sequences of groupoid $C^*$-algebras.*

To any smooth groupoid $G$ is associated two $C^*$-algebras corresponding more or less to two different completions of the involutive convolution algebra $C_c^\infty(G)$, namely the reduced and maximal $C^*$-algebras [5, 7, 27]. We will denote respectively these $C^*$-algebras by $C^*(G)$ and $C^*_{max}(G)$. Recall that the identity on $C_c^\infty(G)$ induces a surjective morphism from $C^*_{max}(G)$ onto $C^*(G)$ which is an isomorphism if the groupoid $G$ is amenable. Moreover in this case the $C^*$ algebra of $G$ is nuclear [1] .

We will use the following usual notations :

Let $G \underset{r}{\overset{s}{\rightrightarrows}} G^{(0)}$ be a smooth groupoid with source $s$ and range $r$. If $U$ is any subset of $G^{(0)}$, we let :

$$ G_U := s^{-1}(U)\ ,\ G^U := r^{-1}(U) \text{ and } G_U^U = G|_U := G_U \cap G^U\ . $$

To an open subset $O$ of $G^{(0)}$ corresponds an inclusion $i_O$ of $C_c^\infty(G|_O)$ into $C_c^\infty(G)$ which induces an injective morphism, again denoted by $i_O$, from $C^*(G|_O)$ into $C^*(G)$.

When $O$ is saturated, $C^*(G|_O)$ is an ideal of $C^*(G)$. In this case, $F := G^{(0)} \setminus O$ is a saturated closed subset of $G^{(0)}$ and the restriction of functions induces a surjective morphism $r_F$ from $C^*(G)$ to $C^*(G|_F)$. Moreover, according to [13], the following sequence of $C^*$-algebras is exact :

$$ 0 \longrightarrow C^*(G|_O) \xrightarrow{i_O} C^*(G) \xrightarrow{r_F} C^*(G|_F) \longrightarrow 0\ \cdot $$

1.1.3. *$KK$-elements associated to deformation groupoids.*
A smooth groupoid $G$ is called a *deformation groupoid* if :

$$G = G_1 \times \{0\} \cup G_2 \times ]0,1] \rightrightarrows G^{(0)} = M \times [0,1],$$

where $G_1$ and $G_2$ are smooth groupoids with unit space $M$. That is, $G$ is obtained by gluing $G_2 \times ]0,1] \rightrightarrows M \times ]0,1]$, which is the cartesian product of the groupoid $G_2 \rightrightarrows M$ with the space $]0,1]$, with the groupoid $G_1 \times \{0\} \rightrightarrows M \times \{0\}$.

In this situation one can consider the saturated open subset $M \times ]0,1]$ of $G^{(0)}$. Using the isomorphisms $C^*(G|_{M\times]0,1]}) \simeq C^*(G_2) \otimes C_0(]0,1])$ and $C^*(G|_{M\times\{0\}}) \simeq C^*(G_1)$, we obtain the following exact sequence of $C^*$-algebras :

$$0 \longrightarrow C^*(G_2) \otimes C_0(]0,1]) \xrightarrow{i_{M\times]0,1]}} C^*(G) \xrightarrow{ev_0} C^*(G_1) \longrightarrow 0$$

where $i_{M\times]0,1]}$ is the inclusion map and $ev_0$ is the *evaluation map* at 0, that is $ev_0$ is the map coming from the restriction of functions to $G|_{M\times\{0\}}$.
We assume now that $C^*(G_1)$ is nuclear. Since the $C^*$-algebra $C^*(G_2) \otimes C_0(]0,1])$ is contractible, the long exact sequence in $KK$-theory shows that the group homomorphism $(ev_0)_* = \cdot \otimes [ev_0] : KK(A, C^*(G)) \to KK(A, C^*(G_1))$ is an isomorphism for each $C^*$-algebra $A$ [9].
In particular with $A = C^*(G)$ we get that $[ev_0]$ is invertible in $KK$-theory : there is an element $[ev_0]^{-1}$ in $KK(C^*(G_1), C^*(G))$ such that $[ev_0] \otimes [ev_0]^{-1} = 1_{C^*(G)}$ and $[ev_0]^{-1} \otimes [ev_0] = 1_{C^*(G_1)}$.
Let $ev_1 : C^*(G) \to C^*(G_2)$ be the evaluation map at 1 and $[ev_1]$ the corresponding element of $KK(C^*(G), C^*(G_2))$.
The *$KK$-element associated to the deformation groupoid* $G$ is defined by :

$$\delta = [ev_0]^{-1} \otimes [ev_1] \in KK(C^*(G_1), C^*(G_2)) \ .$$

One can find examples of such elements related to index theory in [7, 13, 11, 4].

1.2. **Generalities about $K$-duality.** We give in this paragraph some general facts about Poincaré duality in bivariant K-theory. Most of them are well known and proofs are only added when no self contained demonstration could be found in the literature. All $C^*$-algebras are assumed to be separable and $\sigma$-unital.

Let us first recall what means the Poincaré duality in $K$-theory [16, 9, 7]:

**Definition 1.** Let $A, B$ be two $C^*$-algebras. One says that $A$ and $B$ are Poincaré dual, or shortly $K$-dual, when there exists $\alpha \in K^0(A \otimes B) = KK(A \otimes B, \mathbb{C})$ and $\beta \in KK(\mathbb{C}, A \otimes B) \simeq K_0(A \otimes B)$ such that

$$\beta \underset{B}{\otimes} \alpha = 1 \in KK(A,A) \text{ and } \beta \underset{A}{\otimes} \alpha = 1 \in KK(B,B)$$

Such elements are then called Dirac and dual-Dirac elements.

It follows that for $A, B$ two $K$-dual $C^*$-algebras and for any $C^*$-algebras $C, D$, the following isomorphisms hold:

$$\beta \underset{B}{\otimes} \cdot : KK(B \otimes C, D) \longrightarrow KK(C, A \otimes D);$$

$$\beta \underset{A}{\otimes} \cdot : KK(A \otimes C, D) \longrightarrow KK(C, B \otimes D);$$

with inverses given respectively by $\cdot \underset{A}{\otimes} \alpha$ and $\cdot \underset{B}{\otimes} \alpha$.

**Example 1.** *A basic example is* $A = C(V)$ *and* $B = C_0(T^*V)$ *where* $V$ *is a closed smooth manifold (*[16, 9]*, see also* [11] *for a description of the Dirac element in terms of groupoids). This duality allows to recover that the usual quantification and principal symbol maps are mutually inverse isomorphisms in* $K$*-theory :*

$$\Delta_V = (\cdot \underset{C_0(T^*V)}{\otimes} \alpha) : K_0(C_0(T^*V)) \xrightarrow{\simeq} K^0(C(V))$$

$$\Sigma_V = (\beta \cdot \underset{C(V)}{\otimes} \cdot) : K^0(C(V)) \xrightarrow{\simeq} K_0(C_0(T^*V))$$

We observe that:

**Lemma 2.** *Let* $A, B$ *be two* $C^*$*-algebras. Assume that there exists* $\alpha \in KK(A \otimes B, \mathbb{C})$ *and* $\beta, \beta' \in KK(\mathbb{C}, A \otimes B)$ *satisfying*

$$\beta \underset{B}{\otimes} \alpha = 1 \in KK(A, A) \text{ and } \beta' \underset{A}{\otimes} \alpha = 1 \in KK(B, B)$$

*Then* $\beta = \beta'$ *so* $A, B$ *are* $K$*-dual.*

*Proof.* A simple calculation shows that for all $x \in KK(C, A \otimes D)$ we have :

$$\beta \underset{B}{\otimes} (x \underset{A}{\otimes} \alpha) = x \underset{A \otimes B}{\otimes} (\beta \underset{B}{\otimes} \alpha) .$$

Applying this to $C = \mathbb{C}$, $D = A$ and $x = \beta'$ we get:

$$\beta' = \beta \underset{B}{\otimes} (\beta' \underset{A}{\otimes} \alpha) = \beta \underset{B}{\otimes} 1 = \beta$$

□

This implies in particular that given two $K$-dual $C^*$-algebras and a Dirac element $\alpha$, the dual-Dirac element $\beta$ satisfying the definition 1 is unique. The lemma implies also that if there exists $\alpha \in KK(A \otimes B, \mathbb{C})$ such that

$$\cdot \underset{B}{\otimes} \alpha : KK(\mathbb{C}, A \otimes B) \longrightarrow KK(A, A) \text{ and } \cdot \underset{A}{\otimes} \alpha : KK(\mathbb{C}, A \otimes B) \longrightarrow KK(B, B)$$

are onto, then $A, B$ are $K$-dual.

## 2. Stratified pseudomanifolds

We are interested in studying stratified pseudomanifolds [30, 19, 12]. We will use the notations and equivalent descriptions given by A. Verona in [29] or used by J.P. Brasselet, G. Hector and M. Saralegi in [3]. The interested reader should look at [14] for a review of the subject.

### 2.0.1. *Definitions.*

Let $X$ be a locally compact separable metrizable space.

**Definition 1.** *A* $C^\infty$*-stratification of* $X$ *is a pair* $(\mathsf{S}, N)$ *such that :*

*(1)* $\mathsf{S} = \{s_i\}$ *is a locally finite partition of* $X$ *into locally closed subsets of* $X$*, called the strata, which are smooth manifolds and which satisfie :*

$$s_0 \cap \bar{s_1} = \emptyset \text{ if and only if } s_0 \subset \bar{s_1}.$$

*(2)* $N = \{\mathcal{N}_s, \pi_s, \rho_s\}_{s\in\mathsf{S}}$ *is the set of control data or tube system :* $\mathcal{N}_s$ *is an open neighborhood of* $s$ *in* $X$*,* $\pi_s : \mathcal{N}_s \to s$ *is a continuous retraction and* $\rho_s : \mathcal{N}_s \to [0, +\infty[$ *is a continuous map such that* $s = \rho_s^{-1}(0)$*. The map* $\rho_s$ *is either surjective or constant equal to* $0$*.*
*Moreover if* $\mathcal{N}_{s_0} \cap s_1 \neq \emptyset$ *then the map*

$$(\pi_s, \rho_s) : \mathcal{N}_{s_0} \cap s_1 \to s_0 \times ]0, +\infty[$$

*is a smooth submersion.*

*(3) The following compatibility conditions hold :*
*for* $s_0 \neq s_1$ *the equalities*

$$\pi_{s_0} \circ \pi_{s_1} = \pi_{s_0} \text{ and } \rho_{s_0} \circ \pi_{s_1} = \rho_{s_0}$$

*hold everywhere they make sense, that is on* $\mathcal{N}_{s_0} \cap \mathcal{N}_{s_1} \cap \pi_{s_1}^{-1}(\mathcal{N}_{s_0} \cap s_1)$*.*

*(4) For any two strata* $s_0$ *and* $s_1$ *the following equivalences hold :*

$$s_0 \cap \bar{s_1} \neq \emptyset \text{ if and only if } \mathcal{N}_{s_0} \cap s_1 \neq \emptyset \ ,$$

$$\mathcal{N}_{s_0} \cap \mathcal{N}_{s_1} \neq \emptyset \text{ if and only if } s_0 \subset \bar{s_1},\ s_0 = s_1 \text{ or } s_1 \subset \bar{s_0}.$$

To such a stratification is naturally associated a filtration. Precisely, let $X_j$ be the union of strata of dimension $\leq j$. We get the filtration :

$$\emptyset \subset X_0 \subset \cdots \subset X_n = X \ .$$

We call $n$ the *dimension* of $X$ and $X^\circ := X \setminus X_{n-1}$ the *regular* part of $X$. The strata included in $X \setminus X^\circ$ are called *singular*, their set is denoted $\mathsf{S}_{sing}$.
For any subset $A$ of $X$, $A^\circ$ will denote $A \cap X^\circ$.

A crucial notion for our purpose will be the notion of *depth* :

**Definition 2.** *When* $s_0 \subset \bar{s_1}$ *we denote it by* $s_0 \leq s_1$ *and we say that* $s_0$ *is smaller then* $s_1$*. If moreover* $s_0 \neq s_1$ *we note* $s_0 < s_1$*. The depth* $d(s)$ *of a stratum* $s$ *is the biggest* $k$ *such that one can find* $k$ *different strata* $s_0, \cdots, s_{k-1}$ *such that*

$$s_0 < s_1 < \cdots < s_{k-1} < s_k := s.$$

*The depth of the stratification* $(\mathsf{S}, N)$ *of* $X$ *is :*

$$d(X) := sup\{d(s),\ s \in \mathsf{S}\}.$$

*A stratum whose depth is* $0$ *will be called minimal.*

We have followed the terminology of [3], but remark that the opposite convention for the depth also exists [29].

Finally we can define stratified pseudomanifolds :

**Definition 3.** *A stratified pseudomanifold is a triple* $(X, \mathsf{S}, N)$ *where* $X$ *is a locally compact separable metric space,* $(\mathsf{S}, N)$ *is a* $C^\infty$*-stratification on* $X$ *and the regular part* $X^\circ$ *is a dense open subset of* $X$*.*

If $(X, \mathsf{S}_X, N_X)$ and $(Y, \mathsf{S}_Y, N_Y)$ are two stratified pseudomanifolds an homeomorphism $f : X \to Y$ is an isomorphism of stratified pseudomanifold if :

(1) $\mathsf{S}_Y = \{f(s),\ s \in \mathsf{S}_X\}$ and the restriction of $f$ to each stratum is a diffeomorphism onto its image.
(2) $\pi_{f(s)} \circ f = f \circ \pi_s$ and $\rho_s = \rho_{f(s)} \circ f$ for any stratum $s$ of $X$.

Let us make some basic remark on the previous definitions.

