A large Deviation Principle for 2D Stochastic Navier-Stokes Equation

Mathieu Gourcy*

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Abstract

In this paper one specifies the ergodic behavior of the 2D-stochastic Navier-Stokes equation by giving a Large Deviation Principle for the occupation measure for large time. It describes the exact rate of exponential convergence. The considered random force is non-degenerate and compatible with the strong Feller property.

Keywords : Stochastic Navier-Stokes Equation, Large Deviations, Occupation Measure.

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1 Introduction and Results

Let us introduce the two-dimensional incompressible Navier-Stokes equation (NSE in short), which describes the evolution of an incompressible fluid. It is most frequently written in terms of the velocity field u at each point ξ in the domain. Let D be a bounded domain of $\mathbb{R}^2$ with smooth boundary ∂D, we consider the equation

$$\frac{du(t,\xi)}{dt} + (u(t,\xi).\nabla)u(t,\xi) - \Delta u(t,\xi) + \nabla P(t,\xi) = g(\xi) + \eta(t,\xi) \ , \tag{1.1}$$

for $t \geq 0, \xi \in D$, and subject to the incompressibility condition

$$\text{div } u(t,\xi) = 0 \quad , \quad t \geq 0 \quad , \quad \xi \in D \ ,$$

the boundary condition

$$u(t,\xi) = 0 \quad , \quad t \geq 0 \quad , \quad \xi \in \partial D$$

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*Laboratoire de Mathématiques, CNRS-UMR 6620, Université Blaise Pascal, 63177 Aubière, France, mathieu.gourcy@math.univ-bpclermont.fr

and the initial condition $u(\xi, 0) = x(\xi)$ for $\xi \in D$. In (1.1), the function g is a deterministic external forcing and η a random forcing taking form (1.5) detailed later. For simplicity, we have written the equation in dimensionless form, and with the viscosity equal to 1. It is also possible to work with periodic boundary conditions.

In the usual way, by applying to (1.1) the projection to the linear space of divergence free vector fields (often called Leray projector), the pressure P disappears from the equations. Let $\mathcal{V}$ be the space of $\mathcal{C}^\infty$ 2-dimensional vector fields $u(\xi)$ on D with compact support strictly contained in D, and satisfying div $u(\xi) = 0$. We denote by H (respectively V) its closure in the L^2 topology (respectively in the H^1 topology). According to the classical theory of Navier-Stokes equations, we have

$$\begin{array}{rcl} H & = & \{u \in [L^2(D)]^2 \text{ s.t. div } u = 0, \quad \gamma_\nu(u) = 0\} \\ V & = & \{u \in [H_0^1(D)]^2 \text{ s.t. div } u = 0\} \end{array}$$

where $\gamma_\nu(u)$ is a trace that coincides with $u \cdot \nu$ for smooth u, ν being the outer normal to ∂D (see for example the book of Temam [26, Chap.1]).

Moreover $|\cdot|$ and $\langle \cdot, \cdot \rangle$ stand for the norm and inner product in H. Identifying H with its dual H' and identifying H' with a subspace of V' (the dual space of V) we have $V \subset H \subset V'$, and we also denote by $\langle \cdot, \cdot \rangle$ the duality between V and V'. Let us define the linear operator A in H by the formula

$$Au = -P_{div} \Delta u \quad , \quad \forall u \in D(A) = \left(H^2(D)\right)^2 \cap V$$

and the bilinear operator $B : V \times V \to V'$ by $B(u, v) = P_{div}(u \cdot \nabla)v$ where P_{div} is the L^2 projection operator onto the space H of divergence-free vector field. The space V coincides with $D\left(A^{\frac{1}{2}}\right)$ and is endowed with the norm $|x|_V = |A^{\frac{1}{2}} x|$. The unbounded linear operator A is closed, positive and selfadjoint in H, with compact inverse A^{-1}. Following classical spectral theory, we denote by $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ the eigenvalues of A and by $e_1, e_2, ..$ a corresponding complete orthonormal system of eigenvectors. Finally we can define the fractional powers A^α and their domains, the spaces $D(A^\alpha)$ equipped with the norm $|u|_\alpha := |A^\alpha u|$, that correspond to Sobolev spaces $[H^{2\alpha}(D)]^2$ with the suitable conditions. We remark in particular that $D(A^\alpha)$ is dense in $D\left(A^\beta\right)$ for $\alpha > \beta \geq 0$, and that for any $\alpha > 0$,

$$|x| \leq \frac{1}{{\lambda_1}^\alpha} |A^\alpha x|. \tag{1.2}$$

The incompressibility condition implies for any u, v, z in V

$$\langle B(u, v), v \rangle = 0 \quad , \quad \langle B(u, v), z \rangle = -\langle B(u, z), v \rangle. \tag{1.3}$$

By applying to each term of the NSE the projection operator P_{div}, we formally rewrite the system (1.1) in the abstract form :

$$dX(t) + AX(t)dt + B\left(X(t), X(t)\right) dt = f dt + P_{div} \eta(t, \xi) dt \ ; \ X(0) = x \tag{1.4}$$

where $X(t)$ is identified with $u(t, \cdot)$ and $f = P_{div} g$ (the irrotational components of f and η are absorbed in the term $\nabla P(t, \xi)$, see [26]).

In mathematical literature, it is common to assume the random force $P_{div}\eta(t,\xi)dt$ to be random fields that are smooth in x, while as a function of time t they are white noises (see [7] for example). Since we are interested in the long time behavior of equation (1.4), both the forcing terms are assumed to be stationary in order to have an autonomous system (i.e $f \in H$ do not depend on the time variable t, whereas the white noise is by definition a stationary process).

Let us describe the form of the noise. We assume that

$$P_{div}\eta(t,\xi)dt = GdW(t) \tag{1.5}$$

where $W(t)$ is a standard cylindrical Wiener process in H (see [7]) defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $G : H \to H$ is a bounded linear operator satisfying

$$D\left(A^{2\alpha}\right) \subset Im(G) \subset D\left(A^{\frac{1}{2}+\varepsilon}\right) \quad , \text{ for some } \frac{1}{4} < \alpha < \frac{1}{2} \ , \ \varepsilon \in \left(0, 2\alpha - \frac{1}{2}\right] . \tag{1.6}$$

Here, $Im(G)$ is the range of the operator G. Roughly speaking, the first embedding in (1.6) means that the noise is not too degenerate, and the second implies that $tr(G^*G) < \infty$ (i.e, the energy injected by the random force is finite) and also gives us more spatial regularity for the solution to (1.4).

The stochastic NSE has been intensively studied since the work of Bensoussan and Temam [2]. Here we adopt the approach of generalized solutions given by Flandoli [13] (see for instance Flandoli and Gatarek [14] for solutions in law called martingale solutions). On a fixed probability space he built an associated Markovian semigroup of transition with an invariant measure. Under a condition of non-degenerance of type (1.6) for the noise, the uniqueness of the invariant measure was first showed by Flandoli and Maslowski [15] and in a more classical way by Ferrario [12]. Goldys and Maslowski [17] established recently the exponentially fast convergence of transition measures to the invariant measure. More references on the degenerated noise case will be presented in Remarks 1.6.

Under (1.6), it is known (see next section for more details and references) that the solution $X(t)$ of (1.4) is a Markov process with a unique invariant measure μ supported by $D(A^\alpha)$. By the uniqueness (see [7]), μ is ergodic in the sense that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \Psi\left(X(t,x)\right) dt = \int \Psi d\mu \quad \mathbb{P}\text{-a.s.}$$

for all initial conditions x and all continuous and bounded functions Ψ.

In the sequel, $\mathbb{P}_x$ is the law on $C(\mathbb{R}^+, H)$ of the Markov process with $x \in H$ as initial state, and for any initial measure ν on H, let $\mathbb{P}_\nu(\cdot) := \int_H \mathbb{P}_x(\cdot)\nu(dx)$. Our aim is to establish the large deviation principle (LDP in short) for the occupation measure L_t of the solution X to (1.4) given by

$$L_t(A) := \frac{1}{t}\int_0^t \delta_{X(s)}(A)ds, \quad \forall A \in \mathcal{B}(H)$$

where δ_a is the Dirac measure at a, and $\mathcal{B}(H)$ the Borelian σ-field in H. Notice that L_t is a random measure on H but in fact supported on $D(A^\alpha)$, because of the regularity of X, given by Theorem 2.2 below. The LDP for empirical measure is one of the strongest ergodic result for the long time behavior of Markov processes. This is a traditional subject in probability since the pioneering work of Donsker and Varadhan [10], however in our infinite dimensional setting their assumptions are not satisfied (see [8, 9] for an introduction to Large Deviation theory).

Let us begin with some necessary definitions. For $E = H$ or $E = D\left(A^{\alpha}\right)$, let $M_1(E)$ be the space of probability measures (resp. $M_b(E)$ the space of signed σ-additive measures of bounded variation) on E equipped with the Borel σ-field $\mathcal{B}$. On the space $M_b(E)$ (or $M_1(E)$ its subspace), we consider $\sigma\left(M_b(E), b\mathcal{B}(E)\right)$, the so called τ-topology of convergence against measurable and bounded functions which is much stronger than the usual weak convergence topology $\sigma\left(M_b(E), C_b(E)\right)$. The duality relation between $\nu \in M_b(E)$ and $\Psi \in b\mathcal{B}(E)$ will be denoted by

$$\nu(\Psi) := \int_E \Psi d\nu.$$

It is time to state our main result.

