

## Lower bound for the Néron-Tate height

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(joint work with Vincent Bosser)

We propose a totally explicit lower bound for the Néron-Tate height of algebraic points of infinite order of abelian varieties.

Let  $k$  be a number field of degree  $D = [k : \mathbb{Q}]$ . Let  $A$  be an abelian variety defined over a fixed subfield of  $k$ , of dimension  $g$ . Let  $L$  be a polarization of  $A$ . We denote by  $\widehat{h}_L$  the Néron-Tate height on  $A(k)$  relative to  $L$ . It is well-known that, for  $p \in A(k)$ , we have  $\widehat{h}_L(p) = 0$  if and only if  $p$  is a torsion point, i.e.  $np = 0$  for some positive integer  $n$ . The general problem of bounding from below  $\widehat{h}_L(p)$  when  $p \in A(k)$  is not a torsion point has been often tackled in the literature, overall from the point of view of the dependence on  $D$  (Lehmer's problem) or on the Faltings height  $h_F(A)$  of  $A$  (Lang-Silverman conjecture). Moreover most of results concern elliptic curves or abelian varieties with complex multiplication. Let us cite two emblematic results due to David Masser, valid in great generality ( $A_{\text{tors}}$  is the set of torsion points) [3, 4].

**Theorem (Masser, 1985-86)** *In the above setting, there exist positive constants  $c(A, \varepsilon)$  and  $c(k, g)$ , depending only on  $A, \varepsilon$  and on  $k, g$  respectively, such that, for all  $\varepsilon > 0$  and all  $p \in A(k) \setminus A_{\text{tors}}$ , one has*

$$\widehat{h}_L(p)^{-1} \leq c(A, \varepsilon) D^{2g+1+\varepsilon} \quad \text{and} \quad \widehat{h}_L(p)^{-1} \leq c(k, g) \max(1, h_F(A))^{2g+1}.$$

Unfortunately, there is no bound which takes into account both degree and Faltings height (at this level of generality). Up to now, there is only one published bound for  $\widehat{h}_L(p)^{-1}$  which is totally explicit in all parameters. It is due to Bruno Winckler (PhD thesis, 2015, [8]) valid for a CM elliptic curve  $A$ . His bound looks like Dobrowolski-Laurent's one:  $c(A)D(\max(1, \log D)/\max(1, \log \log D))^3$  with a constant  $c(A)$  quite complicated (but explicit). We propose here the following much simpler bound.

**Theorem 1.** *Let  $(A, L)$  be a polarized abelian variety over  $k$  and  $p \in A(k) \setminus A_{\text{tors}}$ . Then we have*

$$\widehat{h}_L(p)^{-1} \leq \max(D + g^g, h_F(A))^{10^5 g}.$$

Note that the bound does not depend on the polarization  $L$ . The proof of this theorem involves two ingredients, namely a generalized period theorem and Minkowski's convex body theorem.

Let us explain the first one. Let  $\sigma: k \hookrightarrow \mathbb{C}$  be a complex embedding. By extending the scalars we get a complex abelian variety  $A_\sigma = A \times_\sigma \text{Spec } \mathbb{C}$  isomorphic to the torus  $t_{A_\sigma}/\Omega_{A_\sigma}$  composed with the tangent space at the origin  $t_{A_\sigma}$  and with the period lattice  $\Omega_{A_\sigma}$  of  $A_\sigma$ . From the Riemann form associated to  $L_\sigma$ , we get an hermitian norm  $\|\cdot\|_{L, \sigma}$  on  $t_{A_\sigma}$  (see for instance [1, § 2.4]). For  $\omega \in \Omega_{A_\sigma}$ , let  $A_\omega$  be the smallest abelian subvariety of  $A_\sigma$  such that  $\omega \in t_{A_\omega}$ . Actually  $A_\omega$  is an abelian variety defined over a number field  $K/k$  of relative degree  $\leq 2(9g)^{2g}$

(Silverberg [7]). A *period theorem* consists of bounding from above the geometrical degree  $\deg_L A_\omega$  in terms of  $g, D, \|\omega\|_{L,\sigma}$  and  $h_F(A)$ . Such a theorem is useful to bound the minimal isogeny degree between two isogeneous abelian varieties ([1, 2, 5, 6]). A *generalized period theorem* consists of replacing  $\omega$  by a logarithm  $u \in t_{A_\sigma}$  of a  $k$ -rational point  $p \in A(k)$  (we have  $\sigma(p) = \exp_{A_\sigma}(u)$ ). In this setting we have the following bound (written in a very simplified form).

**Theorem 2.** *If  $u \neq 0$  then*

$$(\deg_L A_u)^{1/(2 \dim A_u)} \leq \left( D\widehat{h}_L(p) + \|u\|_{L,\sigma}^2 \right) \max(D + g^g, h_F(A))^{50}.$$

The proof of Theorem 2 extends that of the period theorem [1] using Gel'fond-Baker's method with Philippon-Waldschmidt's approach and some adelic geometry. Since it is long enough, we shall only explain in the rest of the exposition how to deduce Theorem 1 from Theorem 2. The very classical argument is to use the pigeonhole principle. Here we replace it by the more convenient Minkowski's first theorem. Let  $E$  be the  $\mathbb{R}$ -vector space  $\mathbb{R} \times t_{A_\sigma}$  endowed with the Euclidean norm

$$\|(a, x)\|^2 := a^2 D\widehat{h}_L(p) + \|a \cdot u + x\|_{L,\sigma}^2.$$

In  $(E, \|\cdot\|)$  stands the lattice  $\mathbb{Z} \times \Omega_{A_\sigma}$  whose determinant is  $D\widehat{h}_L(p)h^0(A, L)^2$ . So, by Minkowski, there exists  $(\ell, \omega) \in \mathbb{Z} \times \Omega_{A_\sigma} \setminus \{0\}$  such that

$$(\star) \quad D\widehat{h}_L(\ell p) + \|\ell u + \omega\|_{L,\sigma}^2 \leq \gamma_{2g+1} \left( D\widehat{h}_L(p)h^0(A, L)^2 \right)^{1/(2g+1)}$$

where  $\gamma_{2g+1} \leq g + 1$  is the Hermite constant. Since  $p$  is assumed to be non-torsion, the logarithm  $\ell u + \omega$  of  $\sigma(\ell p)$  is not 0 and Theorem 2 gives a lower bound for the left-hand side of inequality  $(\star)$ , involving a lower bound for  $\widehat{h}_L(p)$ . Nevertheless, at this stage, the dimension  $h^0(A, L)$  of the global sections space of the polarization is still in the bound. To remove it, we use Zarhin's trick by replacing  $A$  with  $(A \times \widehat{A})^4$  (here  $\widehat{A}$  is the dual abelian variety), endowed with a principal polarization compatible to  $L$ . Then the Néron-Tate height of  $p$  remains unchanged whereas Faltings height and dimension of  $A$  are multiplied by 8, ruining the numerical constant but also making  $h^0(A, L)$  disappear.

## REFERENCES

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