**Remark 1.** *(1) At a first sight, the definition of a stratification given here seems more restrictive than the usual one In fact according to* [29] *these definitions are equivalent.*

*(2) Usually, for example in* [12]*, the extra assumption* $X_{n-1} = X_{n-2}$ *is required in the definition of stratified pseudomanifold. Our constructions remain without this extra assumption.*

*(3) A stratum* $s$ *is regular if and only if* $\mathcal{N}_s = s$ *and then* $\rho_s = 0$*.*

*(4) Pseudomanifolds of depth* $0$ *are smooth manifolds, and the strata are then union of connected components.*

The following simple consequence of the axioms will be usefull enough in the sequel to be pointed out:

**Proposition 1.** *Let* $(X, \mathsf{S}, N)$ *be a stratified pseudomanifold. Any subset* $\{s_i\}_I$ *of distinct elements of* $\mathsf{S}$ *is totally ordered by* $<$ *as soon as the intersection* $\cap_{i\in I}\mathcal{N}_{s_i}$ *is non empty. In particular if the strata* $s_0$ *and* $s_1$ *are such that* $\mathcal{N}_{s_0} \cap \mathcal{N}_{s_1} \neq \emptyset$ *then* $d(s_0) \neq d(s_1)$ *or* $s_0 = s_1$*.*

By a slight abuse of language we will sometime talk about a stratified pseudomanifold $X$ while we only have a partition $\mathsf{S}$ on the space $X$. This means that one can find at least one control data $N$ such that $(X, \mathsf{S}, N)$ is a stratified pseudomanifold in the sense of our definition 3.

2.0.2. *Examples.*

(1) Smooth manifolds are, without other mention, pseudomanifolds of depth 0 and with a single stratum.

(2) Stratified pseudomanifolds of depth one are *wedges* and are obtained as follows. Take $M$ to be a manifold with a boundary $L$ and suppose that you have a submersion $\pi$ of $L$ onto a compact manifold $s$. Consider the *mapping cone* of $(L, \pi)$ :

$$c_\pi L := L \times [0, 1]/ \sim_\pi$$

where $(z, t) \sim_\pi (z', t')$ if and only if $(z, t) = (z', t')$ or $t = t' = 0$ and $\pi(z) = \pi(z')$. The image of $L \times \{0\}$ identifies with $s$ and by a slight abuse of notation we will denote it $S$. Now glue $c_\pi L$ and $M$ along their boundary in order to get $X$. The space $X$ with the partition $\{s, X \setminus s\}$ is a stratified pseudomanifold.

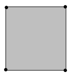
Two extreme examples are obtained by considering $\pi$ either equal to identity, with $s = L$ or equal to the projection on one point $c$. In the first case $X$ is a manifold with boundary $L$ isomorphic to $M$ and the stratification corresponds the partition of $X$ by $\{L, X \setminus L\}$. In the second case $X$ is a *conical manifold* and the stratification corresponds the partition of $X$ by $\{c, X \setminus c\}$, where $c$ is the singular point.

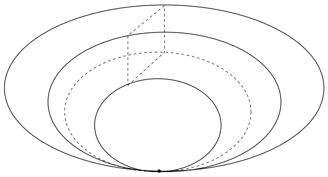
(3) Manifolds with corners with their partition into faces are stratified pseudomanifolds [20, 22].

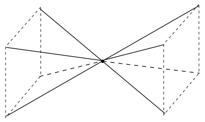
(4) If $(X, \mathsf{S}, N)$ is a pseudomanifold and $M$ is a smooth manifold then $X \times M$ is naturally endowed with a structure of pseudomanifold of same depth as $X$ whose strata are $\{s \times M,\ s \in \mathsf{S}\}$.

(5) If $(X, \mathsf{S}, N)$ is a pseudomanifold of depth $k$ then $CrX := X \times S^1/X \times \{p\}$ is naturally endowed with a structure of pseudomanifold of depth $k + 1$, whose strata are $\{s\times]0, 1[,\ s \in \mathsf{S}\} \cup \{[p]\}$. Here we have identified $S^1 \setminus \{p\}$ with $]0, 1[$ and we have denoted by $[p]$ the image of $X \times \{p\}$ in $CrX$.

For example, if $X$ is the square we get the following picture :







2.0.3. *The unfolding process.*
Let $(X,\mathsf{S},N)$ be a stratified pseudomanifold. If $s$ is a singular stratum, we let $L_s:=\rho_s^{-1}(1)$. Then $L_s$ inherits from $X$ a structure of stratified pseudomanifold. One can then define the *open mapping cone* of $(L_s,\pi_s)$ :

$$c_{\pi_s}L_s := L_s\times[0,+\infty[/\sim_{\pi_s}$$

where $\sim_{\pi_s}$ is as before.

According to [29], see also [3] the open mapping cone is naturally endowed with a structure of stratified pseudomanifold whose strata are $\{(s'\cap L_s)\times]0,+\infty[,\ s'\in \mathsf{S}\}\cup\{s\}$. Here we identifie $s$ with the image of $L_s\times\{0\}$ in $c_{\pi_s}L_s$. Moreover, up to isomorphism , the control data on $X$ can be chosen such that one can find a continuous retraction $f_s:\mathcal{N}_s\setminus s\to L_s$ for which the map

$$\begin{array}{cccc} \Psi_s: & \mathcal{N}_s & \to & c_{\pi_s}L_s \\ & z & \mapsto & \left\{\begin{array}{cc} [f_s(z),\rho_s(z)] & \text{if } z\notin s \\ z & \text{elsewhere}\end{array}\right. \end{array} \tag{2.1}$$

is an isomorphism of stratified pseudomanifolds. Here $[y,t]$ denotes the class in $c_{\pi_s}L_s$ of $(y,t)\in L_s\times[0,+\infty[$.

This result of local triviality around strata will be crucial for our purpose. In particular it enables one to make the unfolding process [3] which consists in replacing each minimal stratum $s$ by $L_s$. Precisely suppose that $d(X)=k>0$ and let $\mathsf{S}_0$ be the set of strata of depth 0. Define $O_0:=\cup_{s\in\mathsf{S}_0}\{z\in X\mid \rho_s(z)<1\}$, $X_b=X\setminus O_0$ and $L:=\cup_{s\in\mathsf{S}_0}\{z\in X\mid\rho_s(z)=1\}\subset X_b$. Notice that it follows from remark 1 that the $L_s$'s where $s\in\mathsf{S}_0$ are disjoint and thus $L=\sqcup_{s\in\mathsf{S}_0}L_s$. We let

$$2X = X_b^-\cup L\times[-1,1]\cup X_b^+$$

where $X_b^\pm=X_b$ and $X_b^-$ (respectively $X_b^+$) is glued along $L$ with $L\times\{-1\}\subset L\times[-1,1]$ (respectively $L\times\{1\}\subset L\times[-1,1]$).

Let $s$ be a stratum of $X$ which is not minimal and wich incounter $O_0$. We define the following subset of $2X$:

$$\tilde{s}:=(s\cap X_b^-)\cup(s\cap L)\times[-1,1]\cup(s\cap X_b^+)$$

We then define

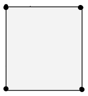
$$\mathsf{S}_{2X}:=\{\tilde{s};\ s\in\mathsf{S}\text{ and } s\cap O_0\neq\emptyset\}\cup\{s^-,s^+;s^\pm=s\in\mathsf{S}\text{ and } s\cap O_0=\emptyset\}.$$

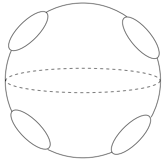
The space $2X$ inherits from $X$ a structure of stratified pseudomanifold of depth $k-1$ whose set of strata is $\mathsf{S}_{2X}$.

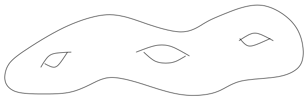
Notice that there is a natural map $p$ from $2X$ onto $X$. The restriction $p$ to any copy of $X_b$ is identity and for $(z,t)\in L_s\times[-1,1]$, $p(z,t)=\Psi_s^{-1}([z,\mid t\mid])$. The strata of $2X$ are the connected component of the preimage by $p$ of the strata of $X$.

The interested reader can find all the details related to the unfolding process in [3] and [29] where it is called decomposition. In particular starting with a compact pseudomanifold $X$ of depth $k$, one can iterates this process $k$ times and obtain a compact smooth manifold $2^kX$ together with a continuous surjective map $\pi : 2^kX \to X$ whose restriction to $\pi^{-1}(X^\circ)$ is a trivial $2^k$-fold covering.

**Example 2.** *Look at the square $C$ with stratification given by its vertices, edges and its interior. It can be endowed with a structure of stratified pseudomanifold of depth 2. Applying once the unfolding process gives a sphere with 4 holes : $S^2 \setminus \{D_1, D_2, D_4\}$ where the $D_i$ are homeomorphic to disjoint open disk. The set of strata of $S^2$ is then $\{S^2, S_1, S_2, S_3, S_4\}$ where $S_i$ is the boundary of $D_i$. Applying the unfolding process once more gives the torus with three holes.*







## 3. The tangent groupoid and $\mathsf{S}$-tangent space of a compact stratified pseudo-manifold

3.1. **The set construction.** We begin by the description at the set level of the *tangent groupoid* and the $\mathsf{S}$-*tangent space* of a compact stratified pseudo-manifold.

We keep the notation of the previous section : $X$ is a compact stratified pseudo-manifold, $\mathsf{S}$ the set of strata, $X^\circ$ the regular part and $N = \{\mathcal{N}_s, \pi_s, \rho_s\}_{s\in\mathsf{S}}$ the set of control data.

For each $s \in \mathsf{S}$ we let

$$O_s := \{z \in X \mid \rho_s(z) < 1\} \text{ and } F_s := O_s \setminus \bigcup_{s_0<s} O_{s_0} \ .$$

Note that $F_s = O_s$ if and only if $s$ is a minimal stratum and $O_s = s$ when $s$ is regular.

**Lemma 3.** *The set $\{F_s\}_{s\in\mathcal{S}}$ form a partition of $X$.*

*Proof.* If $z$ belongs to $X$, let $\mathsf{S}_z := \{s \in \mathsf{S} \mid \rho_s(z) < 1\}$. It follows from proposition 1 that $\mathsf{S}_z$ is a countable set totaly ordered by $<$. Let $s_0^z$ be the minimal element of $\mathsf{S}_z$. Then $z$ belongs to $F_{s_0^z}$. Thus $X \subset \bigcup_{s\in\mathcal{S}} F_s$.
If $z \in F_s \cap F_{s'}$ then $s < s'$, $s = s'$ or $s' < s$. Suppose that $s' < s$, since $z \in F_{s'} \subset O_{s'}$ we have $z \in \bigcup_{s_0<s} O_{s_0}$ which is incompatible with $z \in F_s$. Thus $F_s \cap F_{s'} = \emptyset$ or $s = s'$. □

Recall that $O_s^\circ = O_s \cap X^\circ$. We denote again by $\pi_s : O_s^\circ \to S$ the projection. When $s$ is a stratum, $\pi_s$ is a proper submersion and one can consider the pull-back groupoid ${}^*\pi_s^*(TS) \rightrightarrows O_s^\circ$ of the (usual) tangent space $TS \rightrightarrows S$ by $\pi_s$. It is naturally endowed with a smooth groupoid structure. Notice that when $s$ is regular $S = O_s = O_s^\circ$, $\pi_s$ is identity and ${}^*\pi_s^*(TS) = TO_s^\circ$.

At the set level, the *tangent space* of $X$ is the groupoid :

$$T^{\mathsf{S}}X = \bigcup_{s\in\mathsf{S}} {}^*\pi_s^*(TS)|_{F_s^\circ} \rightrightarrows X^\circ$$

where $F_s^\circ = F_s \cap X^\circ$ and its *tangent groupoid* is as in the smooth case [7] a deformation of the pair groupoid on the tangent space :

$$\mathcal{G}_X^t := T^{\mathsf{S}}X \times \{0\} \cup X^\circ \times X^\circ \times ]0,1] \rightrightarrows X^\circ \times [0,1] \ .$$

**Examples 4.** (1) When $X$ has depth 0, we recover the usual tangent space and tangent groupoid.

(2) Suppose that $X$ is a trivial wedge (see example 2.0.2):

$$X = c_\pi L \cup M$$

where $M$ is a manifold with boundary $L$ and $L$ is the product of two manifolds $L = s \times Q$ with $\pi : L \to s$ being the first projection. We have denoted by $c_\pi L = L \times [0,1]/\sim_\pi$ the mapping cone of $(L, \pi)$. In other word $c_\pi L = s \times cQ$ where $cQ := Q \times [0,1]/Q \times \{0\}$ is the cone over $Q$. We denote again by $s$ the image of $L \times \{0\}$ in $X$. Then $X$ admits two strata : $s$ and $X^\circ = X \setminus s$, $F_s = O_s = L \times ]0,1[$ and $F_{X^\circ} = X^\circ \setminus O_s = M$. The tangent space is

$$T^{\mathsf{S}}X = Ts \times (Q \times ]0,1[) \times (Q \times ]0,1[) \sqcup TM \rightrightarrows X^\circ$$

where $Ts \times (Q \times ]0,1[) \times (Q \times ]0,1[)$ is the product of the tangent space $Ts \rightrightarrows s$ with the pair groupoid over $Q \times ]0,1[$ and $TM$ denotes the restriction of the usual tangent bundle $TX^\circ$ to the sub-manifold with boundary $M$.

**Remark 5.** For any stratum $s$, the restriction of $\mathcal{G}^t_X$ to $F^\circ_s$ is equal to

$${}^*\pi_s^*(TS)|_{F^\circ_s} \times \{0\} \cup F^\circ_s \times F^\circ_s \times ]0,1] \rightrightarrows F^\circ_s \times [0,1]$$

which is also the restriction to $F^\circ_s$ of ${}^*(\pi_s \times \mathrm{Id})^*(\mathcal{G}^t_s)$, the pull-back by $\pi_s \times \mathrm{Id} : O^\circ_s \times [0,1] \to S \times [0,1]$ of the (usual) tangent groupoid of $s$ :

$$\mathcal{G}^t_s = Ts \times \{0\} \cup s \times s \times ]0,1] \rightrightarrows s \times [0,1]\ .$$

In the following, we will denote by $\mathcal{A}^t_{\pi_s \times \mathrm{Id}}$ the Lie algebroid of ${}^*(\pi_s \times \mathrm{Id})^*(\mathcal{G}^t_s)$.