Theorem 1.1. *Let $f \in H$ and $\frac{1}{4} < \alpha < \frac{1}{2}$ a fixed number such that (1.6) holds (throughout this paper). Let $0 < \lambda_0 < \frac{\lambda_1}{2\|Q\|}$, where $\|Q\|$ is the norm of $Q := GG^*$ as an operator in H and*

$$\mathcal{M}_{\lambda_0,L} := \Big\{\nu \in M_1\left(H\right) \ / \ \int_H e^{\lambda_0|x|^2}\nu(dx) \le L\Big\}. \tag{1.7}$$

The family $\mathbb{P}_\nu(L_T \in \cdot)$ as $T \to +\infty$ satisfies the large deviation principle (LDP) with respect to the topology τ, with speed T and the rate function J, uniformly for any initial measure ν in $\mathcal{M}_{\lambda_0,L}$ where $L > 1$ is any fixed number. Here the rate function $J : M_1\left(H\right) \to [0, +\infty]$ is the level-2 entropy of Donsker-Varadhan defined by (3.2) below. More precisely we have:

i) J is a good rate function on $M_1\left(H\right)$ equipped with the topology τ of the convergence against bounded and borelian functions, i.e., $[J \le a]$ is τ-compact for every $a \in \mathbb{R}^+$.

ii) for all open set G in $M_1\left(H\right)$ with respect to the topology τ,

$$\liminf_{T\to\infty} \frac{1}{T} \log \inf_{\nu\in\mathcal{M}_{\lambda_0,L}} \mathbb{P}_\nu(L_T \in G) \ge -\inf_G J$$

iii) for all closed set F in $M_1\left(H\right)$ with respect to the topology τ,

$$\limsup_{T\to\infty} \frac{1}{T} \log \sup_{\nu\in\mathcal{M}_{\lambda_0,L}} \mathbb{P}_\nu(L_T \in F) \le -\inf_F J \ .$$

Furthermore, we have for μ the invariant measure, and $\forall \nu \in M_1(H)$,

$$J(\nu) < +\infty \implies \nu \ll \mu,\ \nu(V) = 1 \text{ and } \int_V |A^{\frac{1}{2}}x|^2 d\nu < +\infty. \tag{1.8}$$

The LDP w.r.t. the topology τ is much stronger than that w.r.t. the usual weak convergence topology as in Donsker-Varadhan [10]. Indeed, this theorem and the estimate (4.3) below have interesting consequences for which the topology τ is crucial. For instance, LDP can be deduced for many non-continuous physical observables of the system such as $|x|_V := |A^{\frac{1}{2}}x|$, the Sobolev norm (which is not continuous on H).

Corollary 1.2. *Let* $(\mathbb{B}\ ,\ \|\cdot\|_{\mathbb{B}})$ *a separable Banach space, and* $f : D\left(A^{\frac{1}{2}}\right) \to \mathbb{B}$ *a measurable function bounded on the balls* $\{x \text{ s.t } |A^{\frac{1}{2}}x| \leq R\}$ *for any* $R > 0$*, and satisfying*

$$\lim_{|A^{\frac{1}{2}}x|\to\infty} \frac{\|f(x)\|_{\mathbb{B}}}{|A^{\frac{1}{2}}x|^2} = 0. \tag{1.9}$$

Then $\mathbb{P}_\nu\left(L_T(f) \in \cdot\right)$ *satisfies the LDP on* $\mathbb{B}$ *with speed* T *and the rate function* I_f *given by*

$$I_f(y) = \inf\{J(\nu);\quad J(\nu) < +\infty,\ \nu(f) = y\},\ \forall y \in \mathbb{B}$$

uniformly over initial distributions ν *in* $\mathcal{M}_{\lambda_0,L}$ *(for any* $L > 1$*).*

As a particular case of Corollary 1.2, we can state the

Proposition 1.3. *As* $T \to \infty$*, the family*

$$\mathbb{P}_\nu\left(\frac{1}{T}\int_0^T X_t dt \in \cdot\right)$$

satisfies a LDP on $D\left(A^{\frac{1}{2}}\right)$ *with speed* T *and the rate function* I *defined by*

$$I(z) = \inf\left\{J(\nu);\quad J(\nu) < +\infty,\ \nu\left(x\right) = z\right\},\ \forall z \in D\left(A^{\frac{1}{2}}\right)$$

uniformly over initial distributions ν *in* $\mathcal{M}_{\lambda_0,L}$ *(for any* $L > 1$*).*

Remarks 1.4. We give now two examples of noises in our class (1.6). Let us first represent the cylindrical Wiener process $W(t)$ as a series with respect to the system $(e_k)_k$ which diagonalizes A on its domain, and define $Ge_k = \sigma_k e_k$, so that

$$GW(t) = \sum_{k=1}^{\infty} \sigma_k \beta_k(t) e_k$$

where $(\beta_k)_{k\in\mathbb{N}}$ is a family of independent real valued standard Brownian motions. The condition (1.6) is

$$\frac{c}{k^{2\alpha}} \leq \sigma_k \leq \frac{C}{k^{\frac{1}{2}+\varepsilon}}$$

for two positive constants c and C, and k large enough since $\lambda_k \sim k$ as $k \to \infty$. Hence, the cylindrical Wiener process with values in $D\left(A^{2\alpha}\right)$, that is when $\sigma_i\lambda_i^{2\alpha} = 1$, is allowed.

A more general example of noise for which our assumption holds for $\frac{1}{4} < \alpha < \frac{1}{2}$ fixed is $G := A^{-\beta}L$ where L is any linear bounded and invertible operator on H and $\frac{1}{2} < \beta \leq 2\alpha$.

Remarks 1.5. The class (1.7) of initial distributions for the uniform LDP is sufficiently rich. For example, choosing L large enough, it includes all the Dirac probability measures δ_x with x in any ball of H.

Remarks 1.6. During the last years, a lot of progresses have been done in the treatment of very degenerated noises (acting only on a finite number of modes as in Kolmogorov's turbulence theory). In this case Bismut-Elworthy-Li formula become irrelevant, and the Strong Feller property does not hold. However a careful analysis of the dynamic allow to obtain uniqueness of invariant measure. We refer to the works of Weinan E, Mattingly and Sinai (see [11, 23]) where only a finite number N of low modes is forced where N depends on the viscosity. More recently, Hairer, Mattingly and Pardoux (see [20]) have removed this last dependence of N on the viscosity and have established the uniqueness when only four lowest modes are forced. Degenerated kick noises have also been considered by Kuksin and Shirikyan among others (see [22]). For all thoses cases, we believe that the LDP w.r.t the topology τ is false. The LDP w.r.t the weak convergence topology in that degenerated noise case is an interesting open problem. Finally the 3D case is much more delicate. But it seems possible, by selecting solutions to build a Makovian semigroup with strong Feller and irreducibility properties (see Da-Prato, Debussche [6] and Odasso [24]). It is hoped that our method will make it possible to treat this $3D$-case with a non degenerate noise.

Remarks 1.7. In another direction, a Wentzell-Freidlin type large deviation principle was proved by Chang [4] for the paths of the solution when the magnitude of the additive noise tends to zero. This result is extended to the multiplicative noise case by Sritharan and Sundar [25] (see also the recent works of Collina, Livi, Mazzino [5] and Amirdjanova, Xiong [1]).

This paper is organized as follows. In section 2, we recall results on existence and uniqueness of solution, and invariant probability measure for the equation (1.4). In section 3, we present some general facts about Large Deviations for strongly Feller and topologically irreducible Markov processes. Then, in section 4, we prove a useful exponential estimate for the solution, and we do some comments on the rate function which governs the LDP. We establish first this LDP on $D(A^\alpha)$ in section 5, and we extend it on H in section 6. Finally, Proposition 1.3 and Corollary 1.2 are investigated in section 7.

2 Existence and uniqueness results for the solution and the invariant measure

Following the literature ([12, 13, 15, 17] among many others), we say that a progressively measurable process $X(t)$ is a generalized solution of equation (1.4) if

$$X \in C([0,T], H) \cap L^2\left([0,T], D(A^{\frac{1}{4}})\right) \qquad \mathbb{P}-a.s.$$

and the equation is satisfied $\mathbb{P}$-a.s. in the weak sense

$$\begin{aligned} & \langle X(t), y\rangle + \int_0^t \langle X(s), Ay\rangle ds - \int_0^t \langle B\left(X(s), y\right), X(s)\rangle ds \\ = \ & \langle x, y\rangle + t\langle f, y\rangle + \langle GW(t), y\rangle \end{aligned}$$

for all $t \in [0,T]$, $y \in D(A)$ and the initial condition $x \in H$. This definition is justified by the properties (1.3) of the non linearity B, and the Sobolev continuous embedding $D(A^{\frac{1}{4}}) \subset [L^4(D)]^2$ since

$$\langle B\left(X(s), y\right), X(s)\rangle \leq C|A^{\frac{1}{2}}y|\ |X(s)|^2_{[L^4(D)]^2} \leq C|A^{\frac{1}{2}}y|\ |A^{\frac{1}{4}}X(s)|^2.$$

Hence all the terms make sense.

Flandoli [13] proved for the first existence of a solution under the weaker assumption $Im(G) \subset D(A^{\frac{1}{4}+\varepsilon})$: the classical definition of solution was not adapted here because of the low regularity of the noise. However, under our condition (1.6), the noise is more regular and his result can be read as

Theorem 2.1. [13] *Assume that (1.6) holds for* $\frac{1}{4} < \alpha < \frac{1}{2}$.

For all $x \in H$, $f \in D\left(A^{-\frac{1}{2}}\right)$, *there exists a unique generalized solution* X^x *of the equation (1.4) such that* $\mathbb{P}$*-a.e.*

$$X^x \in C([0,T], H) \cap L^2\left([0,T], D(A^{\frac{1}{2}})\right). \tag{2.1}$$

and $X^x - Z \in L^2\left([0,T], D(A^{\frac{1}{2}})\right)$ *where* Z *is the solution to the auxiliary Ornstein-Uhlenbeck equation*

$$dZ(t) + AZ(t) = GdW(t).$$

The family of solutions X^x *for* $x \in H$ *forms a Markov family which admits an invariant measure* μ.

The following step consists in analyzing whether μ is unique. For this purpose topological irreducibility and strong Feller property were investigated. We recall first the definitions.

Denote by E a generic space. Given the solution X^x, a E-valued continuous process starting from $x \in E$, the transition functions $P(t,x,\Gamma) := \mathbb{P}\left(X^x(t) \in \Gamma\right)$ are well defined for any $t \leq T, x \in E$ and Γ any measurable subset of E. The topological irreducibility in E means that $P(t,x,O) > 0$ for some $t > 0, x \in E$ and any non-empty open subset O of E, and P_t is strongly Feller if $P_t : b\mathcal{B}(E) \to C_b(E)$.

In the case of the stochastic Navier-Stokes equation, Flandoli and Maslowski [15] proved the topological irreducibility in H and the Strong Feller property in $D(A^{\frac{1}{4}})$. They obtained thus the uniqueness. But, for the investigation of a large deviation principle, we have a powerful criterion if the semigroup is topologically irreducible and strongly Feller on the same space. So, our beginning is the following theorem for solutions starting from a regular initial condition due to Ferrario [12].

Theorem 2.2. *[12] Assume that (1.6) holds for* $\frac{1}{4} < \alpha < \frac{1}{2}$.

(i) If $x \in D\left(A^{\alpha}\right)$, $f \in D\left(A^{\alpha-\frac{1}{2}}\right)$, *the unique solution* X^x *of equation (1.4) given by Theorem 2.1 satisfies in fact* $\mathbb{P}$*-a.e.*

$$X^x \quad \in \quad C\left([0,T], D\left(A^{\alpha}\right)\right) \cap L^2\left([0,T], D(A^{\frac{1}{2}})\right) \cap L^{\frac{4}{1-2\alpha}}\left([0,T], D\left(A^{\frac{1}{4}+\frac{\alpha}{2}}\right)\right)$$

and also $X^x - Z \quad \in \quad L^2\left([0,T], D\left(A^{\alpha+\frac{1}{2}}\right)\right)$ $\mathbb{P}$*-a.e.*

(ii) The process (X^x) *is Markovian, and its transition probability* $P_t f(x) := \mathbb{E} f(X_t^x)$ *is topologically irreducible and strongly Feller in* $D\left(A^{\alpha}\right)$. *In particular, the invariant measure* μ *is unique.*

(iii) Moreover, for every $t_0 > 0$, *and every* $x \in H$, *the corresponding solution satisfies* $\mathbb{P}$*-a.e.* $X^x \in C\left([t_0,T], D\left(A^{\alpha}\right)\right)$.

In fact the original assumption of Ferrario was

$$D\left(A^{2\alpha}\right) \subset Im(G) \subset D\left(A^{\frac{1}{4}+\frac{\alpha}{2}+\varepsilon'}\right) \tag{2.2}$$

for $\alpha \in [\frac{1}{4}, \frac{1}{2})$ and some $\varepsilon' > 0$. However, the second embedding in (2.2) is clearly implied by the second embedding in (1.6). But our condition implies also that the energy injected in the system is finite. More precisely, we recall the

Lemma 2.3. *If the linear and continuous operator* $G : H \to H$ *satisfies for some* $\varepsilon > 0$

$$Im(G) \subset D(A^{\frac{1}{2}+\varepsilon}) \tag{2.3}$$

then the symmetric nonnegative operator $Q := GG^*$ *is of trace class.*

The above finite trace property is crucial in the application of Itô's formula for establishing our exponential estimates, and implies the usual regularity (2.1) for the solution.

3 General results about large deviations

In this section, we recall general results on the Large Deviation Principle for strong Feller and topologically irreducible Markov processes. We follow [27] and [28].

3.1 Notations and entropy of Donsker-Varadhan

Here we consider a general E-valued continuous Markov process,

$$\left(\Omega, \left(\mathcal{F}_t\right)_{t\geq 0}, \mathcal{F}, \left(X_t\right)_{t\geq 0}, \left(\mathbb{P}_x\right)_{x\in E}\right)$$

whose semigroup of Markov transition kernels is denoted by $\left(P_t(x,dy)\right)_{t\geq 0}$, where : $\Omega = C(\mathbb{R}^+, E)$ is the space of continuous functions from $\mathbb{R}^+$ to E equipped with the compact convergence topology; the natural filtration is $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$ for any $t \geq 0$ and $\mathcal{F} = \sigma(X_s, 0 \leq s)$. As usual, the law of the Markov process with initial state x in E is $\mathbb{P}_x$, and for any initial measure ν on E, let $\mathbb{P}_\nu(\cdot) = \int_E \mathbb{P}_x(\cdot)\nu(dx)$.

The empirical measure of level-3 (or process level) is given by

$$R_t := \frac{1}{t}\int_0^t \delta_{\theta_s X} ds$$

where $(\theta_s X)_t = X_{s+t}$ for all $t, s \geq 0$ are the shifts on Ω. Thus, R_t is a random element of $M_1(\Omega)$, the space of all probability measure on Ω.

The level-3 entropy functional of Donsker-Varadhan $H : M_1(\Omega) \to [0, +\infty]$ is defined by

$$H(Q) := \begin{cases} \mathbb{E}^{\bar{Q}} h_{\mathcal{F}_1}\left(\bar{Q}_{\omega(-\infty,0]}; \mathbb{P}_{w(0)}\right) & \text{if } Q \in M_1^s(\Omega) \\ +\infty & \text{otherwise} \end{cases} \tag{3.1}$$

where: $M_1^s(\Omega)$ is the space of those elements in $M_1(\Omega)$ which are moreover stationary; $\bar{Q}$ is the unique stationary extension of $Q \in M_1^s(\Omega)$ to $\bar{\Omega} := C(\mathbb{R}, E)$; the filtration is extended on $\bar{\Omega}$ with $\mathcal{F}_t^s = \sigma(X(u); s \leq u \leq t)$, $\forall s, t \in \mathbb{R}$; finally $\bar{Q}_{X(-\infty,t]}$ is the regular conditional distribution of $\bar{Q}$ knowing $\mathcal{F}_t^{-\infty}$ and $h_{\mathcal{G}}(\nu, \mu)$ is the usual relative entropy or Kullback information of ν with respect to μ restricted on the σ-field $\mathcal{G}$, given by

$$h_{\mathcal{G}}(\nu, \mu) := \begin{cases} \int \frac{d\nu}{d\mu}|_{\mathcal{G}} \log\left(\frac{d\nu}{d\mu}|_{\mathcal{G}}\right) d\mu, \text{ if } \nu \ll \mu \text{ on } \mathcal{G} \\ +\infty \text{ otherwise.} \end{cases}$$

Now, the level-2 entropy functional $J : M_1(E) \to [0, \infty]$ which governs the LDP in our main result is

$$J(\beta) = \inf\{H(Q) \mid Q \in M_1^s(\Omega) \text{ and } Q_0 = \beta\}, \ \forall \beta \in M_1(E), \tag{3.2}$$

where $Q_0(\cdot) = Q(X(0) \in \cdot)$ is the marginal law at $t = 0$.

Proposition 3.1. *For our model, $J(\nu) < +\infty \implies \nu \ll \mu$. Furthermore, a necessary and sufficient condition for $J(\nu) = 0$ is $\nu = \mu$.*

Proof. Here we take $E := D(A^\alpha)$, where X_t is strongly Feller and topologically irreducible by Theorem 2.2. Consider ν such that $J(\nu) < \infty$. By definition, there exists some $Q \in M_1^s(\Omega)$ such that $Q_0 = \nu$, and $H(Q) < \infty$ (see the expression (3.1) giving the Level-3 entropy).

For such Q and all $t > 0$, we have by stationarity (see [27, App. B])

$$\begin{aligned} H(Q) &= \mathbb{E}^{\bar{Q}} h_{\mathcal{F}_1}\left(\bar{Q}_{X(-\infty,0]}; \mathbb{P}_{X_0}\right) \\ &= \frac{1}{t}\mathbb{E}^{\bar{Q}} h_{\mathcal{F}_t}\left(\bar{Q}_{X(-\infty,0]}; \mathbb{P}_{X_0}\right). \end{aligned}$$

By Jensen inequality we obtain

$$\frac{1}{t}\mathbb{E}^{\bar{Q}} h_{\mathcal{F}_t}\left(\bar{Q}_{X(-\infty,0]}; \mathbb{P}_{X_0}\right) \geq \frac{1}{t} h_{\mathcal{F}_t}\left(Q; \mathbb{P}_\nu\right).$$

Then, noting that entropy of marginal measures is not larger than global entropy,

$$\begin{aligned} H(Q) &\geq \frac{1}{t} h_{\sigma(X_t)}\left(Q;\mathbb{P}_\nu\right) \\ &\geq \frac{1}{t} h_{\mathcal{B}(E)}\left(\nu;\nu P_t\right) \end{aligned}$$

and taking infinimum over such Q, we get

$$J(\nu) \geq \frac{1}{t} h_{\mathcal{B}(E)}\left(\nu;\nu P_t\right). \tag{3.3}$$

So the Kullback information of ν with respect to νP_t is finite, which implies by definition that $\nu \ll \nu P_t$. Since P_t is topologically irreducible and strongly Feller, all the measures $P_t(x,dy), t>0, x\in E$ are equivalent to μ (see [7, p. 42]), and we have

$$\nu P_t = \int_E P_t(x,.)\nu(dx) \ll \mu.$$

Thus $\nu \ll \nu P_t \ll \mu$.