3.2. **The smooth structure.** In this subsection we prove that the tangent space and the tangent groupoid of a stratified pseudomanifold can be endowed with a smooth structure in such a way that the underlying manifolds has the same connectedness than $X^\circ$).

To do so , we first choose a *glueing function*. It is a smooth decreasing and positive function $\tau : \mathbb{R} \to \mathbb{R}$ such that $\tau([0,+\infty[) = [0,1]$, $\tau^{-1}(0) = [1,+\infty[$ and $\tau'$ doesn't vanish on $]0,1[$.
If $s$ is any singular stratum, we let $\tau_s = \tau \circ \rho_s$. Thus $\tau_s = 0$ outside $O^\circ_s$.

Roughly speaking, the topology on $T^{\mathsf{S}}X$ will be of the following form. A sequence $(x_n, V_n, y_n) \in {}^*\pi^*_{s_n}(Ts_n)|_{F_{s_n}}$ where $n$ belongs to $\mathbb{N}$, will go to $(x, V, y) \in {}^*\pi^*_s(Ts)|_{F_s}$ if and only if :

$$x_n \to x,\ y_n \to y,\ V_n + \frac{\pi_s(x_n) - \pi_s(y_n)}{\tau_{s_n}(x_n)} \to V$$

To understand the last convergence, first note that for $n$ big enough we have $s_n < s$ and $T_y s \simeq T_{\pi_{s_n}(y)} s_n \times \mathsf{K}_n$ where $\mathsf{K}_n$ is the kernel of the differential of the restriction of $\pi_{s_n}$ from $Ts \to Ts_n$. Recall that $\pi_{s_n}(x_n) = \pi_{s_n}(y_n)$ and $\pi_{s_n} \circ \pi_s = \pi_{s_n}$. Under suitable identifications, $\frac{\pi_s(x_n) - \pi_s(y_n)}{\tau_{s_n}(x_n)}$ gives an element of $\mathsf{K}_n$ while $v_n$ belongs to $T_{\pi_{s_n}(y)} s_n$.

We will present the smooth structure with two different and independent point of view. Before this, we look in more details at the local structure of $X^\circ$. Then, in the second part we will produce an atlas for the tangent space. In the third part we will describe the smooth structure of the tangent space by giving the Lie algebroid of the tangent groupoid. These parts are quite technical and can be left out as soon as you believe that the tangent space and the tangent groupoid can be endowed with a smooth structure compatible with the topology described above.

### 3.2.1. *The local structure of $X^\circ$.*

Let us choose a riemanian metric on $X^\circ$. We denote by $n$ the dimension of $X^\circ$ and by $n_s$ the dimension of the stratum $s$. Thus $n = n_s$ when $s$ is regular.

When $s$ is a stratum, recall that $\pi_s : \mathcal{N}_s^\circ \to s$ is a smooth submersion. Let $K_s \subset T\mathcal{N}_s^\circ$ be the kernel of the differential $T\pi_s$ and $q_s : T\mathcal{N}_s^\circ \to T\mathcal{N}_s^\circ$ the orthogonal projection on $K_s$. Thus

$$(T\pi_s, q_s) : T\mathcal{N}_s^\circ \to \pi_s^*(Ts) \oplus K_s \tag{3.1}$$

is an isomorphism which enables us to identifie $\pi_s^*(Ts)$ with a subbundle of $TX^\circ$. Notice that when $s$ is regular, $s = \mathcal{N}_s$ and $\pi_s$ is identity, thus $K_s = s \times \{0\}$.

If $z$ belongs to $X^\circ$, let $\mathsf{S}_z := \{s \in \mathsf{S} \mid \rho_s(z) \leq 1\}$. It is a finite set which is totally ordered according to proposition 1. Suppose that $\mathsf{S}_z = \{s_0, \cdots, s_\kappa\}$ with

$$s_0 < s_1 < \cdots < s_\kappa$$

where $s_\kappa \subset X^\circ$ is regular. One can find an open neighborhood $U_z$ of $z$ in $X^\circ$ such that $U_z \subset \cap_{i=0}^{\kappa} \mathcal{N}_{s_i}$ and for any singular stratum $s$ we have $U_z \cap O_s \neq \emptyset$ if and only if $s \in \mathsf{S}_z$.

Moreover with no loss of generality, with the help of a reparametrisation of the control data if necessary, we can suppose that for $0 \leq i \leq j \leq \kappa$, $z$ belongs to $\mathcal{N}_{s_i} \cap \mathcal{N}_{s_j} \cap \pi_{s_j}^{-1}(\mathcal{N}_{s_i} \cap s_j)$ and thus choose $U_z \subset \mathcal{N}_{s_i} \cap \mathcal{N}_{s_j} \cap \pi_{s_j}^{-1}(\mathcal{N}_{s_i} \cap s_j)$ . So the following equalities hold on $U_z$ :

$$\text{for } 0 \leq i \leq j \leq \kappa : \pi_{s_i} \circ \pi_{s_j} = \pi_{s_i} \text{ and } \rho_{s_i} \circ \pi_{s_j} = \rho_{s_i} \ . \tag{3.2}$$

By taking a possible smaller $U_z$, for $0 \leq i \leq \kappa$ one can find a smooth submersion $f_z^{s_i} : U_z \to \mathbb{R}^{n-n_{s_i}}$, such that the map

$$\Psi_z^i := (\pi_{s_i}, f_z^{s_i}) : U_z \to (s_i \cap \pi_{s_i}(U_z)) \times \mathbb{R}^{n-n_{s_i}}$$

is a diffeomorphism. Moreover it follows from 3.2 that for $0 \leq i \leq j \leq \kappa$, $\Psi_z^i \circ (\Psi_z^j)^{-1}$ is of the form $(z,t) \mapsto (\pi_{s_i}(z), h_{ji}(z,t))$.

Thus, when $\kappa \geq 1$, by taking canonical forms of the submersions $\pi_{s_\kappa}, \cdots, \pi_{s_0}$ successively and a possible smaller $U_z$, there are diffeomorphisms : $\phi : U_z \to \mathbb{R}^n$, $\phi_{s_i} : \pi_{s_i}(U_z) \to \mathbb{R}^{n_{s_i}}$ such that the diagram :

$$\begin{tikzcd}
U_z \arrow[r, "\phi"] \arrow[d, "\pi_{s_i}"] & \mathbb{R}^n \arrow[d, "\sigma_{n_{s_i}}"] \\
\pi_{s_i}(U_z) \arrow[r, "\phi_{s_i}"] \arrow[d, "\pi_{s_j}"] & \mathbb{R}^{n_{s_i}} \arrow[d, "\sigma_{n_{s_j}}"] \\
\pi_{s_j}(U_z) \arrow[r, "\phi_{s_j}"] & \mathbb{R}^{n_{s_j}}
\end{tikzcd} \tag{3.3}$$

commutes for all $i \geq j$, where $\sigma_d : \mathbb{R}^n \to \mathbb{R}^d$ denotes the projection onto the last $d^{\text{th}}$ coordinates.

A chart of $X^\circ$ of this type will be called a *distinguished chart.*

In the same time, the compatibility conditions 3.2 ensure that

$$K_{s_\kappa}|_{U_z} = \{0\} \times U_z \subset K_{s_{\kappa-1}}|_{U_z} \subset \cdots \subset K_{s_0}|_{U_z} \subset TU_z \ . \tag{3.4}$$

So in particular we have for $0 \leq i \leq j \leq \kappa$ :

$$q_{s_i} \circ q_{s_j} = q_{s_j} \circ q_{s_i} = q_{s_j} \text{ on } TU_z. \tag{3.5}$$

Let $T_z^{s_i}$ be the orthogonal complement of $K_{s_i}|_{U_z}$ into $K_{s_{i-1}}|_{U_z}$ for all $i = 1, \ldots, \kappa$ and $T_z^{s_0}$ the orthogonal complement of $K_{s_0}|_{U_z}$ into $TU_z$, thus the filtration (3.4) leads to the following graduation:

$$TU_z = \underbrace{T_z^{s_\kappa}}_{=K_{s_{\kappa-1}}|_{U_z}} \oplus T_z^{s_{\kappa-1}} \oplus T_z^{s_{\kappa-2}} \oplus \cdots \oplus T_z^{s_0} \ . \tag{3.6}$$

It will be usefull for our purpose to understand how the differential map of a distinguished chart behaves.
According to
(3.7)

$$\begin{aligned} \mathbb{R}^n \quad &= \mathbb{R}^{n-n_\kappa} \times \mathbb{R}^{n_\kappa - n_{\kappa-1}} \times \cdots \times \mathbb{R}^{n_1 - n_0} \times \mathbb{R}^{n_0}, \\ &\simeq \mathbb{R}^{n-n_\kappa} \oplus \mathbb{R}^{n_\kappa - n_{\kappa-1}} \oplus \cdots \oplus \mathbb{R}^{n_1 - n_0} \oplus \mathbb{R}^{n_0} \\ &\quad \text{where } \mathbb{R}^{n_j - n_{j-1}} \simeq \{0\} \times \mathbb{R}^{n_j - n_{j-1}} \times \{0\} \subset \mathbb{R}^{n-n_j} \times \mathbb{R}^{n_j - n_{j-1}} \times \mathbb{R}^{n_{j-1}} = \mathbb{R}^n \end{aligned}$$

and with the previous notation, notice that 3.3 implies that the differential map:
(3.8)

$$d\phi : K_{s_{\kappa-1}}|_{U_z} \oplus T_z^{s_{\kappa-1}} \oplus T_z^{s_{\kappa-2}} \oplus \cdots \oplus T_z^{s_0} \longrightarrow \mathbb{R}^n \times (\mathbb{R}^{n-n_{s_{\kappa-1}}} \oplus \cdots \oplus \mathbb{R}^{n_{s_1} - n_{s_0}} \oplus \mathbb{R}^{n_{s_0}})$$

is upper triangular with respect to these decompositions. Equivalently, one can say that $d\phi$ respects the filtrations:

$$L_0 \subset L_1 \subset \cdots L_{l-1} \subset TU_z, \quad L_i := \oplus_{j=0}^{i} T_z^{s_i} \simeq TU_z / K_{s_i} \simeq \pi_{s_i}^*(Ts_i)|_{U_z}$$

and

$$\mathbb{R}^{s_{s_0}} \subset \mathbb{R}^{n_{s_1}} \subset \cdots \subset \mathbb{R}^{n_{s_\kappa}} = \mathbb{R}^n.$$

In particular, we can consider the diagonal blocks of $d\phi$ with respect to (3.6) and 3.7. Since $d\phi$ is invertible, they provide isomorphisms

$$[d\phi]^j : T_z^{s_j} \longrightarrow \mathbb{R}^{n_{s_j} - n_{s_{j-1}}} \ ; \ j = 1, \ldots, \kappa \ \text{ and } [d\phi_x]^0 : T_z^{s_0} \longrightarrow \mathbb{R}^{n_{s_0}}. \tag{3.9}$$

Of course, the inverse of $d\phi$ is upper triangular with diagonal blocks given by $([d\phi]^j)^{-1}$, $j = 0, 1, \ldots, \kappa + 1$.

Let us see how these decompositions project to the strata.
If $U_z^i$ denotes $\pi_{s_i}(U_z)$, the graduation 3.6 behaves correctely with the inclusion $\pi_{s_i}^*(TU_z^i) \subset TU_z$ and provides each $TU_z^i$ with the graduation :

$$TU_z^i = T_{z,i}^{s_i} \oplus T_{z,i}^{s_{i-1}} \oplus \cdots \oplus T_{z,i}^{s_0}, \quad T_{z,i}^{s_j} = d\pi_{s_i}(T_z^{s_j}). \tag{3.10}$$

Again, each differential map:

$$d(\phi_{s_i}) : T_{z,i}^{s_i} \oplus T_{z,i}^{s_{i-1}} \oplus \cdots \oplus T_{z,i}^{s_0} \longrightarrow \mathbb{R}^{n_{s_i}} \times (\mathbb{R}^{n_{s_i} - d_{n_{i-1}}} \oplus \cdots \oplus \mathbb{R}^{n_{s_1} - n_{s_0}} \oplus \mathbb{R}^{n_{s_0}}) \tag{3.11}$$

is upper triangular with diagonal blocks corresponding to those of $d\phi$ via the appropriate composition with $d\pi_{s_i}$ and diagram (3.3).

Now, with all these ingredients in hand we can define appropriate charts for the tangent space of $X$.

3.2.2. *An atlas for* $T^{\mathsf{S}}X$.
The atlas will contain two kinds of local charts. The kind of these charts will depend on the fact that their domains meet or not a glueing between the differences pieces composing the tangent space $T^{\mathsf{S}}X$, that is the boundary of some $F_s$.

The first kind of charts, called *regular charts* are charts whose domain is contained in $T^{\mathsf{S}}X|_{\overset{\circ}{F_s}}$ for a given stratum $s$ of the stratification. We observe that $T^{\mathsf{S}}X|_{\overset{\circ}{F_s}}$ is a smooth groupoid as an open subgroupoid of ${}^*\pi_s^*(Ts) \rightrightarrows \mathcal{N}_s^\circ$. Thus, regular charts have domains contained in

$$\sqcup_{s\in\mathcal{S}}\overset{\circ}{F_s}$$

and coincide with the usual local charts of the (disjoint) union of the smooth groupoids $\sqcup_{s\in\mathcal{S}}{}^*\pi_s^*(Ts)$.