If the probability measure ν is such that $J(\nu)=0$ then $h_{\mathcal{B}(E)}\left(\nu;\nu P_t\right)=0$ using (3.3). By the well known property of the Kullback information, we obtain $\nu = \nu P_t$ for every $t\geq 0$. Finally, the uniqueness of the invariant measure for P_t in Theorem 2.2 implies $\nu=\mu$ and the proof is finished. □

3.2 The hyper-exponential recurrence criterion

The following criterion of the so-called hyper-exponential recurrence was established by Wu [28, Theorem 2.1] in the study of a SDE modelling a stochastically forced Hamiltonian system and is true for general polish space E.

Theorem 3.2. *[28] Let $\mathcal{A}\subset M_1(E)$ and assume that*

$$P_t \text{ is strong Feller and topologically irreducible on } E. \tag{3.4}$$

If $\forall\lambda>0$ there exists some compact $K\subset\subset E$, such that

$$\sup_{\nu\in\mathcal{A}} \mathbb{E}^\nu e^{\lambda\tau_K} < \infty \tag{3.5}$$

and

$$\sup_{x\in K} \mathbb{E}^x e^{\lambda\tau_K^{(1)}} < \infty \tag{3.6}$$

where $\tau_K := \inf\{t\geq 0 \ s.t\ X_t\in K\}$ and $\tau_K^{(1)} := \inf\{t\geq 1\ s.t\ X_t\in K\}$, then the family $\mathbb{P}_\nu(L_t\in\cdot)$ satisfies the LDP on $M_1(E)$ w.r.t to the τ-topology with the rate function J defined by (3.2), and uniformly for initial measures ν in the subset $\mathcal{A}$.

More precisely, the three properties hold:

(a1) $J: M_1(E)\to[0,+\infty]$ is inf-compact w.r.t the τ-topology

(a2)(the lower bound) for any τ-open G in $M_1(E)$,

$$\liminf_{T\to\infty} \frac{1}{T} \log \inf_{\nu\in\mathcal{A}} \mathbb{P}_\nu(L_T \in G) \geq -\inf_G J \tag{3.7}$$

(a3)(the upper bound) for any τ-closed F in $M_1(E)$,

$$\limsup_{T\to\infty} \frac{1}{T} \log \sup_{\nu\in\mathcal{A}} \mathbb{P}_\nu(L_T \in F) \leq -\inf_F J. \tag{3.8}$$

This theorem is in fact a slight extension of the result in [28] to a uniform LDP over a non-empty family of initial measures. Let us recall briefly the main steps of the proof and the corresponding references (see however [19] for a complete proof). At first, a pointwise level-3 lower bound can be deduced from the properties (3.4) via the notion of μ-essential irreducibility (see [27]). This pointwise lower bound yields the uniform lower bound (3.7) if the uniform upper bound (3.8) is satisfied (as in [18]).

So, essential part of the proof is the uniform upper bound (3.8) for the strong topology τ. Indeed, the upper bound for the weak topology may be proved by the exponential tightness implied by (3.5) and (3.6) (see section 4.1), but the strong Feller property is crucial for the τ topology. By an extension of the Gartner-Ellis theorem (see [27]), it is sufficient to prove that $\forall\ (f_n) \in B_b(E)$ decreasing to zero pointwisely on E, we have

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \log \sup_{\nu\in\mathcal{A}} \mathbb{E}^\nu \int_0^T f_n(X_s)ds = 0.$$

This last assertion follows from the Markovian and the strong Feller property, also from (3.5) and (3.6) and can be proved as in [28]. Actually this last point is a problem for establishing the LDP for degenerated noise with a unique invariant measure.

4 Exponential estimates for the solution and some comments on the rate function J

In this section we establish the following crucial exponential estimates for the solution.

Proposition 4.1. *For any fixed $0 < \lambda_0 < \frac{\lambda_1}{2\|Q\|}$ and any $x \in H$, the process X satisfies*

$$\mathbb{E}^x \exp\left(\lambda_0|X(t)|^2 + \int_0^t \lambda_0|A^{\frac{1}{2}}X(s)|^2ds\right) \leq e^{\left(\lambda_0\left(Tr(Q)+\frac{|f|^2}{\lambda_1-2\|Q\|\lambda_0}\right)t\right)} e^{\lambda_0|x|^2}.$$

In particular, the following estimates hold

$$\mathbb{E}^x \exp\left(\lambda_0|X(t)|^2ds\right) \leq \exp\left(\lambda_0 t\left(tr(Q) + \frac{|f|^2}{\lambda_1 - 2\|Q\|\lambda_0}\right)\right) e^{\lambda_0|x|^2} \tag{4.1}$$

and

$$\mathbb{E}^x \exp\left(\lambda_0 \int_0^t |A^{\frac{1}{2}}X(s)|^2 ds\right) \le \exp\left(\lambda_0 t \left(tr(Q) + \frac{|f|^2}{\lambda_1 - 2\|Q\|\lambda_0}\right)\right) e^{\lambda_0 |x|^2}. \quad (4.2)$$

Moreover, for any fixed $L > 1$*, we have*

$$\sup_{\nu \in \mathcal{M}_{\lambda_0,L}} \mathbb{E}^\nu \exp\left(\lambda_0 \int_0^t |A^{\frac{1}{2}}X(s)|^2 ds\right) \le e^{\left(\lambda_0 t \left(tr(Q) + \frac{|f|^2}{\lambda_1 - 2\|Q\|\lambda_0}\right)\right)} L \quad (4.3)$$

where $\mathcal{M}_{\lambda_0,L}$ *is the set of initial measures defined by (1.7).*

Before proving this property at the end of this section, we first give some consequences of these estimates, and some comments about the entropy J of Donsker-Varadhan.

4.1 First consequences of the exponential estimates

The first one is the

Corollary 4.2. *Under the estimate (4.2), the family of law* $\mathbb{P}_\nu(L_t \in \cdot)$ *is uniformly exponentially tight over* $\mathcal{M}_{\lambda_0,L}$*. More precisely, for any* $\varepsilon > 0$*, there is some compact subset* $K = K_\varepsilon$ *in* $M_1(H)$ *in the weak convergence topology such that*

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{\nu \in \mathcal{M}_{\lambda_0,L}} \mathbb{P}_\nu(L_t \notin K) \le -\frac{1}{\varepsilon}.$$

Consequently for any closed set F *in* $M_1(H)$ *equipped with the weak convergence topology* $\sigma(M_1(H), C_b(H))$*, we have*

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{\nu \in \mathcal{M}_{\lambda_0,L}} \mathbb{P}_\nu(L_t \in F) \le -\inf_F J \quad (4.4)$$

where the entropy of Donsker-Varadhan $J : \nu \in M_1(H) \to J(\nu) \in [0, +\infty]$ *satisfies*

$$\lambda_0 \int_H |A^{\frac{1}{2}}x|^2 d\nu \le J(\nu) + \lambda_0 \left(tr(Q) + \frac{|f|^2}{\lambda_1 - 2\|Q\|\lambda_0}\right) \quad (4.5)$$

for any $0 < \lambda_0 < \frac{\lambda_1}{2\|Q\|}$.

Proof. The proof of the exponential tightness from (4.2) is given in [18, Prop. 5.1]. According to the general theory, it yields the upper bound (4.4) by using a general weak upper bound on compact subsets (see [8]).

For the proof of (4.5), let us recall the definition of a Cramer functional

$$\Lambda^0(V) := \sup_{x\in H} \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}^x \exp(tL_t(V)), \quad \forall V \in b\mathcal{B}(H) \quad (4.6)$$

and the definition of its Legendre transformation:

$$(\Lambda^0)^*(\nu) = \sup_{V\in b\mathcal{B}(H)} \int_H V d\nu - \Lambda^0(V), \quad \forall \nu \in M_1(H)). \tag{4.7}$$

It is known that $(\Lambda^0)^* = J$ (see for instance [27, prop. B.13]). Let us consider the function $V_n(x) := \lambda_0 |A^{\frac{1}{2}}x|^2 \wedge n$ which is bounded and measurable on H (Here $a \wedge b$ is the minimum of two real numbers a and b). For $0 < \lambda_0 < \frac{\lambda_1}{2\|Q\|}$, by the definitions (4.6), (4.7), and the exponential estimate (4.2), we obtain

$$\begin{aligned} \nu(V_n) &\leq (\Lambda^0)^*(\nu) + \Lambda^0(V_n) \\ &\leq J(\nu) + \lambda_0 \left(tr(Q) + \frac{|f|^2}{\lambda_1 - 2\|Q\|\lambda_0} \right). \end{aligned}$$

Hence, we obtain (4.5) by Fatou's lemma.

□

In fact, this kind of estimates provides us an alternative way for proving the existence of an invariant measure based on the large deviations theory. More precisely, we have the

Corollary 4.3. *Assume that a Feller-Markov process X on H satisfies the exponential estimate (4.2), then X admits at least one invariant measure.*

Proof. It is sufficient to prove that for any $a > 0$, the subset $[J \leq a]$ is tight. In that case, by Prokorov's criterion, its closure is compact in $M_1(H)$ w.r.t the weak topology. Hence, the l.s.c function $J : M_1(H) \to [0, +\infty]$ admits compact level subsets, w.r.t the weak topology $\sigma(M_1(H), C_b(H))$. Moreover, considering the closed subset $F = M_1(H)$ in the good upper bound (4.4), we obtain the existence of $\nu \in M_1(H)$ satisfying $J(\nu) = 0$, so ν is an invariant measure (as in the proof of Proposition 3.1).

Now, for any $a > 0$, let us show that the tightness of $[J \leq a]$ is a simple consequence of (4.5). Let $\varepsilon > 0$ fixed, and consider the finite number

$$M_{a,\varepsilon} := \frac{a + \lambda_0 \left(tr(Q) + \frac{|f|^2}{\lambda_1 - 2\|Q\|\lambda_0} \right)}{\lambda_0 \varepsilon}.$$

By the compact embedding $D(A^{\frac{1}{2}}) \subset H$, the subset

$$K_\varepsilon = \left\{ x \in D(A^{\frac{1}{2}}) \ s.t. \ |A^{\frac{1}{2}}x|^2 \leq M_{a,\varepsilon} \right\}$$

is compact in H, and by using (4.5) we obtain for all β in $[J \leq a]$,

$$\begin{aligned} \beta(K_\varepsilon^c) &\leq \int_{K_\varepsilon^c} \frac{|A^{\frac{1}{2}}x|^2}{M_{a,\varepsilon}} \, d\beta(x) \leq \frac{1}{M_{a,\varepsilon}} \int_H |A^{\frac{1}{2}}x|^2 \, d\beta(x) \\ &\leq \frac{1}{M_{a,\varepsilon}} \frac{a + \lambda_0 \left(tr(Q) + \frac{|f|^2}{\lambda_1 - 2\|Q\|\lambda_0} \right)}{\lambda_0} \\ &\leq \varepsilon \end{aligned}$$

and so $[J \leq a]$ is tight. □

In the following paragraph we focus on the entropy J which governs the LDP.

4.2 Some comments on the entropy J defined by (3.2)

In fact $J(\nu)$ admits a closed form only in the case when the unique invariant measure μ is known, and the Markov process X_t is symmetric w.r.t μ. For our model, Theorem 1.1 describes the exact rate of exponential convergence, but the expression of this exact rate given by J is more qualitative than quantitative. How to obtain estimates on $J(\nu)$ is an important and very interesting question. Usually, we can proceed by using functional inequalities such as logarithmic Sobolev or spectral gap inequalities as in Deuschel-Stroock [9]. Unfortunately, for the 2D Stochastic Navier-Stokes equations, those inequalities are actually unknown (see however the very recent works of Goldys-Maslowski [17] and Hairer-Mattingly [21] for the existence of a spectral gap in a space of weighted bounded or weighted Lipschitz functions, which is different from the Poincaré inequality). In this section, we consider the case $E := H$.

At first, under the Feller assumption, we know that (see Lemma B.7 in [27])

$$J(\nu) = \sup\left\{-\int \frac{\mathcal{L}u}{u} d\nu \ ; \ 1 \le u \in D_e(\mathcal{L})\right\}, \qquad \nu \in M_1(H) \tag{4.8}$$

where $D_e(\mathcal{L})$ is the extended domain of the generator $\mathcal{L}$ of P_t in $C_b(H)$. We recall that $u \in D_e(\mathcal{L})$ and $v := \mathcal{L}u$ if $u \in C_b(E)$ and there exists $v \in C_b(H)$ such that $P_t u - u = \int_0^t P_s v ds$, for all $t \ge 0$. For the 2D-stochastic Navier-Stokes equation, we recall also that $\mathcal{L}$ is given by

$$\mathcal{L}f(x) := \frac{1}{2} tr(GG^* D^2 f)(x) \ + \ \langle -Ax - B(x,x) + f, \nabla^H f(x)\rangle \tag{4.9}$$

at least for f cylindrical, i.e $f(x) = g\left(\langle x, e_1\rangle, \langle x, e_2\rangle, \ldots, \langle x, e_n\rangle\right)$. In this expression, we denote by ∇^H the gradient in H, and $D^2 f := \left(\partial_{e_i}\partial_{e_j} f\right)_{i,j\ge 1}$. Since f is cylindrical, the gradient $\nabla^H f(x)$ is in $H^k := D(A^{\frac{k}{2}})$, for any $k \ge 0$ and the left-hand side in (4.9) is well defined by $\langle B(x,x), \nabla^H f(x)\rangle = -\langle B(x, \nabla^H f(x)), x\rangle$ and the inequality

$$|\langle B(x, \nabla^H f(x)), x\rangle| \le C|x|^2 |A\nabla^H f(x)| \left(1 + \log \frac{|A^{\frac{3}{2}}\nabla^H f(x)|^2}{\lambda_1 |A\nabla^H f(x)|^2}\right)^{\frac{1}{2}}$$

established for instance in [16, p 100].

In this paragraph, we introduce some class of measures $\mu^h \in M_1(H)$ for which it is possible to give a more explicit form than (3.2) or (4.8) for $J(\mu^h)$. Here, we assume that $G = G^* = Q^{\frac{1}{2}}$. Let (X_t^x) be the solution of the 2D stochastic Navier-Stokes equation (1.4) with initial position x, defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and let us consider the Girsanov perturbation defined by : for any $T > 0$, and any $x \in H$,

$$\frac{d\mathbb{Q}_x^h}{d\mathbb{P}} |_{\mathcal{F}_T} \ = \ \exp\left(\int_0^T \sqrt{GG^*}\nabla^H h(X_s^x) dW_s - \frac{1}{2}\int_0^T \left|\sqrt{GG^*}\nabla^H h(X_s^x)\right|^2 ds\right). \tag{4.10}$$

In the above expression, we take $h \in C^1(H)$ satisfying $\langle GG^* \nabla^H h, \nabla^H h \rangle \le C < \infty$ so that

$$\mathbb{E} \exp\left(\frac{1}{2} \int_0^T \left| \sqrt{GG^*} \nabla^H h(X_s^x) \right|^2 ds \right) < \infty, \ \ \forall T \ge 0, \ \forall x \in H. \tag{4.11}$$

A simple sufficient condition on h is $h(x) = g\left(\langle x, e_1 \rangle, \langle x, e_2 \rangle, \ldots, \langle x, e_n \rangle\right)$ where the function $g \in C^1(\mathbb{R}^n)$ has a bounded gradient.

In the case when (4.11) is true, $L_t^x := \int_0^t \sqrt{GG^*} \nabla^H h(X_s^x) dW_s$ is a continuous martingale under $\mathbb{P}$, and also $M_t^x := \exp(L_t^x - \frac{1}{2}\langle L^x \rangle_t)$, the exponential local martingale given in (4.10), becomes a true martingale by Novikov's criterion.

Hence, $(\mathbb{Q}_x^h)_{x \in H}$ given in (4.10) defines a new Markov family with the transition semigroup $Q_t^h f(x) = \mathbb{E}^{\mathbb{Q}_x^h} f(X_t^x)$. By Girsanov's formula, the generator of Q_t^h takes the form $\mathcal{L}_h u = \mathcal{L} u + 2\Gamma(h,u)$ where

$$\Gamma(h,u) = 1/2 \ \langle GG^* \nabla^H h, \nabla^H u \rangle$$

is the *carré du champ* of $\mathcal{L}$, and under $\mathbb{Q}_x^h$ the process (X_t^x) satisfies in a weak sense (i.e. in law) the following perturbation of the 2D-stochastic Navier-Stokes equation

$$dX_t + AX_t dt + B\left(X_t, X_t\right) dt = f dt + \sqrt{GG^*} \nabla^H h\left(X_t\right) dt + G d\tilde{W}_t \ \ , \ \ X_0 = x \tag{4.12}$$

where $\tilde{W}_t$ is a cylindrical Wiener process under $\mathbb{Q}_x^h$.

For the existence and the uniqueness of an invariant measure $\mu^h \in M_1(H)$ for Q_t^h, let us first give the

Lemma 4.4. *Let* $0 < \delta < \frac{\lambda_1}{4\|Q\|}$ *and* $h \in C^1(H)$ *such that*

$$\langle GG^* \nabla^H h(x), \nabla^H h(x) \rangle \le C < \infty, \tag{4.13}$$

then there is some constant $K(\delta) > 0$ *such that*

$$\mathbb{E}^{\mathbb{Q}_x^h} \exp\left(\delta \int_0^t |A^{\frac{1}{2}} X_s^x|^2 ds \right) \le e^{tK(\delta)} e^{2\delta |x|^2}. \tag{4.14}$$

Proof. By using (4.10) and Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}_x^h} \exp\left(\delta \int_0^t |A^{\frac{1}{2}} X_s^x|^2 ds \right) &= \mathbb{E}^{\mathbb{P}} \exp\left(\delta \int_0^t |A^{\frac{1}{2}} X_s^x|^2 ds \right) \exp\left(L_t^x - \frac{1}{2}\langle L^x \rangle_t \right) \\ &= \mathbb{E}^{\mathbb{P}} \exp\left(\delta \int_0^t |A^{\frac{1}{2}} X_s^x|^2 ds + \frac{1}{2}\langle L^x \rangle_t \right) \exp\left(L_t^x - \langle L^x \rangle_t \right) \\ &\le \left(\mathbb{E}^{\mathbb{P}} \exp\left(2\delta \int_0^t |A^{\frac{1}{2}} X_s^x|^2 ds + \langle L^x \rangle_t \right) \right)^{\frac{1}{2}} \end{aligned}$$

since the exponential local martingale $\exp\left(2L_t^x - 2\langle L^x \rangle_t\right)$ is a supermartingale. Hence, noting that $\langle L \rangle_t \le Ct$ by (4.13), and using estimate (4.2) with $\lambda_0 = 2\delta$, we obtain

$$\mathbb{E}^{\mathbb{Q}_x^h} \exp\left(\delta \int_0^t |A^{\frac{1}{2}} X_s^x|^2 ds \right) \ \le \ \exp\left(\delta t \left(tr(Q) + \frac{|f|^2}{\lambda_1 - 4\|Q\|\delta} \right) \right) e^{\frac{Ct}{2}} e^{\delta |x|^2}$$

i.e the estimate (4.14). □

For $x, y \in H$, the following control is well known (see [26])

$$|X_t^x - X_t^y| \leq |x-y| \exp\left(C \int_0^t |A^{\frac{1}{2}} X_s^x|^2 ds \right)$$

which implies the convergence in probability $X_t^y \to X_t^x$ and $f(X_t^y) \to f(X_t^x)$ for any function $f \in C_b(H)$, as $y \to x$. In particular, the Feller property for P_t follows by Lebesgue Theorem. We prove now that Q_t^h is also a Feller semigroup.