The second kind of charts, called *deformation charts* (adapted to a stratum $s$), are charts whose domain meets $T^{\mathsf{S}}X|_{\partial F_s}$ for a given stratum $s$, that is, charts around points $\gamma_0 = (p,u,q) \in T^{\mathsf{S}}X|_{\partial F_s}$, $s \in \mathcal{S}$. Their description is much more difficult.
Let $(p,u,q) \in T^{\mathsf{S}}X$ and $s$ such that $p \in \partial F_s$. This means exactly that $\rho_{s_i}(p) = 1$ for some $s_i < s$. Let

$$s_l = s > s_{l-1} > \cdots > s_0 \tag{3.12}$$

be the ordered sequence of strata such that $\rho_{s_i}(p) = 1$, $i = 0,\dots,l-1$. Thanks to the compatibility conditions 3.2 this sequence only depends on $\pi_s(p)$ thus it is the same for $q$. Note that since $p$ belongs to $F_s = O_s \setminus \cup_{s'<s}O_{s'}$, the sequence above correspond exactly to the first $l+1$ terms of $\mathsf{S}_p = \{s \in \mathsf{S} \mid \rho_s(p) \leq 1\}$.

Let us take distinguished charts $\phi : U_p \to \mathbb{R}^n$ around $p$ and $\phi' : U_q \to \mathbb{R}^n$ around $q$. Since $\pi_s(p) = \pi_s(q)$, we can also ask that the corresponding induced charts on the strata satisfy :

$$\pi_{s_i}(U_p) = \pi_{s_i}(U_q) \text{ and } \phi_{s_i} = \phi'_{s_i} \text{ for } i = 0,\dots,l. \tag{3.13}$$

We will note in the sequel $U_i = \pi_{s_i}(U_p)$ and $\phi_i = \phi_{s_i}$ for $i = 0,\dots,l$ and $n_i = n_{s_i}$. According to (3.7) with $l$ instead of $k$ we will write $\phi(x) = (x^{l+1}, x^l, \dots, x^1, x^0)$ and $\phi'(y) = (y^{l+1}, y^l, \dots, y^1, y^0)$.
As in the previous part 3.4 we let : $K_i := K_{s_i}|_{U_p} = \ker(T\pi_{s_i})|_{U_p}$ for $i = 0\cdots l$. We have the filtration:

$$K_l \subset K_{l-1} \subset \cdots \subset K_0 \subset TO_p \tag{3.14}$$

which leads to the corresponding graduation 3.6:

$$TU_p = K_l \oplus T^l \oplus T^{l-1} \oplus \cdots \oplus T^0 \tag{3.15}$$

where $T^i = T_p^{s_i}$ is the orthogonal complement of $K_i$ into $K_{i-1}$ for all $i = 1,\dots,l-1$ and $T^0 = T_p^{s_0}$ the orthogonal complement of $K_0$ into $TU_p$.
As in 3.16 for all $x \in U_p$ the differential map:

$$d\phi_x : K_l \oplus T^l \oplus T^{l-1} \oplus \cdots \oplus T^0 \longrightarrow \mathbb{R}^{n-n_l} \oplus \mathbb{R}^{n_l-n_{l-1}} \oplus \cdots \oplus \mathbb{R}^{n_1-n_0} \oplus \mathbb{R}^{n_0} \tag{3.16}$$

is upper triangular with respect to these decompositions and induces the isomorphisms :

$$[d\phi_x]^j : T^j \longrightarrow \mathbb{R}^{d_j-d_{j-1}} \ ;\ j = 0,1,\dots,l \ \text{ and } [d\phi_x]^{l+1} : K_l \longrightarrow \mathbb{R}^{n-d_l} \tag{3.17}$$

Let us also introduce the positive smooth functions:

$$t_i = \sum_{j=0}^{i} \tau \circ \rho_{s_j} \ , \ i = 0, 1, \dots, l \ ; \qquad \theta_i = \prod_{j=i-1}^{l} t_j \ , \ i = 1, \dots, l$$

Note that $t_j$ (resp. $\theta_j$) is strictly positive on $F_{s_i}$ if $j \geq i$ (resp. $j > i$) and vanishes identically if $j < i$ (resp. $j \leq i$).

We are ready to define a deformation chart around the point $(p, u, q)$. The domain will be:

$$\widetilde{U} = T^{\mathsf{S}}X|_{U_q}^{U_p} \tag{3.18}$$

where $U_p$ and $U_q$ are chosen as above, and the chart itself:

$$\widetilde{\phi} : \widetilde{U} \to \mathbb{R}^{2n} \tag{3.19}$$

is defined as follows. For all $(x, v, y) \in \widetilde{U}$, there exists a unique $i$ such that $x \in F_{s_i}$, then $(x, v) \in \pi_{s_i}^*(TU_i)$ can be immersed in $TU_p$ using 3.1 and decomposed as:

$$v = v_i + v_{i-1} + \cdots + v_0 \in T^I \oplus \cdots \oplus T^0.$$

We can set :

$$d\phi_x^{[i]}(v) = [d\phi]^i(v_i) + [d\phi]^{i-1}(v_{i-1}) + \cdots + [d\phi]^0(v_0) \in \{0\} \times \mathbb{R}^{n_i} \subset \mathbb{R}^n \tag{3.20}$$

and define:

$$\widetilde{\phi}(x, v, y) = (\phi(x), \frac{x^{l+1} - y^{l+1}}{\theta_{l+1}(x)}, \dots, \frac{x^{i+1} - y^{i+1}}{\theta_{i+1}(x)}, d\phi_x^{[i]}(v)) \tag{3.21}$$

The map $\widetilde{\phi}$ is clearly injective with inverse defined as follows. For $(x, w) \in \widetilde{\phi}(\widetilde{U})$ and $i$ such that $\phi^{-1}(x) \in F_{s_i}$:

$$\widetilde{\phi}^{-1}(x, w) = \Big(\phi^{-1}(x), (d\phi^{[i]}_{\phi^{-1}(x)})^{-1}(w), \phi'^{-1}(x - \Theta^{[i+1]}(\phi^{-1}(x)) \cdot w)\Big)$$

where, using the decomposition (**??**) for $w = (w^{l+1}, \cdots, w_0) \in \mathbb{R}^n$:

$$\Theta^{[i+1]}(x) \cdot w = \theta_{l+1}(x) w^{l+1} + \cdots + \theta_{i+1}(x) w^{i+1} \in \mathbb{R}^{n-n_i} \times \{0\} \subset \mathbb{R}^n \ .$$

To ensure that $(\widetilde{\phi}, \widetilde{U})$ is a local chart, it remains to check that $\widetilde{\phi}(\widetilde{U})$ is an open subset of $\mathbb{R}^{2n}$. It is easy to see that $\widetilde{\phi}(\overset{\circ}{F}_{s_i})$ is open for every $i \in \{0, \dots, l\}$ so we consider $(p, u, q) \in \widetilde{U}$ such that $p \in \partial F_{s_i}$ for some integer $i$. Let $J = \{i_0, \dots, i_k\} \subset \{0, 1, \dots, i-1\}$ such that:

$$\forall j \in J, \quad \rho_{s_j}(p) = 1 \ .$$

Thus we have:

$$\rho_{s_i}(p) < 1; \quad \forall j \in J, \ \rho_{s_j}(p) = 1; \quad \forall j \notin J \text{ and } j < i, \ \rho_{s_j}(p) > 1 \tag{3.22}$$

by construction, $q$ satisfies the same relations. Set $\widetilde{\phi}(p, u, q) = (\mathbf{x}_0, \mathbf{v}_0)$. Using the Taylor formula and the fact that $\theta_{j+1}$ is negligible with respect to $1 - \rho_{s_j}$ at the region $\rho_{s_j} = 1$, noting also the invariance of $\rho_{s_k}$ with respect to perturbations of points along the fibers of $\pi_{s_{k+1}}, \pi_{s_{k+2}}, \dots$; we prove that there exist an open ball $B_1$ of $\mathbb{R}^n$ centered at $\mathbf{x}_0$ and an open ball $B_2$ of $\mathbb{R}^n$ centered at 0 and containing $\mathbf{v}_0$ such that for all $(\mathbf{x}, \mathbf{v}) \in B_1 \times B_2$, if

$$x = \phi^{-1}(\mathbf{x}) \in F_{s_j} \text{ for } j \in J \text{ or } j = i, \text{ then } y = \phi'^{-1}(\mathbf{x} - \Theta^{[j+1]}(x) \cdot \mathbf{v}) \in F_{s_j} \ .$$

This proves that $(\mathbf{x},\mathbf{v}) \in \operatorname{Im}\widetilde{\phi}$, thus

$$\widetilde{\phi}(p,u,q) \in B_1 \times B_2 \subset \operatorname{Im}\widetilde{\phi}$$

and the required assertion is proved. We end with:

**Theorem 1.** *The collection of regular and deformation charts provides $T^{\mathsf{S}}X$ with a structure of smooth groupoid.*

*Proof.* The compatibility between a regular and a deformation chart contains no issue and is ommitted. We need only to check the compatibility between a deformation chart adapted to a stratum $s$ and a deformation chart adapted to a stratum $t$, when their domains overlap, which implies automatically that $s < t$ or $s > t$ or $s = t$.

Let us work out only the case $s = t$, since the other case is similar. We have here to compare two charts $\widetilde{\phi}$ and $\widetilde{\psi}$ with common domain $\widetilde{U}$ and involving the same chain of strata $s = s_l > s_{l-1} > \cdots > s_0$. The whole notations are as before and $\psi, \psi'$ are the underlying charts of $X^\circ$ allowing the definition of $\widetilde{\psi}$. We note, for the sake of concision, $u^k$ (resp. $u'^k$), $k = l+1, \dots, 0$, the coordinate functions of $u := \psi \circ \phi^{-1}$ (resp. $\psi' \circ \phi'^{-1}$) with respect to the decomposition (3.7) of $\mathbb{R}^n$. Observe, thanks to the particular assumptions made on $\phi, \phi', \psi, \psi'$ (cf.(3.3), (3.13)), that $u^k(\mathbf{x})$ only depends on $\mathbf{x}_k := (\mathbf{x}^k, \mathbf{x}^{k-1}, \dots, \mathbf{x}^0) \in \mathbb{R}^{d_k}$ and that $u^k = u'^k$ for all $k < l+1$. Let $(\mathbf{x},\mathbf{v}) \in \operatorname{Im}\widetilde{\phi}$ and $i$ such that $x = \phi^{-1}(\mathbf{x}) \in F_{s_i}$. Then:

$$\begin{aligned}\widetilde{\psi} \circ \widetilde{\phi}^{-1}(\mathbf{x},\mathbf{v}) &= \Big( u(\mathbf{x}), \tfrac{u^{l+1}(\mathbf{x}) - u'^{l+1}(\mathbf{x} - \Theta^{[i+1]}\cdot\mathbf{v})}{\theta^{l+1}}, \tfrac{u^{l}(\mathbf{x}) - u^{l}(\mathbf{x} - \Theta^{[i+1]}\cdot\mathbf{v})}{\theta^{l}}, \dots \\ &\qquad \dots, \tfrac{u^{i+1}(\mathbf{x}) - u^{i+1}(\mathbf{x} - \Theta^{[i+1]}\cdot\mathbf{v})}{\theta^{i+1}}, d\psi^{[i]} \circ (d\phi^{[i]})^{-1}(\mathbf{v}) \Big)\end{aligned} \tag{3.23}$$

We need to check that the above expression matches smoothly with the corresponding expression for an integer $k \in [i,l]$ when $\theta_k(x)$ (and thus $\theta_{k-1}, \dots, \theta_{i+1}$) goes to zero. For that, the Taylor formula applied to $u^r$, $k \geq r \geq i+1$, shows that the map defined belows is smooth in $(\mathbf{x},\mathbf{v},t)$ where $(\mathbf{x},\mathbf{v})$ are as before and $t = (t_l, t_{l-1}, \dots, t_0) \in \mathbb{R}^{l+1}$ is this time an arbitrary $(l+1)$-uple close to 0:

$$\begin{cases} \frac{u^r(\mathbf{x}) - u^r(\mathbf{x} - \Theta^{[i+1]}\cdot\mathbf{v})}{\theta^r} & \text{if } \theta_r = \Pi_{r-1}^{l} t_j \neq 0 \\ d(u^r)_{\mathbf{x}}(\mathbf{v}^r + t_{r-2}\mathbf{v}^{r-1} + \cdots + t_i \mathbf{v}^{i+1}) & \text{if } \exists j \in \{r-1, r, \dots, l\} \text{ such that } t_j = 0, \end{cases}$$

In our case, $t_j = t_j(x)$ and $t_{k-1}, \dots, t_i$ go to zero, so the second line in the previous expression is just:

$$d(u^r)_{\mathbf{x}}(\mathbf{v}^r)$$

and for obvious matricial reasons:

$$d(u^r)_{\mathbf{x}}(\mathbf{v}^r) = (d\psi)^r \circ ((d\phi)^r)^{-1}(\mathbf{v}^r) = d\psi^{[k]} \circ (d\phi^{[k]})^{-1}(\mathbf{v}^r)$$

Summing up these relations for $r = i+1, \dots, k$, we arrive at the desired identity.

Thus, $T^{\mathsf{S}}X$ is endowed with a structure of smooth manifold. Moreover, in this local charts, the smoothness of all algebraic operations associated with this groupoid is straightforward. □

### 3.2.3. *The Lie algebroid of the tangent groupoid.*

We define

$$\begin{array}{rccc} Q_s : & TX^\circ & \longrightarrow & TX^\circ \\ & (z,V) & \mapsto & \left\{ \begin{array}{ll} (z, \tau_s(z) q_s(z,V)) & \text{if } z \in \mathcal{N}_s^\circ \\ 0 & \text{elsewhere} \end{array} \right. \end{array}$$

By a slight abuse of notation, we will keep the notations $q_s$ and $Q_s$ for the corresponding maps induced on the set of local tangent vector fields on $X^\circ$.