We must show that for any sequence $x_n \to x$ in H, and for any $f \in C_b(H)$, the convergence $Q_t^h f(x_n) \to Q_t^h f(x)$ holds. By using the density given in (4.10), it is equivalent to prove that $\mathbb{E}^{\mathbb{P}} f(X_t^{x_n}) M_t^{x_n} \to \mathbb{E}^{\mathbb{P}} f(X_t^x) M_t^x$. The quadratic variation process of the $\mathbb{P}$-martingale $L_t^{x_n} - L_t^x$ satisfies

$$\begin{aligned}
\langle L^{x_n} - L^x \rangle_t &= \int_0^t \langle GG^*(\nabla^H h(X_s^{x_n}) - \nabla^H h(X_s^x)), \nabla^H h(X_s^{x_n}) - \nabla^H h(X_s^x) \rangle ds \\
&\leq 4Ct
\end{aligned}$$

by polarization under our assumption (4.13). Hence, the convergence $X_t^{x_n} \to X_t^x$ in probability implies the convergence in L^2 for the martingale $L_t^{x_n} \to L_t^x$, and in particular the convergence in probability for the exponential martingale $M_t^{x_n} \to M_t^x$. Since $\mathbb{E}^{\mathbb{P}} M_t^{x_n} = \mathbb{E}^{\mathbb{P}} M_t^x = 1$, and $M_t^{x_n}, M_t^x \geq 0$ we obtain the convergence in L^1 for $M_t^{x_n} \to M_t^x$ by a well known lemma. Finally, since $f(X_t^{x_n}) \to f(X_t^x)$ in probability and is bounded, we obtain $\mathbb{E}^{\mathbb{P}} f(X_t^{x_n}) M_t^{x_n} \to \mathbb{E}^{\mathbb{P}} f(X_t^x) M_t^x$ as desired.

Hence the exponential estimate (4.14) implies the existence of an invariant measure for the Feller semigroup Q_t^h by Corollary 4.3. Moreover, by (4.10), we know that $Q_t^h(x, \cdot) \sim P_t(x, \cdot) \sim P_t(y, \cdot) \sim Q_t^h(y, \cdot)$. So, the semigroup Q_t^h is regular and its invariant measure μ^h is unique (see Doob's theorem in [7]).

In that case, we have the following simple expression for $J(\mu^h)$, where μ^h can be seen as the unique invariant measure for the solution to the equation (4.12).

Proposition 4.5. *For $h \in C^1(H)$ such that $\langle GG^* \nabla^H h(x), \nabla^H h(x) \rangle \leq C < \infty$, we have*

$$J(\mu^h) = \frac{1}{2} \int_H \langle GG^* \nabla^H h, \nabla^H h \rangle d\mu^h = \int_H \Gamma(h,h) d\mu^h$$

Proof. For the "$\geq$", let us consider a nice approximating sequence of cylindrical functions h_n for h such that $h_n(x) \to h(x)$, $\nabla^H h_n(x) \to \nabla^H h(x)$ and $|\nabla^H h_n(x)| \leq M$. Since $\mathcal{L} e^{h_n} = e^{h_n} (\mathcal{L} h_n + \Gamma(h_n, h_n))$, we have using (4.8)

$$\begin{aligned}
J(\mu^h) &\geq -\int \frac{\mathcal{L} e^{h_n}}{e^{h_n}} d\mu^h \\
&= -\int \mathcal{L}_h h_n d\mu^h + \int 2\Gamma(h, h_n) - \Gamma(h_n, h_n) d\mu^h \\
&= \int \left[2\Gamma(h, h_n) - \Gamma(h_n, h_n) \right] d\mu^h
\end{aligned}$$

because μ^h is the invariant measure for the semigroup Q_t^h generated by $\mathcal{L}_h$. Now, letting $n \to \infty$, we obtain $J(\mu^h) \geq \int \Gamma(h,h) d\mu^h$ by dominated convergence.

For the "$\leq$", we denote by $\mathbb{Q}^h_{\mu^h}$ the law of the unique stationary Markov process with μ^h as initial distribution and the transition semigroup Q^h_t. By our assumption on h, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^h_{\mu^h}}\langle L\rangle_t &= 2\mathbb{E}^{\mathbb{Q}^h_{\mu^h}}\int_0^t \Gamma(h,h)(X_s)ds\\ &= 2t\int \Gamma(h,h)d\mu^h\\ &\leq Ct < \infty\end{aligned}$$

where the second equality follows from the stationarity of $\mathbb{Q}^h_{\mu^h}$. Hence, $L^x_t - \langle L^x\rangle_t$ being a $\mathbb{Q}^h_{\mu^h}$-local martingale by Girsanov, is in fact a true $\mathbb{Q}^h_{\mu^h}$-martingale, and the definition (3.1) of the level-3 entropy gives

$$\begin{aligned}H(\mathbb{Q}^h_{\mu^h}) &= \mathbb{E}^{\mathbb{Q}^h_{\mu^h}}\log M_1\\ &= \mathbb{E}^{\mathbb{Q}^h_{\mu^h}}\left(L^x_1 - \frac{1}{2}\langle L^x\rangle_1\right)\\ &= \frac{1}{2}\mathbb{E}^{\mathbb{Q}^h_{\mu^h}}\langle L^x\rangle_1.\end{aligned}$$

So, again with the stationarity of $\mathbb{Q}^h_{\mu^h}$, we obtain

$$\begin{aligned}H(\mathbb{Q}^h_{\mu^h}) &= \frac{1}{2}\mathbb{E}^{\mathbb{Q}^h_{\mu^h}}\int_0^1 \left|\sqrt{GG^*}\nabla^H h(X^x_s)\right|^2 ds\\ &= \frac{1}{2}\int_H \left|\sqrt{GG^*}\nabla^H h(x)\right|^2 d\mu^h\end{aligned}$$

Finally, by the definition (3.2) of the level-2 entropy, we have

$$J(\mu^h) \leq H(\mathbb{Q}^h_{\mu^h}) = \frac{1}{2}\int_H \left|\sqrt{GG^*}\nabla^H h(x)\right|^2 d\mu^h = \int \Gamma(h,h)d\mu^h.$$

□

4.3 Proof of Proposition 4.1

We finish this section by giving the proof of the exponential estimates. Let us introduce the finite dimensional approximations system associated with equation (1.1). Let Π_n the orthogonal projections on the finite dimensional space spanned by the first n eigenvectors $(e_1, \ldots, e_n)$, and set, for $n \geq 1$,

$$B_n(x) = \Pi_n B(\Pi_n x, \Pi_n x), \quad G_n = \Pi_n G \Pi_n, \quad f_n = \Pi_n f$$

and $Q_n = G_n G_n^*$. We will consider the finite dimensional equations

$$dX_n(t) + AX_n(t)dt + B_n(X_n(t)) = f_n dt + G_n dW(t)\ ;\ X_n(0) = x_n := \Pi_n x. \quad (4.15)$$

Note that equation (4.15) is a finite-dimensional stochastic equation. Hence, there exists a solution, and $X_n(t)$ is smooth in space. Moreover, the following convergence was proved by Capinski and Gatarek [3] (see also Goldys and Maslowski [17]).

Theorem 4.6. *[17] For any $\delta > 0$, solution X_n of (4.15) converge in distribution to the solution X of (1.4) on the space $C([0,T], H^{-\delta})$, where $H^{-\delta}$ is the dual space of H^{δ}.*

The first aim of this paragraph is to prove some estimates on X_n, the finite dimensional approximations. Recall that $\langle \ , \ \rangle$ denote the scalar product in H. Let us apply Itô's formula to the finite dimensional diffusion X_n. Since by (1.3),

$$\langle B_n\left(X_n(t)\right), X_n(t)\rangle = 0$$

we obtain:

$$\begin{aligned} d|X_n(t)|^2 &= 2\langle X_n(t), dX_n(t)\rangle + tr(Q_n)dt \\ &= \left[-2|A^{\frac{1}{2}}X_n(t)|^2 + 2\langle X_n(t), f_n\rangle + tr(Q_n)\right]dt + 2\langle X_n(t), G_n dW(t)\rangle. \end{aligned}$$

Hence, for $U_n(t) := |X_n(t)| + \int_0^t |A^{\frac{1}{2}}X_n(s)|ds$, it yields

$$dU_n(t) = \left[-|A^{\frac{1}{2}}X_n(t)|^2 + 2\langle X_n(t), f_n\rangle + tr(Q_n)\right]dt + 2\langle X_n(t), G_n dW(t)\rangle.$$

Same manner, denoting by $d[U_n, U_n]_t$ the quadratic variation process of U_n, we can also compute with the Itô formula

$$\begin{aligned} de^{\lambda_0 U_n(t)} &= e^{\lambda_0 U_n(t)}\left[\lambda_0 dU_n(t) + \frac{\lambda_0^2}{2}d[U_n, U_n]_t\right] \\ &= \lambda_0 e^{\lambda_0 U_n(t)}\left[-|A^{\frac{1}{2}}X_n(t)|^2 + 2\langle X_n(t), f_n\rangle + tr(Q_n) + 2\lambda_0|G_n^* X_n(t)|^2\right]dt \\ &\quad +2\lambda_0 e^{\lambda_0|X_n(t)|^2}\langle X_n(t), G_n dW(t)\rangle. \end{aligned} \tag{4.16}$$