Let $\mathcal{A}$ be the smooth vector bundle $\mathcal{A} := TX^\circ \times [0,1]$ over $X^\circ \times [0,1]$. We define the following morphism of vector bundle :

$$\begin{array}{rccc} \Phi : & \mathcal{A} = TX^\circ \times [0,1] & \longrightarrow & TX^\circ \times T[0,1] \\ & (z,V,t) & \mapsto & (z, tV + \sum_{s \in \mathsf{S}_{sing}} Q_s(z,V); t, 0) \end{array}$$

In the sequel we will give an idea of how one can show that there is a unique structure of Lie algebroid on $\mathcal{A}$ such that $\Phi$ is its anchor map. The Lie algebroid $\mathcal{A}$ is almost injective and so it is integrable, moreover we will see that at a set level $\mathcal{G}_X^t$ must be a groupoid which integrates it [8, 10]. In particular $\mathcal{G}_X^t$ can be equipped with a unique smooth structure such that it integrates the Lie algebroid $\mathcal{A}$.
Now we can state the following :

**Theorem 2.** *There exists a unique structure of Lie algebroid on the smooth vector bundle $\mathcal{A} = TX^\circ \times [0,1]$ over $X^\circ \times [0,1]$ with $\Phi$ as anchor.*

To prove this theorem we will need several lemmas :

**Lemma 1.** *Let $s_0$ and $s_1$ be two strata such that $d(s_0) \leq d(s_1)$.*

*(1) For any tangent vector field $W$ on $X^\circ$, $Q_{s_1}(W)(\tau_{s_0}) = 0$.*
*(2) For any $(z,V) \in TX^\circ$, the following equality holds :*

$$Q_{s_1} \circ Q_{s_0}(z,V) = Q_{s_0} \circ Q_{s_1}(z,V) = \tau_{s_0}(z) Q_{s_1}(z,V) \ .$$

*Proof.* First notice that outside $O_{s_0} \cap O_{s_1}$ either $Q_{s_1}$ hence $Q_{s_1}(W)$ or $\tau_{s_0}$ and $Q_{s_0}$ vanish thus the equalities of (1) and (2) are simply $0 = 0$.
(1) According to the compatibility conditions 3.2 we have $\rho_{s_0} \circ \pi_{s_1} = \rho_{s_0}$ on $O_{s_0} \cap O_{s_1}$. Thus $\rho_{s_0}$ is constant on the fibers of $\pi_{s_1}$ and since $\tau_{s_0} = \tau \circ \rho_{s_0}$, $\tau_{s_0}$ is also constant on the fibers of $\pi_{s_1}$. For any tangent vector field $W$, and any $z \in O_{s_1}^\circ$ the vector $\widetilde{Q_{s_1}}(W)(z)$ is tangent to the fibers of $\pi_{s_1}$ thus $Q_{s_1}(V)(\tau_{s_0}) = 0$ on $O_{s_0} \cap O_{s_1}$.
(2) The result follows from the first remark and the equality 3.5 of the part above.
□

The next lemma ensures that $\Phi$ is almost injective, in particular it is injective in restriction to $X^\circ \times ]0,1]$. A simple calculation shows the following :

**Lemma 6.** *For any $t \in ]0,1]$ the bundle map $\Phi_t$ is bijective, moreover*

$$\Phi_t^{-1}(z) = \frac{1}{t} V - \sum_{s \in \mathsf{S}_{sing}} \frac{1}{(t + t_s(z)) \cdot (t + t_s(z) - \tau_s(z))} Q_s(z,V)$$

*where for any singular stratum $s$ the map $t_s$ is defined as follows :*

$$t_s : X^\circ \to \mathbb{R} \ , \ t_s(z) = \sum_{s_0 \leq s}^{l} \tau_{s_0}(z) \ .$$

Thus in order to prove the theorem 2 it is enough to show that locally the image of the map induced by $\Phi$ from the set of smooth local sections of $\mathcal{A}$ to the set of smooth local tangent vector fields on $X^\circ \times [0,1]$ is stable under the Lie bracket.

*Idea of the proof of Theorem 2.* First notice that outside the closure of $\cup_{s_i \in \mathsf{S}_{sing}} O^\circ_{s_i}$ the image under $\tilde{\Phi}$ of local tangent vector fields is clearly stable under Lie Bracket. Thus using decomposition of the form 3.6 described in the last part and standard arguments it remains to show that if $s_a$ and $s_b$ are strata of depth respectively $a$ and $b$ with $s_a \leq s_b$, if $U$ is an open subset of $X^\circ$, as small as we want contained in $\mathcal{N}_{s_a} \cap \mathcal{N}_{s_b}$, and if $W^\perp, V^\perp,$ $V_a$ and $W_b$ are tangent vector fields on $U$, satisfying :

$V^\perp$ and $V_a$ can be porjected by $\pi_{s_b}$,

$Q_s(W^\perp) = Q_s(V^\perp) = 0$ for any $s \in \mathsf{S}$ ,

$$Q_s(V_a) = \left\{ \begin{array}{l} \tau_s V_a \text{ when } s \leq s_a \\ 0 \text{ elsewhere} \end{array} \right. \text{ and } Q_s(W_a) = \left\{ \begin{array}{l} \tau_s W_a \text{ when } s \leq s_b \\ 0 \text{ elsewhere} \end{array} \right. ,$$

then $[\Phi(W^\perp + W_b), \Phi(V^\perp)]$ and $[\Phi(W_b), \Phi(V_a)]$ are in the image of $\Phi$. In other word, we have to show that the maps $(z,t) \in X^\circ \times ]0,1] \mapsto (\Phi_t^{-1}([\Phi(W^\perp + W_b), \Phi(V^\perp)](z)), t)$ and $(z,t) \in X^\circ \times ]0,1] \mapsto (\Phi_t^{-1}([\Phi(W_b), \Phi(V_a)](z)), t)$ can be extended into smooth local section of $\mathcal{A}$. The result follows from our preceding lemmas and usual calculations. □

Now we can state :

**Theorem 7.** *The groupoid $\mathcal{G}^t_X$ can be equipped with a smooth structure such that its Lie algebroid is $\mathcal{A}$ with $\Phi$ as anchor.*

*Proof.* According to proposition 2 and lemma 6, the Lie algebroid $\mathcal{A}$ is almost injective. Thus according to [10] there is a unique quasi-graphoid $\mathcal{G}(\mathcal{A}) \rightrightarrows X^\circ \times [0,1]$ which integrates $\mathcal{A}$. Suppose for simplicity that for each stratum $s$, $O_s$ is connected (which will ensure that $\mathcal{G}^t_X|_{F^\circ_s \times [0,1]}$ is a quasi-graphoid).

Moreover the map $\Phi$ satisfies :

(i) $\Phi$ induces an isomorphism from $\mathcal{A}_{]0,1]} := \mathcal{A}|_{X^\circ \times ]0,1]}$ to $TX^\circ \times ]0,1]$,

(ii) for any stratum $s$, the Lie algebroid $\mathcal{A}$ restricted over $F^\circ_s \times [0,1]$ to a Lie algebroid $\mathcal{A}_s := \mathcal{A}|_{F^\circ_s \times [0,1]}$ which is isomorphic to the restriction of $\mathcal{A}^t_{\pi_s \times \mathrm{Id}}$ over $F^\circ_s \times [0,1]$.

Thus, again by using the uniqueness of quasi-graphoid integrating a given almost injective Lie algebroid, we obtain :

(i) the restriction of the $\mathcal{G}(\mathcal{A})$ over $X^\circ \times ]0,1]$ is isomorphic to $X^\circ \times X^\circ \times ]0,1] \rightrightarrows X^\circ \times ]0,1]$, the pair groupoid on $X^\circ$ parametrized by $]0,1]$,

f

(ii) or each stratum $s$ the restriction over $F^\circ_s \times [0,1]$ is equal to $\mathcal{G}^t_X|_{F^\circ_s \times [0,1]}$.

Finally $\mathcal{G}(\mathcal{A}) = \mathcal{G}^t_X$ and there is a unique smooth structure on $\mathcal{G}^t_X$ such that $\mathcal{A}$ is its Lie algebroid.
If some $O_s$ is not connected, replace in the construction of the tangent space the groupoid ${}^*({}^*\pi_s)(Ts)|_{F_s}$ by its $s$-connected component. Let $CT^{\mathsf{S}}X$ and $C\mathcal{G}^t_X$ be the corresponding groupoids. Then the previous proof applied to equip $C\mathcal{G}^t_X$ with a unique smooth structure such that $\mathcal{A}$ is its Lie algebroid. One can then show that there is a unique smooth structure on $\mathcal{G}^t_X$ such that $C\mathcal{G}^t_X$ is its $s$-connected component. □

Thus $T^{\mathsf{S}}X$ which is the restriction of $\mathcal{G}_X^t$ to the saturated set $X^\circ \times \{0\}$ can be also endowed with a smooth structure.

3.2.4. *Standard projection from the tangent space onto the space.* The space of orbits of $X^\circ/T^{\mathsf{S}}X$ is equivalent to $X$ in the sense that there is a canonical isomorphism $C_0(X^\circ/T^{\mathsf{S}}X) \simeq C(X)$.

Moreover one can define a continuous surjective map : $p : X^\circ \to X$ such that $p \circ r = p \circ s$.

To do so, let us first choose a smooth non decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f([0,1]) = 0$ and $f|_{[2,+\infty[} = \mathrm{Id}$.

If $X$ has depth 0, $X^\circ = X$ and we just take $p = id$.

Let us assume that the depth of $X$ is $k > 0$. For each singular stratum $s$ recall that there is an isomorphism 2.1 :

$$\Psi_s : \mathcal{N}_s \to c_{\pi_s} L_s = L_s \times [0, +\infty[/ \sim_s \quad .$$

we define the map

$$p_s : \mathcal{N}_s \longrightarrow \mathcal{N}_s$$

given by

$$\Psi_s \circ p_s \circ \Psi_s^{-1}[x,t] = [x, f(t)]$$

For each integer $i \in [0, k-1]$, we define a continuous map:

$$p_i : X \longrightarrow X$$

by setting $p_i(z) = p_s(z)$ if $z$ belongs to $\mathcal{N}_s$ for some singular stratum of depth $i$ and $p_i(z) = z$ elsewhere. In particular, $p_i|_{O_s} = \pi_s$ for every stratum $s$ of depth $i$. We define now the map :

$$p = p_0 \circ p_1 \circ \cdots \circ p_{k-1}$$

it satisfies the following properties :

**Lemma 8.** *1) Let $r, s : T^{\mathsf{S}}X \to X^\circ$ be the target and source maps of the* S*-tangent space of $X$. Then:*

$$p \circ r = p \circ s$$

*2) The restriction of $p$ from $X^\circ$ to $X$ is onto.*

*Proof.* 1) Let $\gamma \in T^{\mathsf{S}}X$. There exists a unique stratum $s$ such that $\gamma \in {}^*\pi_s^*(TS)$. If $s$ is regular, then $r(\gamma) = s(\gamma)$ so the result is trivial here. Let us assume that $s$ is singular and let $i < k$ its depth. By definition, $r(\gamma)$ and $s(\gamma)$ belong to $O_s$. For each stratum $t \geq s$ of depth $j \geq i$, we have everywhere it makes sense:

$$\pi_t \circ p_t = \pi_t, \ \pi_s \circ \pi_t = \pi_s, \ \rho_s \circ \pi_t = \rho_s$$

thus :

$$\rho_s \circ p_t = \rho_s \circ \pi_t \circ p_t = \rho_s \circ \pi_t = \rho_s$$

which proves that $p_j(O_s) = O_s$, and moreover:

$$\pi_s \circ p_t = \pi_s \circ \pi_t \circ p_t = \pi_s \circ \pi_t = \pi_s$$

Recalling that $p_i|_{O_s} = \pi_s|_{O_s}$, this last relation implies:

$$p_i \circ \cdots \circ p_{k-1}|_{O_s} = \pi_s \circ p_{i+1} \circ \cdots \circ p_{k-1}|_{O_s} = \pi_s|_{O_s}$$

Since by definition we also have $\pi_s(r(\gamma)) = \pi_s(s(\gamma))$, we conclude that:

$$p(r(\gamma)) = p_0 \circ \cdots p_{i-1} \circ \pi_s(r(\gamma)) = p_0 \circ \cdots p_{i-1} \circ \pi_s(s(\gamma)) = p(s(\gamma))$$

2) We have $p_{k-1}(X^\circ) = X^\circ \bigcup_{d(s)=k-1} s$ and for all $j$ $p_{j-1}(X^\circ \cup \bigcup_{d(s)\geq j} s) = X^\circ \cup \bigcup_{d(s)\geq j-1} s$. $\square$

Such a map $p$ will be called a *standard projection* for $T^{\mathsf{S}}X$ on $X$.

## 4. Poincaré duality for stratified pseudo-manifolds

Let $X$ be a closed stratified pseudomanifold of depth $k \geq 0$. The tangent groupoid $\mathcal{G}^t_X$ is a deformation groupoid which gives rise to a $KK$-element, called a *pre-Dirac*, in the $K$-homology of $T^{\mathsf{S}}X$:

$$\delta_X = [e_0]^{-1} \otimes [e_1] \in KK(C^*(T^{\mathsf{S}}X), \mathbb{C}). \tag{4.1}$$

Here $e_0 : C^*(\mathcal{G}^t_X) \to C^*(T^{\mathsf{S}}X)$ and $e_1 : C^*(\mathcal{G}^t_X) \to \mathcal{K}(L^2(X^o))$ are the usual evaluation morphisms. This $K$-homology class will be pulled-back using a morphism:

$$\Psi_X : C^*(T^{\mathsf{S}}X) \otimes C(X) \to C^*(T^{\mathsf{S}}X),$$

wich is defined in the sequel.
If $k = 0$, $X$ is smooth and we have:

$$\Psi_X(a \otimes b)(V) = b(x).a(V) \tag{4.2}$$

for every $V \in T_xX$, $x \in X$.
In order to extend it for arbitrary pseudomanifolds we will use a standard projection $p : X^\circ \to X$ for $T^{\mathsf{S}}X$ onto $X$ as in paragraph 3.2.4. We obtain imediately:

**Lemma 9.** *The following formula:*

$$\forall a \in C^*(T^{\mathsf{S}}X),\ b \in C(X),\ \gamma \in T^{\mathsf{S}}X, \quad \Psi_X(a \otimes b)(\gamma) = b(p \circ r(\gamma)).a(\gamma)$$

*defines a homomorphism:*

$$\Psi_X : C^*(T^{\mathsf{S}}X) \otimes C(X) \to C^*(T^{\mathsf{S}}X)$$

*where $p : X^\circ \to X$ is a standard projection for $T^{\mathsf{S}}X$.*

We set:

$$D_X = \Psi_X^*(\delta_X) = [\Psi_X] \otimes \delta_X \in KK(C^*(T^{\mathsf{S}}X) \otimes C(X), \mathbb{C}). \tag{4.3}$$

This section is devoted to the proof of the main theorem:

**Theorem 3.** *Let $X$ be a closed stratified pseudomanifold. The $K$-homology class $D_X$ is a Dirac element, that is, it provides a Poincaré duality between the algebras $C^*(T^{\mathsf{S}}X)$ and $C(X)$.*

The proof is done thanks to a recursive argument based on the unfolding process to reduce the depth of the stratification.

If depth$(X) = 0$ the result is well known, and that $D_X$ is a Dirac element is a consequence of [11]. Let $k \geq 0$ and assume that the theorem 3 holds for all closed stratified pseudomanifolds with depth $\leq k$.
Let $X$ be a closed stratified pseudomanifold of depth $k+1$. We let $\mathcal{S}_0$ denotes the set of minimal strata in $X$.

4.0.5. *The first step.* We will first consider

$$W = \bigcup_{s\in\mathcal{S}_0} \rho_s^{-1}([0,2[), \quad \overline{W} = \bigcup_{s\in\mathcal{S}_0} \rho_s^{-1}([0,2]),$$

$$X_b = X \setminus W, \quad L := \cup_{s\in\mathsf{S}_0} \rho_s^{-1}(2) \quad \text{and} \quad O_b = X_b \setminus L.$$

$L$ is a pseudomanifold of depth $k$ and $X_b$ is of depth $k$.
Now we state :

$$T^{\mathsf{S}} X_b = T^{\mathsf{S}} X|_{X_b^\circ} \text{ and } T^{\mathsf{S}} W = T^{\mathsf{S}} X|_{W^\circ} \ .$$

We are going to define Dirac elements associated to $O_b$ and $\overline{W}$.
The inclusion

$$i' : T^{\mathsf{S}} O_b \hookrightarrow T^{\mathsf{S}} X$$

provides a $K$-homological element $\delta_{O_b} = i'^*(\delta_X) \in K^0(C^*(T^{\mathsf{S}} O_b))$ which coincides with the $K$-class coming with the deformation groupoid:

$$\mathcal{G}^t_{O_b} = \mathcal{G}^t_X|_{O_b^\circ \times [0,1]} \overset{i'}{\hookrightarrow} \mathcal{G}^t_X$$

Next, if $a \in C^*(T^{\mathsf{S}} X_b)$ and $b \in C_0(O_b)$, then $\Psi_X(\widetilde{a} \otimes \widetilde{b})$, where $f(\widetilde{a}) = a$ and $\widetilde{b} \in C(X)$ is the extension by 0 of $b$, does not depend on the choice of $\widetilde{a}$ and lies in the ideal $C^*(T^{\mathsf{S}} O_b)$. In other words, $\Psi_X$ induces a homomorphism:

$$\Psi_{O_b} : C^*(T^{\mathsf{S}} X_b) \otimes C_0(O_b) \longrightarrow C^*(T^{\mathsf{S}} O_b) \tag{4.4}$$

We thus get:

$$D_{O_b} = [\Psi_{O_b}] \otimes \delta_{O_b} \in KK(C^*(T^{\mathsf{S}} X_b) \otimes C_0(O_b), \mathbb{C}). \tag{4.5}$$

We can do similar things with $T^{\mathsf{S}} W$ and $\overline{W}$. Indeed the inclusions $i : T^{\mathsf{S}} W \hookrightarrow T^{\mathsf{S}} X$ and $\mathcal{G}^t_W = \mathcal{G}^t_X|_{W^\circ \times [0,1]} \hookrightarrow \mathcal{G}^t_X$ show that $i^*\delta_X$ coincides with the element $\delta_W$ associated with the deformation groupoid $\mathcal{G}^t_W$. Moreover, $\Psi_X$ induces as before a homomorphism:

$$\Psi_{\overline{W}} : C^*(T^{\mathsf{S}} W) \otimes C(\overline{W}) \longrightarrow C^*(T^{\mathsf{S}} W) \tag{4.6}$$

So we can set:

$$D_{\overline{W}} = [\Psi_{\overline{W}}] \otimes \delta_W \in KK(C^*(T^{\mathsf{S}} W) \otimes C_0(\overline{W}), \mathbb{C}). \tag{4.7}$$

Since $X_b^\circ$ is a closed saturated subspace of $X^\circ$, we have the following exact sequence of $C^*$-algebras:

$$0 \longrightarrow C^*(T^{\mathsf{S}} W) \xrightarrow{i} C^*(T^{\mathsf{S}} X) \xrightarrow{f} C^*(T^{\mathsf{S}} X_b) \longrightarrow 0 \tag{4.8}$$

We also have

$$0 \longrightarrow C_0(O_b) \xrightarrow{j} C(X) \xrightarrow{g} C(\overline{W}) \longrightarrow 0 \tag{4.9}$$

We have the following proposition.

**Proposition 2.** *The elements* $D_{O_b} \in KK(C^*(T^{\mathsf{S}} X_b) \otimes C_0(O_b), \mathbb{C})$ *and* $D_{\overline{W}} \in KK(C^*(T^{\mathsf{S}} W) \otimes C(\overline{W}), \mathbb{C})$ *satisfy :*
*1) The following equalities hold:*

$$[i] \underset{C^*(T^{\mathsf{S}} X)}{\otimes} D_X = [g] \underset{C(\overline{W})}{\otimes} D_{\overline{W}} \tag{4.10}$$

$$[f] \underset{C^*(T^{\mathsf{S}} X_b)}{\otimes} D_{O_b} = [j] \underset{C(X)}{\otimes} D_X \tag{4.11}$$

*2) If $\partial_f \in KK(C_0(\mathbb{R}) \otimes C^*(T^{\mathsf{S}}X_b), C^*(T^{\mathsf{S}}W))$ and $\partial_g \in KK(C_0(\mathbb{R}) \otimes C(\overline{W}), C(O_b))$ denote the boundary elements in $KK$-theory associated respectively with (4.8) and (4.9), then:*

$$\partial_g \underset{C^*(T^{\mathsf{S}}X_b)}{\otimes} D_{O_b} = \partial_f \underset{C(\overline{W})}{\otimes} D_{\overline{W}} \tag{4.12}$$

This proposition has the following obvious corollary:

**Corollary 1.** *The following diagrams commute for all $C^*$-algebras $C$ and $D$:*

(4.13)

$$\begin{tikzcd}
\cdots \arrow[r] & KK_i(C, D \otimes C_0(O_b)) \arrow[r] \arrow[d, "\cdot \underset{C_0(O_b)}{\otimes} D_{O_b}"] & KK_i(C, D \otimes C(X)) \arrow[r] \arrow[d, "\cdot \underset{C(X)}{\otimes} D_X"] & KK_i(C, D \otimes C(\overline{W})) \arrow[r] \arrow[d, "\cdot \underset{C(\overline{W})}{\otimes} D_{\overline{W}}"] & KK_{i+1}(C, D \otimes C_0(O_b)) \arrow[r] \arrow[d, "\cdot \underset{C_0(O_b)}{\otimes} D_{O_b}"] & \cdots \\
\cdots \arrow[r] & KK_i(C \otimes C^*(T^{\mathsf{S}}X_b), D) \arrow[r] & KK_i(C \otimes C^*(T^{\mathsf{S}}X), D) \arrow[r] & KK_i(C \otimes C^*(T^{\mathsf{S}}W), D) \arrow[r] & KK_{i+1}(C \otimes C^*(T^{\mathsf{S}}X_b), D) \arrow[r] & \cdots
\end{tikzcd}$$

(4.14)

$$\begin{tikzcd}
\cdots \arrow[r] & KK_i(C, D \otimes C^*(T^{\mathsf{S}}W)) \arrow[r] \arrow[d, "\cdot \underset{C^*(T^{\mathsf{S}}W)}{\otimes} D_{\overline{W}}"] & KK_i(C, D \otimes C^*(T^{\mathsf{S}}X)) \arrow[r] \arrow[d, "\cdot \underset{C^*(T^{\mathsf{S}}X)}{\otimes} D_X"] & KK_i(C, D \otimes C^*(T^{\mathsf{S}}X_b)) \arrow[r] \arrow[d, "\cdot \underset{C^*(T^{\mathsf{S}}X_b)}{\otimes} D_{O_b}"] & KK_{i+1}(C, D \otimes C^*(T^{\mathsf{S}}W)) \arrow[r] \arrow[d, "\cdot \underset{C^*(T^{\mathsf{S}}W)}{\otimes} D_{\overline{W}}"] & \cdots \\
\cdots \arrow[r] & KK_i(C \otimes C(\overline{W}), D) \arrow[r] & KK_i(C \otimes C(X), D) \arrow[r] & KK_i(C \otimes C_0(O_b), D) \arrow[r] & KK_{i+1}(C \otimes C(\overline{W}), D) \arrow[r] & \cdots
\end{tikzcd}$$

Before going to the proof of this proposition, lets us see how it will enables us to show our theorem.

The following proposition is obvious once you have noticed that for any minimal stratum $s$ the inclusion of $C^*$-algebras $C^*(T^{\mathsf{S}}X|_{\rho_s^{-1}([0,2[)}) \subset C^*(T^{\mathsf{S}}X|_{\rho_s^{-1}([0,1[)})$ is a $KK$-equivalence, and that the groupoid $T^{\mathsf{S}}X|_{\rho_s^{-1}([0,1[)} = T^{\mathsf{S}}X|_{O_s} = {}^*\pi_s^*(Ts)$ is Morita equivalent to $Ts$.

**Proposition 3.** *There is a $KK$-equivalence $I = C^*(T^{\mathsf{S}}W) \sim C^*(TS)$ under which the element $D_{\overline{W}}$ corresponds to the Dirac element $D_S$ of the disjoint union of closed smooth manifolds $S = \cup_{s \in \mathcal{S}_{min}} s$. In particular, $D_{\overline{W}}$ is a Dirac element.*

At this stage, we have the commutativity of (4.13) and (4.14). We already know that $D_{\overline{W}}$ is a Dirac element, so if we are able to prove that $D_{O_b}$ is also a Dirac element then the five lemma and the lemma 2 will end the proof of theorem 3.

4.0.6. *The second step : $D_{O_b}$ is a Dirac element.* Let us go back to the compact pseudomanifold of depth $k$ coming from the unfolding process : $2X$. Recall from 2.0.3 that : $2X = X_+ \cup L_1 \times [-1,1] \cup X_+$ where $X_+ = X \setminus \{z \in X \mid \exists s \in \mathsf{S}_0, \rho_s(z) < 1\}$ and $L_1 := \cup_{s \in \mathsf{S}_0} \{z \in X \mid \rho_s(z) = 1\} \subset X_+$. Notes that $X_+ \simeq X_b$ and $L_1 \simeq L$ thus under obvious identification, we will set

$$2X = X_b^- \cup L \times [-1,1] \cup X_b^+$$

where $X_b^{\pm}$ are two disjoint copies of $X_b$. We consider the exact sequences of $C^*$-algebras:

$$0 \longrightarrow C^*(T^{\mathsf{S}}2X|_{L \times ]-1,1[}) \xrightarrow{\mathsf{i}} C^*(T^{\mathsf{S}}2X) \xrightarrow{\mathsf{f}} C^*(T^{\mathsf{S}}2X|_{X_b^- \sqcup X_b^+}) \longrightarrow 0 \tag{4.15}$$

$$0 \longrightarrow C_0(O_b^- \sqcup O_b^+) \xrightarrow{\mathsf{j}} C(2X) \xrightarrow{\mathsf{g}} C(L \times [-1,1]) \longrightarrow 0 \tag{4.16}$$

Let us define three $K$-homology classes related to these algebras and allowing commutative diagrams analogous to (4.13), (4.14).
Since $2X$ is a compact stratified pseudomanifold of depth $k$, we know that $D_{2X}$ is a Dirac element, and it will be our first $K$-homology class. The homomorphism $\Psi_{2X}$ used in $D_{2X}$ induces a homomorphism:

$$\Psi_{L\times[-1,1]} : C^*(T^{\mathsf{S}}2X|_{L\times]-1,1[}) \otimes C(L\times[-1,1]) \longrightarrow C^*(T^{\mathsf{S}}2X|_{L\times]-1,1[}) \tag{4.17}$$

while the subgroupoid:

$$\mathcal{G}^t_{L\times]-1,1[} := \mathcal{G}^t_{2X}|_{L\times]-1,1[\cap(2X)^\circ} \tag{4.18}$$

of the tangent groupoid $\mathcal{G}^t_{2X}$ gives rise to an element $\delta_{L\times]-1,1[} \in KK(C^*(T^{\mathsf{S}}2X|_{L\times]-1,1[}),\mathbb{C})$.
The second $K$-homology class will be:

$$D_{L\times[-1,1]} := [\Psi_{L\times[-1,1]}]\otimes\delta_{L\times]-1,1[} \in KK(C^*(T^{\mathsf{S}}2X|_{L\times]-1,1[})\otimes C(L\times[-1,1]),\mathbb{C}) \tag{4.19}$$

The third $K$-homology class is built by doubling $D_{O_b}$. Denoting $J^\pm = C_0(O_b^\pm))$ and $P^\pm = C^*(T^{\mathsf{S}}2X|_{X_b^\pm})$, we can write:

$$D_{O_b} =: D^\pm_{O_b} \in K^0(C^*(T^{\mathsf{S}}X_b)^\pm \otimes J^\pm),$$

and we set:

$$D_{2O_b} = D^-_{O_b}\oplus D^+_{O_b} \in K^0(P^-\otimes J^-)\oplus K^0(P^+\otimes J^+) \subset K^0(C^*(T^{\mathsf{S}}2X|_{X_b^-\sqcup X_b^+})\otimes C_0(O_b^-\sqcup O_b^+)) \tag{4.20}$$