The following inequalities are clear

$$tr(Q_n) \leq tr(Q) \quad , \quad |x_n| \leq |x| \quad , \quad |f_n| \leq |f|. \tag{4.17}$$

Moreover, it is easy to see that

$$|G_n^* X_n(t)|^2 \leq \|Q\| \ |X_n(t)|^2, \tag{4.18}$$

and, by Young's inequality, that

$$2\langle X_n(t), f_n\rangle \leq \varepsilon|X_n(t)|^2 + \frac{|f_n|^2}{\varepsilon}, \quad \forall\varepsilon > 0. \tag{4.19}$$

For $\varepsilon > 0$ fixed later, let us estimate the drift of the process

$$V_n(t) := e^{-\lambda_0\left(Tr(Q) + \frac{|f|^2}{\varepsilon}\right)t} e^{\lambda_0 U_n(t)}.$$

By Itô Formula and using (4.16), (4.17), (4.18) and (4.19) we have,

$$dV_n(t) = e^{-\lambda_0\left(Tr(Q) + \frac{|f|^2}{\varepsilon}\right)t} de^{\lambda_0 U_n(t)} - \lambda_0\left(Tr(Q) + \frac{|f|^2}{\varepsilon}\right) e^{-\lambda_0\left(Tr(Q) + \frac{|f|^2}{\varepsilon}\right)t} e^{\lambda_0 U_n(t)}dt$$

$$\begin{aligned}&\leq \lambda_0 e^{-\lambda_0\left(Tr(Q)+\frac{|f|^2}{\varepsilon}\right)t} e^{\lambda_0 U_n(t)}\left(-|A^{\frac{1}{2}}X_n(t)|^2+\varepsilon|X_n(t)|^2+2\lambda_0\|Q\|\;|X_n(t)|^2\right)dt\\ &\quad +2\lambda_0 e^{-\lambda_0\left(Tr(Q)+\frac{|f|^2}{\varepsilon}\right)t} e^{\lambda_0|X_n(t)|^2}\langle X_n(t),G_n dW(t)\rangle.\end{aligned}$$

Remarking that, by (1.2), for a constant λ_1 depending on the domain D,

$$|X_n(t)|^2 \leq \frac{|A^{\frac{1}{2}}X_n(t)|^2}{\lambda_1},$$

we obtain

$$\begin{aligned}dV_n(t) &\leq e^{-\lambda_0\left(Tr(Q)+\frac{|f|^2}{\varepsilon}\right)t} e^{\lambda_0 U_n(t)}\left(-|A^{\frac{1}{2}}X_n(t)|^2\left(1-\frac{\varepsilon+2\lambda_0\|Q\|}{\lambda_1}\right)\right)dt\\ &\quad +2\lambda_0 e^{-\lambda_0\left(Tr(Q)+\frac{|f|^2}{\varepsilon}\right)t} e^{\lambda_0|X_n(t)|^2}\langle X_n(t),G_n dW(t)\rangle.\end{aligned}$$

Hence, for $0<\lambda_0\leq\frac{\lambda_1-\varepsilon}{2\|Q\|}$, the drift of $V_n(t)$ is non positive. More precisely, for all $0<\lambda_0<\frac{\lambda_1}{2\|Q\|}$, the positive number $\varepsilon:=\lambda_1-2\|Q\|\lambda_0$ satisfies the above condition, and it is our choice in (4.19). Thus, we have

$$dV_n(t) \leq 2\lambda_0\exp\left(-\lambda_0\left(Tr(Q)+\frac{|f|^2}{\varepsilon}\right)t\right)e^{\lambda_0|X_n(t)|^2}\langle X_n(t),G_n dW(t)\rangle.$$

Since $V_n(t)\geq 0$, we obtain by Fatou's lemma $\mathbb{E}^x V_n(t)\leq\mathbb{E}^x V_n(0)$, and this proves in particular the following crucial exponential estimate.

Lemma 4.7. *For $0<\lambda_0<\frac{\lambda_1}{2\|Q\|}$ and any x in H, we have*

$$\begin{aligned}&\mathbb{E}^x\exp\left(\lambda_0\int_0^t|A^{\frac{1}{2}}X_n(s)|^2ds+\lambda_0|X_n(t)|^2\right)\\ &\leq \exp\left(\lambda_0 t\left(tr(Q)+\frac{|f|^2}{\lambda_1-2\|Q\|\lambda_0}\right)\right)e^{\lambda_0|x|^2}.\end{aligned}\tag{4.20}$$

Let us now finish the proof of Proposition 4.1 by using Theorem 4.6. Since the function

$$F(X):=e^{\left(-\lambda_0 t\left(tr(Q)+\frac{|f|^2}{\lambda_1-2\|Q\|\lambda_0}\right)\right)}\exp\left(\lambda_0\int_0^t|A^{\frac{1}{2}}X(s)|^2ds+\lambda_0|X(t)|^2\right)$$

is lower semi continuous on $C([0,T],H^{-\delta})$ as an increasing limit of the continuous functions

$$F_m(X) := e^{\left(-\lambda_0 t\left(tr(Q)+\frac{|f|^2}{\lambda_1-2\|Q\|\lambda_0}\right)\right)}\exp\left(\lambda_0\int_0^t|A^{\frac{1}{2}}\Pi_m X(s)|^2ds+\lambda_0|\Pi_m X(t)|^2\right),$$

we obtain, using Theorem 4.6 for $n\to\infty$ in (4.20),

$$\mathbb{E}^x F(X)\leq\liminf_{n\to\infty}\mathbb{E}^x F(X_n)\leq e^{\lambda_0|x|^2}$$

and the desired estimates (4.1) and (4.2) follow.

$\square$

5 The Large Deviation Principle on $M_1(D(A^\alpha))$

The proof of our Theorem 1.1 consists in two steps. As a first step, we prove this section the LD principle for initial measures in $E := M_1(D(A^\alpha))$. We finish the proof of Theorem 1.1 in the following section by extending the LDP for initial conditions, open and closed subsets in $M_1(H)$, and by establishing the claim (1.8).

The aim of this section is to prove the

Lemma 5.1. *Let $f \in H$ and $\frac{1}{4} < \alpha < \frac{1}{2}$ a fixed number such that (1.6) holds. Let $0 < \lambda_0 < \frac{\lambda_1}{2\|Q\|}$, where $\|Q\|$ is the norm of Q as an operator in H and*

$$\Phi(x) = e^{\lambda_0|x|^2},\ \mathcal{M}^*_{\lambda_0,L} := \Big\{\nu \in M_1(D(A^\alpha)) \ / \ \int \Phi(x)\nu(dx) \le L\Big\}, \tag{5.1}$$

then the family $\mathbb{P}_\nu(L_T \in \cdot)$ as $T \to +\infty$ satisfies the LDP on $M_1(D(A^\alpha))$ w.r.t the topology τ, with speed T and the rate function J, uniformly for any initial measure in $\mathcal{M}^*_{\lambda_0,L}$ where $L > 1$ is any fixed number, and J is the level-2 entropy of Donsker-Varadhan. More precisely, the statements (i),(ii) and (iii) of Theorem 1.1 hold with $M_1(H)$ replaced by $M_1(D(A^\alpha))$.

Proof. By Theorem 3.2, since X_t is strongly Feller and topologically irreducible in $D(A^\alpha)$ (Theorem 2.2), it is sufficient to establish the estimates (3.5) and (3.6) for our model.

For K, we take

$$K := \Big\{x \in D(A^{\frac{1}{2}}) \ s.t \ |A^{\frac{1}{2}}x| \le M\Big\} \tag{5.2}$$

where the real M will be fixed later. Since the embedding $D\left(A^{\frac{1}{2}}\right) \subset D(A^\alpha)$ is compact for $\alpha < \frac{1}{2}$, it is clear that K is a compact subset in $D(A^\alpha)$.

The definition of the occupation measure implies that

$$\mathbb{P}_\nu\left(\tau_K^{(1)} > n\right) \le \mathbb{P}_\nu\left(L_n(K) \le \frac{1}{n}\right) = \mathbb{P}_\nu\left(L_n(K^c) \ge 1 - \frac{1}{n}\right).$$

With our choice for K, we have $L_n(K^c) \le \frac{1}{M^2}L_n(|A^{\frac{1}{2}}x|^2)$. Hence, for any fixed λ_0 such that $0 < \lambda_0 < \frac{\lambda_1}{2\|Q\|}$, we obtain by Chebychev's inequality

$$\begin{aligned}\mathbb{P}_\nu\left(\tau_K^{(1)} > n\right) &\le \mathbb{P}_\nu\left(L_n(|A^{\frac{1}{2}}x|^2) \ge M^2\left(1 - \frac{1}{n}\right)\right)\\ &\le \exp\left(-n\lambda_0 M^2\left(1 - \frac{1}{n}\right)\right)\mathbb{E}^\nu \exp\left(\lambda_0\int_0^n |A^{\frac{1}{2}}X(s)|^2 ds\right).\end{aligned}$$

For any initial measure $\nu \in M_1(D(A^\alpha))$, integrating (4.2) w.r.t $\nu(dx)$, and plugging it in the above estimate yields

$$\mathbb{P}_\nu\left(\tau_K^{(1)} > n\right) \le \nu\left(e^{\lambda_0|\cdot|^2}\right) e^{-n\lambda_0 C}, \quad \forall n \ge 2$$

where

$$C := \frac{M^2}{2} - tr(Q) - \frac{|f|^2}{\lambda_1 - 2\|Q\|\lambda_0}. \tag{5.3}$$