**Proposition 4.** *(1)The element $D_{L\times[-1,1]}$ is a Dirac element.*
*(2) The following equalities hold :*

$$[\mathsf{i}] \underset{C^*(T^{\mathsf{S}}2X)}{\otimes} D_{2X} = [\mathsf{g}] \underset{C(L\times[-1,1])}{\otimes} D_{L\times[-1,1]} \tag{4.21}$$

$$[\mathsf{f}] \underset{C^*(T^{\mathsf{S}}2X|_{X_b^-\sqcup X_b^+})}{\otimes} D_{2O_b} = [\mathsf{j}] \underset{C(2X)}{\otimes} D_{2X} \tag{4.22}$$

*2) If $\partial_{\mathsf{f}} \in KK(C_0(\mathbb{R})\otimes C^*(T^{\mathsf{S}}2X|_{X_b^-\sqcup X_b^+}), C^*(T^{\mathsf{S}}2X|_{L\times]-1,1[}))$ and $\partial_{\mathsf{g}} \in KK(C_0(\mathbb{R})\otimes C(L\times[-1,1]), C(O_b^-\sqcup O_b^+))$ denote the boundary elements in $KK$-theory associated respectively with (4.15) and (4.16), then:*

$$\partial_{\mathsf{g}} \underset{C^*(T^{\mathsf{S}}2X|_{X_b^-\sqcup X_b^+})}{\otimes} D_{2O_b} = \partial_{\mathsf{f}} \underset{C(L\times[-1,1])}{\otimes} D_{L\times[-1,1]} \tag{4.23}$$

As previously, we get immediately :

**Corollary 2.** *The following two diagrams commute for all $C^*$-algebras $C$ and $D$:*

$$\begin{tikzcd}
\cdots \arrow[r] & KK_i(C, D\otimes C_0(O_b^-\sqcup O_b^+)) \arrow[r] \arrow[d, "\cdot \underset{C_0(O_b^-\sqcup O_b^+)}{\otimes} D_{2O_b}"] & KK_i(C, D\otimes C(2X)) \arrow[r] \arrow[d, "\cdot \underset{C(2X)}{\otimes} D_{2X}"] & \cdots \\
 & & & \cdots \\
\cdots \arrow[r] & KK_i(C\otimes C^*(T^{\mathsf{S}}2X|_{X_b^-\sqcup X_b^+}), D) \arrow[r] & KK_i(C\otimes C^*(T^{\mathsf{S}}2X, D)) \arrow[r] & \cdots
\end{tikzcd}$$

(4.24)

$$\begin{tikzcd}
 & \cdots \arrow[r] & KK_i(C, D\otimes C(L\times[-1,1])) \arrow[r] \arrow[dd, "\cdot \underset{C(L\times[-1,1])}{\otimes} D_{L\times[-1,1]}"] & KK_{i+1}(C, D\otimes C_0(O_b^- \sqcup O_b^+)) \arrow[r] \arrow[dd, "\cdot \underset{C_0(O_b^- \sqcup O_b^+)}{\otimes} D_{2O_b}"] & \cdots \\
\cdots & & & & \\
 & \cdots \arrow[r] & KK_i(C\otimes C^*(T^{\mathsf{S}}2X|_{L\times]-1,1[}), D) \arrow[r] & KK_{i+1}(C\otimes C^*(T^{\mathsf{S}}2X|_{X_b^- \sqcup X_b^+}), D) \arrow[r] & \cdots
\end{tikzcd}$$

*and :*

$$\begin{tikzcd}
\cdots \arrow[r] & KK_i(C, D\otimes C^*(T^{\mathsf{S}}2X|_{L\times]-1,1[})) \arrow[r] \arrow[dd, "\cdot \underset{C^*(T^{\mathsf{S}}2X|_{L\times]-1,1[})}{\otimes} D_{L\times[-1,1]}"] & KK_i(C, D\otimes C^*(T^{\mathsf{S}}2X)) \arrow[r] \arrow[dd, "\cdot \underset{C^*(T^{\mathsf{S}}2X)}{\otimes} D_{2X}"] & \cdots & \\
 & & & & \cdots \\
\cdots \arrow[r] & KK_i(C\otimes C(L\times[-1,1], D) \arrow[r] & KK_i(C\otimes C(2X), D) \arrow[r] & \cdots &
\end{tikzcd}$$

(4.25)

$$\begin{tikzcd}
 & \cdots \arrow[r] & KK_i(C, D\otimes C^*(T^{\mathsf{S}}2X|_{X_b^- \sqcup X_b^+})) \arrow[r] \arrow[dd, "\cdot \underset{C^*(T^{\mathsf{S}}2X|_{X_b^- \sqcup X_b^+})}{\otimes} D_{2O_b}"] & KK_{i+1}(C, D\otimes C^*(T^{\mathsf{S}}2X|_{L\times]-1,1[})) \arrow[r] \arrow[dd, "\cdot \underset{C^*(T^{\mathsf{S}}2X|_{L\times]-1,1[})}{\otimes} D_{L\times[-1,1]}"] & \cdots \\
\cdots & & & & \\
 & \cdots \arrow[r] & KK_i(C\otimes C_0(O_b^- \sqcup O_b^+), D) \arrow[r] & KK_{i+1}(C\otimes C(L\times[-1,1]), D) \arrow[r] & \cdots
\end{tikzcd}$$

Thus, since $D_{2X}$ and $D_{L\times[-1,1]}$ are Dirac elements, the five lemma yields that $D_{2O_b}$ is a Dirac element, and this implies that $D^+_{O_b} = D_{O_b}$ is also a Dirac element. The proof of theorem 3 is thus done.

Finally it remains to proove the propositions 2 and 4 above.

4.0.7. *Proof of propositions 2 and 4.* The only tricky point is the commutativity with the boundary maps, that is assertion 2 of proposition 2 and assertion 3 of proposition 2.

Let us start with the proof of proposition 2.

1) We check only (4.21), since (4.22) is proved similarly. We have:

$$\begin{aligned}
[i] \underset{C^*(T^{\mathsf{S}}X)}{\otimes} D_X &= [(i\otimes \mathrm{Id}_X)\circ \Psi_X]\otimes \delta_X \\
&= [(\mathrm{Id}_{T^{\mathsf{S}}X}\otimes g)\circ \Psi_{\overline{W}}\circ i]\otimes \delta_X \\
&= [g] \underset{C(\overline{W})}{\otimes} ([\Psi_{\overline{W}}]\otimes [i]\otimes \delta_X)
\end{aligned}$$

and $[i]\otimes \delta_X = \delta_W$ as noted before.

2) Recall that $\partial\overline{W} = \partial X_b = L$ and insert the exact sequence (4.8) into the following commutative diagram of $C^*$-algebras, where the new morphisms are made with obvious inclusions of open subgroupoids and restrictions onto closed saturated subgroupoids:

(4.26)
$$\begin{tikzcd}
0 \arrow[r] & C^*(T^{\mathsf{S}}W) \arrow[r] \arrow[d, "="] & C^*(T^{\mathsf{S}}X) \arrow[r, "f"] \arrow[d, "p'"] & C^*(T^{\mathsf{S}}X_b) \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & C^*(T^{\mathsf{S}}W) \arrow[r] & C^*(T^{\mathsf{S}}\overline{W}) \arrow[r, "f'"] & C^*(T^{\mathsf{S}}L\times\mathbb{R}) \arrow[r] & 0
\end{tikzcd}$$

Let us call $\partial_{f'}$ the boundary element in $KK$-theory associated with the bottom exact sequence. By functoriality, we have:

$$p^*(\partial_{f'}) = [p] \otimes \partial_{f'} = \partial_f$$

Similarly we may consider:

$$\begin{tikzcd}
0 \arrow[r] & C_0(O_b) \arrow[r] \arrow[d, "="] & C(X) \arrow[r, "g"] \arrow[d, "q'"] & C(\overline{W}) \arrow[r] \arrow[d, "q"] & 0 \\
0 \arrow[r] & C_0(O_b) \arrow[r] & C(X_b) \arrow[r, "g'"] & C(L) \arrow[r] & 0
\end{tikzcd} \tag{4.27}$$

which leads to the relation:

$$q^*(\partial_{g'}) = [q] \otimes \partial_{g'} = \partial_g$$

Thus, the following diagram commutes:

$$\begin{tikzcd}
 & & K^0(C^*(T^{\mathsf{S}}X_b) \otimes C_0(O_b)) \arrow[rr, "\tau_{C^*(T^{\mathsf{S}}X_b)}(\partial_{g'})"] \arrow[d, "\tau_{C^*(T^{\mathsf{S}}X_b)}(\partial_g)"'] & & K^1(C^*(T^{\mathsf{S}}X_b) \otimes C(L)) \arrow[dll, "(1\otimes q)^*"] \\
K^0(C^*(T^{\mathsf{S}}W) \otimes C(\overline{W})) \arrow[rr, "\tau_{C(\overline{W})}(\partial f)"] \arrow[d, "\tau_{C(\overline{W})}(\partial f')"] & & K^1(C^*(T^{\mathsf{S}}X_b) \otimes C(\overline{W})) & & \\
K^1(C^*(T^{\mathsf{S}}L \times \mathbb{R}) \otimes C(\overline{W})) \arrow[urr, "(p\otimes 1)^*"'] & & & & K^1(C^*(T^{\mathsf{S}}L \times \mathbb{R}) \otimes C(L)) \arrow[llll, "(1\otimes q)^*"] \arrow[ull, "(p\otimes q)^*"] \arrow[uu, "(p\otimes 1)^*"']
\end{tikzcd} \tag{4.28}$$

The proof of the assertion 2) in the proposition 2 is thus reduced to the next lemma.

**Lemma 10.** *There exists* $\alpha_L \in K^1(C^*(T^{\mathsf{S}}L \times \mathbb{R}) \otimes C(L))$ *such that*

$$\tau_{C(\overline{W})}(\partial f') \otimes D_{\overline{W}} = (1 \otimes q)^*(\alpha_L) \tag{4.29}$$

*and*

$$\tau_{C^*(T^{\mathsf{S}}X_b)}(\partial_{g'}) \otimes D_{O_b} = (p \otimes 1)^*(\alpha_L) \tag{4.30}$$

*Moreover,* $\alpha_L$ *corresponds to the Dirac element* $D_L$ *through the Bott periodicity isomorphism* $K^1(C^*(T^{\mathsf{S}}L \times \mathbb{R}) \otimes C(L)) \simeq K^0(C^*(T^{\mathsf{S}}L) \otimes C(L))$.

*Proof of the lemma.* Firstly, the morphism $\Psi_X$ induces by restriction the morphisms:

$$\Psi_L : \ C^*(T^{\mathsf{S}}L) \otimes C(L) \longrightarrow C^*(T^{\mathsf{S}}L), \tag{4.31}$$

and

$$\Psi'_{\overline{W}} : \ C^*(T^{\mathsf{S}}\overline{W}) \otimes C(\overline{W}) \longrightarrow C^*(T^{\mathsf{S}}\overline{W}). \tag{4.32}$$

Of course $\Psi'_{\overline{W}}$ extends $\Psi_{\overline{W}}$. These morphisms fit into the following commutative diagram:
(4.33)

$$\begin{tikzcd}
0 \arrow[r] & C^*(T^{\mathsf{S}}W)\otimes C(\overline{W}) \arrow[r] \arrow[d, "\Psi_{\overline{W}}"] & C^*(T^{\mathsf{S}}\overline{W})\otimes C(\overline{W}) \arrow[r, "f'\otimes 1"] \arrow[d, "\Psi'_{\overline{W}}"] & C^*(T^{\mathsf{S}}L\times\mathbb{R})\otimes C(\overline{W}) \arrow[r] \arrow[d, "\Psi_L\circ(1\otimes q)"] & 0 \\
0 \arrow[r] & C^*(T^{\mathsf{S}}W) \arrow[r] & C^*(T^{\mathsf{S}}\overline{W}) \arrow[r, "f'"] & C^*(T^{\mathsf{S}}L\times\mathbb{R}) \arrow[r] & 0
\end{tikzcd}$$

We note that $\tau_{C(\overline{W})}(\partial f')$ is precisely the boundary element $\partial_{f'\otimes 1}$ of the top exact sequence above and it follows by functoriality again that the left hand side of (4.29) can be rewritten as follows:

$$\begin{aligned}
\tau_{C(\overline{W})}(\partial_{f'})\otimes D_{\overline{W}} &= \partial_{f'\otimes 1}\otimes[\Psi_{\overline{W}}]\otimes\delta_W \\
&= (\Psi_L\circ(1\otimes q))^*(\partial_{f'}\otimes\delta_W) \\
&= (1\otimes q)^*\circ\Psi_L^*(\partial_{f'}\otimes\delta_W)
\end{aligned}$$

Similarly,
(4.34)

$$\begin{tikzcd}
0 \arrow[r] & C^*(T^{\mathsf{S}}X_b)\otimes C_0(O_b) \arrow[r] \arrow[d, "\Psi_{O_b}"] & C^*(T^{\mathsf{S}}X_b)\otimes C(X_b) \arrow[r, "1\otimes g'"] \arrow[d, "\Psi_{X_b}"] & C^*(T^{\mathsf{S}}X_b)\otimes C(L) \arrow[r] \arrow[d, "\Psi_L\circ(p\otimes 1)"] & 0 \\
0 \arrow[r] & C^*(T^{\mathsf{S}}O_b) \arrow[r] & C^*(T^{\mathsf{S}}X_b) \arrow[r, "p"] & C^*(T^{\mathsf{S}}L\times\mathbb{R}) \arrow[r] & 0
\end{tikzcd}$$

This gives by functoriality for the left hand side of (4.30):

$$\begin{aligned}
\tau_{C^*(T^{\mathsf{S}}X_b)}(\partial_{g'})\otimes D_{O_b} &= \partial_{g'\otimes 1}\otimes[\Psi_{O_b}]\otimes\delta_{O_b} \\
&= (p\otimes 1)^*\circ\Psi_L^*(\partial_p\otimes\delta_{O_b})
\end{aligned}$$

We are now going to check that

$$\partial_{f'}\otimes\delta_W = \partial_p\otimes\delta_{O_b} \tag{4.35}$$

by computing explicitely these elements. To understand the left hand side above, we glue the groupoids $\mathcal{G}^t_W$ and $\mathcal{G}^t_L\times\mathbb{R}$ into a new smooth groupoid:

$$\mathcal{G}^t_{\overline{W}} := \mathcal{G}^t_W \sqcup \mathcal{G}^t_L\times\mathbb{R},$$

Let us explain this glueing. Consider the following deformation groupoid associated with the locally trivial bundle $\pi_L : L\to S$:

$$\widetilde{\mathcal{G}}^t_{\pi_L} = ]-\infty,0[\times {}^*\pi_L^*(TS)|_{L^\circ}\sqcup[0,+\infty[\times T^{\mathsf{S}}L$$

It can be considered as an extended version (because of the presence of positive values of the deformation parameter) of a *fibered tangent groupoid*, or Thom groupoid, associated with the bundle $\pi_L$. This groupoid is provided with a smooth structure, compatible with those of ${}^*\pi_L^*(TS)|_{L^\circ}$ and $T^{\mathsf{S}}L$, using the function $\tau$ already used in the previous glueings. On the other hand, $\tau$ gives us an action $\phi$ of $\mathbb{R}$ onto itself by the (complete) flow of the vector field $\tau(h)\partial_h$. This gives a smooth isomorphism of groupoids:

$$\mathbb{R}\rtimes_\phi\mathbb{R}\simeq]-\infty,0[\times]-\infty,0[\sqcup[0,+\infty[\times\mathbb{R} = T^{\mathsf{S}}\mathbb{R}_+$$

where $\mathbb{R}_+$ is regarded as a conical pseudomanifold with vertex 0. This action makes sense on the deformation parameter in $\widetilde{\mathcal{G}}^t_{\pi_L}$, it is trivial when the deformation parameter is positive and thus can be arbitrarily restricted in this part, leading to a new presentation of $T^{\mathsf{S}}\overline{W}$. Indeed, there is a smooth isomorphism:

$$\widetilde{\mathcal{G}}^t_{\pi_L}|_{]-\infty,1]} \rtimes_\phi \mathbb{R} \simeq {}^*\pi^*(TS)|_{O_S^\circ} \sqcup T^{\mathsf{S}}L \times [1,2] \times \mathbb{R} \simeq T^{\mathsf{S}}\overline{W}$$

where $O_S = \cup_{s\in\mathcal{S}_{min}} O_s$, $\pi$ is the projection map $O_S \to S$ coinciding with $\pi_s$ on each $O_s$ and a reparametrization $]-\infty,1] \simeq ]0,2]$ is implicitly used. In fact we can describe $\mathcal{G}^t_{\overline{W}}$ in the same way by replacing $\widetilde{\mathcal{G}}^t_{\pi_L}$ by a *two deformation variables* fibered tangent groupoid:

$$\widetilde{\mathcal{G}}^T_{\pi_L} = (\mathbb{R}\times[0,1]) \setminus \{(h,0) \mid h \geq 0\} \times {}^*\pi_L^*(TS)|_{L^\circ} \sqcup \{(h,0) \mid h\geq 0\} \times T^{\mathsf{S}}L$$

We replace $\tau : \mathbb{R} \to [0,1]$ by a smooth function $T : \mathbb{R}\times[0,1] \to [0,1]$ in such a way that $T(h,0) = \tau(h)$, $T(h,k) > 0$ if $k > 0$ and $h < 1$, $T(h,k) = 0$ if $h \geq 1$, and $T(h,k) = 1$ as soon as $k \geq \varepsilon$ and $h \leq 1-\varepsilon$ for some small fixed positive number $\varepsilon$. We can also ask that $T(h,k)$ does not depend on $k$ when $k \geq \varepsilon$ and that $T(h,\varepsilon) = \tau(h-1)$.

We then replace $\phi$ by the (complete) flow $\Phi$ of the vector field $T(h,k)\partial_h$, this gives us an action of $\mathbb{R}$ onto $\widetilde{\mathcal{G}}^T_{\pi_L}$ and an isomorphism:

$$\widetilde{\mathcal{G}}^T_{\pi_L} \rtimes_\Phi \mathbb{R}|_{]-\infty,1]\times[0,1]} \simeq \mathcal{G}^t_{\overline{W}}$$

which allows us to provide $\mathcal{G}^t_{\overline{W}}$ with the smooth structure of $\widetilde{\mathcal{G}}^T_{\pi_L} \rtimes_\Phi \mathbb{R}$. Now, the restriction at the value 1 of the remaining deformation parameter $k$ in $\mathcal{G}^t_{\overline{W}}$ gives, using the identification $W^\circ \simeq ]0,+2[\times L^\circ$ coming from the stratification data:

$$\mathcal{G}^t_{\overline{W}}|_{t=1} = W^\circ \times W^\circ \sqcup L^\circ \times L^\circ \times \mathbb{R} \simeq (L^\circ \times L^\circ) \times (]-\infty,0] \rtimes_\phi \mathbb{R}) =: G_b(\overline{W})$$

The groupoid $G_b(\overline{W})$ is analogous to the defined in [23] for manifolds with boundary. In particular, the arguments in [22] prove that $G_b(\overline{W})$ has vanishing $K$-theory. This implies that the boundary element

$$\partial_{f''} \in KK(C_0(\mathbb{R}) \otimes C^*(L^\circ \times L^\circ \times \mathbb{R}), C^*(W^\circ \times W^\circ))$$

associated with the exact sequence:

$$0 \longrightarrow C^*(W^\circ \times W^\circ) \longrightarrow C^*(G_b(\overline{W})) \xrightarrow{f''} C^*(L^\circ \times L^\circ \times \mathbb{R}) \longrightarrow 0 \tag{4.36}$$

is an isomorphism and it is easy to check that it coincides with the inverse of the Bott projector $b$ after the obvious identifications:

$$KK(C_0(\mathbb{R}) \otimes C^*(L^\circ \times L^\circ \times \mathbb{R}), C^*(W^\circ \times W^\circ)) \simeq KK(C_0(\mathbb{R}^2), \mathbb{C})$$

We invoke now the commutative diagram of exact sequences:

(4.37)

$$\begin{tikzcd}
0 \arrow[r] & C^*(T^{\mathsf{S}}W) \arrow[r] & C^*(T^{\mathsf{S}}\overline{W}) \arrow[r, "f'"] & C^*(T^{\mathsf{S}}L \times \mathbb{R}) \arrow[r] & 0 \\
0 \arrow[r] & C^*(\mathcal{G}^t_W) \arrow[r] \arrow[u, "e_0^W"'] \arrow[d, "e_1^W"] & C^*(\mathcal{G}^t_{\overline{W}}) \arrow[r] \arrow[u, "e_0^{\overline{W}}"'] \arrow[d, "e_1^{\overline{W}}"] & C^*(\mathcal{G}^t_L \times \mathbb{R}) \arrow[r] \arrow[u, "e_0^L\otimes 1"'] \arrow[d, "e_1^L\otimes 1"] & 0 \\
0 \arrow[r] & C^*(W^\circ \times W^\circ) \arrow[r] & C^*(G_b(\overline{W})) \arrow[r, "f''"] & C^*(L^\circ \times L^\circ \times \mathbb{R}) \arrow[r] & 0
\end{tikzcd}$$

which gives by functoriality, using the identification $C_0(\mathbb{R}) \otimes C^*(L^\circ \times L^\circ \times \mathbb{R}) \simeq C_0(\mathbb{R}^2) \otimes C^*(L^\circ \times L^\circ)$:

$$\partial_{f'} \otimes [e_0^W]^{-1} \otimes [e_1^W] = \tau_{C_0(\mathbb{R}^2)}([e_0^L]^{-1} \otimes [e_1^L]) \otimes \partial_{f''}$$

and, after inserting the Morita equivalences $C^*(W^\circ \times W^\circ) \sim \mathbb{C}$, $C^*(L^\circ \times L^\circ) \sim \mathbb{C}$:

$$\partial_{f'} \otimes \delta_W = \delta_L \underset{\mathbb{C}}{\otimes} b^{-1}$$

We proceed in the same way to manage the right hand side $\partial_p \otimes \delta_{O_b}$ of (4.35): we consider the tangent groupoid

$$\mathcal{G}^t_{X_b} = \mathcal{G}^t_{O_b} \sqcup \mathcal{G}^t_L \times \mathbb{R}$$

and the same arguments as above lead to

$$\partial_p \otimes \delta_{O_b} = \delta_L \underset{\mathbb{C}}{\otimes} b^{-1}$$

which completes the proof of the lemma. □

*Proof of proposition 4.* (1)The groupoid isomorphism

$$T^{\mathsf{S}}2X|_{L\times]-1,1[} \simeq T^{\mathsf{S}}L \times T]-1,1[$$

yields an isomorphism:

$$C^*(T^{\mathsf{S}}2X|_{L\times]-1,1[}) \simeq C^*(T^{\mathsf{S}}L) \otimes C^*(T]-1,1[) \tag{4.38}$$

Under this isomorphism, we see that:

$$[\Psi_{L\times[-1,1]}] \otimes \delta_{L\times]-1,1[} = ([\Psi_L] \otimes \delta_L) \underset{\mathbb{C}}{\otimes} ([\Psi_{[-1,1]}] \otimes \delta_{]-1,1[}) \tag{4.39}$$

where $\delta_L \in KK(C^*(T^{\mathsf{S}}L), \mathbb{C})$ comes from the tangent groupoid of $L$ while $\delta_{]-1,1[} \in KK(C^*(T]-1,1[), \mathbb{C})$ comes from the tangent groupoid of $]-1,1[$. One easily see that $\delta_{]-1,1[}$ corresponds to the inverse of the Bott projector $b \in K_0(C_0(\mathbb{R}^2))$ via the isomorphisms $C^*(T]-1,1[) \simeq C^*(T\mathbb{R}) \simeq C_0(\mathbb{R}^2)$. In particular, using the $KK$-equivalence $C([-1,1]) \sim \mathbb{C}$, one gets that:

$$D_{[-1,1]} := [\Psi_{[-1,1]}] \otimes \delta_{]-1,1[} \in KK(C^*(T]-1,1[) \otimes\ C([-1,1]), \mathbb{C})$$

is a Dirac element. Since $L$ is a stratified pseudomanifold of depth $k$, we know that $D_L$ is a Dirac element, so is its product over $\mathbb{C}$ with $D_{[-1,1]}$, which proves using (4.39) that $D_{L\times[-1,1]}$ defined by (4.19) is a Dirac element.
The proof for (2) and (3) is similar to the one of proposition 4. □

4.0.8. *Stratified pseudomanifold with boundary.* As a byproduct of this proof, we have proved that Poincaré duality also hold for compact stratified pseudomanifolds with boundary. Precisely a stratified pseudomanifold with boundary is $(X_b, L, \mathsf{S}_b, N_b)$ where :

1. $X_b$ is a compact separable metrizable space and $L$ is a compact subspace of $X_b$.
2. $\mathsf{S}_b = \{s_i\}$ is a finite partition of $X_b$ into locally closed subset of $X_b$, which are smooth manifolds possibly with boundary. Moreover for each $s_i$ we have
$$s_i \cap L = \partial s_i\ .$$
3. $N_b = \{\mathcal{N}_s, \pi_s, \rho_s\}_{s\in\mathsf{S}_b}$, where $\mathcal{N}_s$ is an open neighborhood of $s$ in $X$, $\pi_s : \mathcal{N}_s \to s$ is a continuous retraction and $\rho_s : \mathcal{N}_s \to [0,+\infty[$ is a continuous map such that $s = \rho_s^{-1}(0)$.

(4) The *double* :

$$X = X_b^- \underset{L}{\cup} X_b^+$$

obtained by gluing two copies of $X_b$ along $L$ together with the partition $\mathsf{S} := \{s_i \ |\partial s_i = \emptyset\} \cup \{s_i \underset{\partial s_i}{\cup} s_i\} \cup \{s_i \ |\partial s_i = \emptyset\}$ and the set of control data $N = \{\tilde{\mathcal{N}_s}, \tilde{\pi}_s, \tilde{\rho}_s\}_{s \in \mathsf{S}}$ where

$$\mathcal{N}_s = \mathcal{N}_{s_i}, \ \pi_s = \pi_{s_i}, \ \rho_s = \rho_{s_i} \text{ if } s = s_i \text{ with } \partial s_i = \emptyset$$

and

$$\mathcal{N}_s = \mathcal{N}_{s_i} \underset{\mathcal{N}_{s_i} \cap L}{\cup} \mathcal{N}_{s_i}, \ \pi_s|_{\mathcal{N}_{s_i} \setminus L} = \pi_{s_i}, \ \rho_s|_{\mathcal{N}_{s_i} \setminus L} = \rho_{s_i} \text{ elsewhere}$$

is a stratified pseudomanifold.

We let $O_b := X_b \setminus L$. According to the previous work, one can define the tangent spaces :

$$T^{\mathsf{S}} X_b := T^{\mathsf{S}} X|_{X_b} \text{ and } T^{\mathsf{S}} O_b := T^{\mathsf{S}} X|_{O_b}$$

We deduce the following :

**Theorem 4.** *The $C^*$-algebras $C^*(T^{\mathsf{S}} X_b)$ and $C_0(O_b)$ are Poincaré Dual as well as the $C^*$-algebras $C^*(T^{\mathsf{S}} O_b)$ and $C(X_b)$.*

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