Let $\lambda > 0$ be fixed. By the integration by parts formula, we have

$$\begin{aligned}
\mathbb{E}^\nu e^{\lambda\tau_K^{(1)}} &= 1 + \int_0^{+\infty} \lambda e^{\lambda t}\mathbb{P}_\nu\left(\tau_K^{(1)} > t\right) dt \\
&\leq e^{2\lambda} + \sum_{n\geq 2} \lambda e^{\lambda(n+1)}\mathbb{P}_\nu\left(\tau_K^{(1)} > n\right) \\
&\leq e^{2\lambda}\left(1 + \lambda\ \nu\left(e^{\lambda_0|\cdot|^2}\right)\ \sum_{n\geq 2} e^{-n(\lambda_0 C - \lambda)}\right).
\end{aligned}$$

Now, by (5.3), we can choose M such that $\lambda_0 C - \lambda \geq 1$ in the definition (5.2) of K. Then, taking the supremum over $\{\nu = \delta_x, x \in K\}$, we get

$$\sup_{x\in K}\mathbb{E}^x e^{\lambda\tau_K^{(1)}} \leq e^{2\lambda}\left(1 + \lambda e^{\frac{\lambda_0 M^2}{\lambda_1}}\sum_{n\geq 1} e^{-n(\lambda_0 C-\lambda)}\right) < \infty$$

since for $x \in K$, $|x|^2 \leq \frac{|A^{\frac{1}{2}}x|^2}{\lambda_1} \leq \frac{M^2}{\lambda_1}$. So the bound (3.5) holds true. We obtain (3.6) in the same way: since $\tau_K \leq \tau_K^{(1)}$, we have

$$\begin{aligned}
\sup_{\nu\in\mathcal{M}^*_{\lambda_0,L}} \mathbb{E}^\nu e^{\lambda\tau_K} &\leq \sup_{\nu\in\mathcal{M}^*_{\lambda_0,L}} \mathbb{E}^\nu e^{\lambda\tau_K^{(1)}} \\
&\leq e^{2\lambda}\left(1 + \lambda L \sum_{n\geq 2} e^{-n(\lambda_0 C-\lambda)}\right) \\
&< \infty
\end{aligned}$$

which finishes the proof of the Lemma.

□

6 Proof of the Theorem 1.1

In this section we finish the proof of Theorem 1.1. It remains to extend Lemma 5.1 for initial conditions, open and closed subsets in $M_1(H)$, and to establish the claim (1.8). In fact, the first claim "$J(\nu) < +\infty \implies \nu \ll \mu$" in (1.8) was established in Proposition 3.1 and the second claim that for ν such that $J(\nu) < \infty$, we have also $\nu(|A^{\frac{1}{2}}x|^2) < \infty$ follows from (4.5).

For the extension of the LDP to $M_1(H)$, a first remark is the

Lemma 6.1. *Let $L > 1$, $0 < \lambda_0 < \frac{\lambda_1}{2\|Q\|}$ be fixed numbers and $\nu \in \mathcal{M}_{\lambda_0,L}$, i.e $\nu \in M_1(H)$ and $\int_H e^{\lambda_0|x|^2}\nu(dx) \leq L$. Then the probability measure $\tilde{\nu} := \mathbb{P}_\nu\left(X(1) \in dy\right)$ satisfies*

(a) $\tilde{\nu}\left(D\left(A^\alpha\right)\right) = 1$

(b)$\int_H e^{\lambda_0|x|^2}\tilde{\nu}(dx) \le e^{\lambda_0 C} L$*, where* $C = tr(Q) + \frac{|f|^2}{\lambda_1 - 2\|Q\|\lambda_0}$

In particular $\tilde{\nu} \in \mathcal{M}^*_{\lambda_0, e^{\lambda_0 C} L}$*, and* $\mathbb{P}_\nu(L_T \circ \theta_1 \in \cdot) = \mathbb{P}_{\tilde{\nu}}(L_T \in \cdot)$*.*

Proof. Notice that $\tilde{\nu}(dy) = \int_H \mathbb{P}_x\left(X(1) \in dy\right)\nu(dx)$, and $\mathbb{P}_x\left(X(1) \in D\left(A^\alpha\right)\right) = 1$, by Theorem 2.2, (iii) the statement (a) is clear. For (b), by using the estimate (4.1) in Proposition 4.1, we get

$$\begin{aligned}
\int_H e^{\lambda_0|y|^2}\tilde{\nu}(dy) &= \int_H \int_H e^{\lambda_0|y|^2} P_1(x,dy)\nu(dx) = \int_H \mathbb{E}^x e^{\lambda_0|X(1)|^2}\nu(dx) \\
&\le \int_H e^{\lambda_0\left(tr(Q) + \frac{|f|^2}{\lambda_1 - 2\|Q\|\lambda_0}\right)} e^{\lambda_0|x|^2}\nu(dx) \\
&\le e^{\lambda_0\left(tr(Q) + \frac{|f|^2}{\lambda_1 - 2\|Q\|\lambda_0}\right)} L.
\end{aligned}$$

□

6.1 Extension of the lower bound

Let G be an open subset in $(M_1(H), \tau)$. For any fixed $\beta_0 \in M_1(H)$, we can take a τ neighborhood of β_0 in $M_1(H)$ of form

$$N(\beta_0, \delta) := \{\beta \in M_1(H), |\beta(f_i) - \beta_0(f_i)| < \delta, \forall i = 1...d\}$$

contained in G, where $\delta > 0$, $1 \le d \in \mathbb{N}$ and $f_i \in b\mathcal{B}(H)$. For establishing the lower bound (ii) in Theorem 1.1, it is sufficient to establish that for every $\beta_0 \in G$ such that $J(\beta_0) < \infty$,

$$\liminf_{T\to\infty} \frac{1}{T} \log \inf_{\nu \in \mathcal{M}_{\lambda_0, L}} \mathbb{P}_\nu(L_T \in N(\beta_0, \delta)) \ge -J(\beta_0).$$

Notice that for $\nu \in \mathcal{M}_{\lambda_0, L}$,

$\mathbb{P}_\nu\left(L_T \in N(\beta_0, \delta)\right) \ge \mathbb{P}_\nu\left(L_T \circ \theta_1 \in N(\beta_0, \delta/2);\ |L_T \circ \theta_1(f_i) - L_T(f_i)| \le \delta/2,\ \forall i = 1..d\right)$

and

$$|L_T \circ \theta_1(f_i) - L_T(f_i)| \le \frac{2\|f_i\|_\infty}{T}$$

so, we obtain for $T \ge \frac{4}{\delta}\max_{1\le i\le d}\|f_i\|_\infty$

$$\begin{aligned}
\mathbb{P}_\nu\left(L_T \in N(\beta_0, \delta)\right) &\ge \mathbb{P}_\nu\left(L_T \circ \theta_1 \in N(\beta_0, \delta/2)\right) \\
&\ge \mathbb{P}_{\tilde{\nu}}\left(L_T \in N(\beta_0, \delta/2)\right).
\end{aligned}$$

We conclude by using the uniform lower bound on $\mathcal{M}^*_{\lambda_0, e^{\lambda_0 C} L}$, obtained in the preceding section.

6.2 Extension of the upper bound

Let F closed in $(M_1(H),\tau)$ such that $\inf_F J = a > 0$ (else the upper bound is clear). We define

$$F_\delta := \Big\{\beta \in M_1(H) \ : \ d_{\|\cdot\|_{var}}(\beta, F) < \delta\Big\}$$

where $d_{\|\cdot\|_{var}}(\beta, F) := \inf_{\lambda\in F}\|\beta-\lambda\|_{var}$, and the total variation norm of λ is

$$\|\lambda\|_{var} := \sup_{f\in b\mathcal{B}(H),\ \|f\|_\infty\le 1}\left|\int f(x)\lambda(dx)\right|. \tag{6.1}$$

Since $\|L_t - L_t\circ\theta_1\|_{var} \le \frac{2}{t}$, we obtain for $t > 2/\delta$,

$$\begin{array}{rcl}\mathbb{P}_\nu\left(L_t \in F\right) & \le & \mathbb{P}_\nu\left(L_t\circ\theta_1 \in F_\delta\right) \\ & = & \mathbb{P}_{\tilde\nu}\left(L_t \in F_\delta\right) \\ & = & \mathbb{P}_{\tilde\nu}\left(L_t \in F_\delta \cap M_1(D(A^\alpha))\right)\end{array}$$

by the regularity properties of the solution under $\tilde\nu$, defined by Lemma 6.1.

Let us fix $0 < b < a$. Since $[J \le b]$ is contained in the open F^c (the complement of F), for each $\nu_i \in [J\le b]$, we can take a neighborhood $N(\nu_i,\delta_i)$ of ν_i included in F^c. Moreover $N(\nu_i,\delta_i)$ can be chosen of form

$$N(\nu_i,\delta_i) := \Big\{\beta\in M_1(H),\ |\nu_i(f_{i,j}) - \beta(f_{i,j})| < \delta_i,\ \forall j = 1\dots d_i\Big\}$$

for a finite number d_i of bounded and measurable $f_{i,j}$ with $\|f_{i,j}\|_\infty \le 1$ for $1\le j\le d_i$. In particular $F\subset N(\nu_i,\delta_i)^c$.

Now, by Lemma 5.1, for any $b<a$, $[J\le b]$ is compact in $(M_1(D(A^\alpha)),\tau)$ and so in $(M_1(H),\tau)$ since $M_1(D(A^\alpha))$ is just a borelian subset of $M_1(H)$. So, we can extract a finite number p of $\nu_i\in[J\le b]$ such that

$$[J\le b] \ \subset\ \cup_{i=1}^{i=p} N(\nu_i,\delta_i/2) \ \subset\ \cup_{i=1}^{i=p} N(\nu_i,\delta_i) \ \subset\ F^c.$$

We now prove that if $\delta \le \min_{i=1\dots p}\delta_i/2$, then

$$\cup_{i=1}^{i=p} N(\nu_i,\delta_i/2) \subset {F_\delta}^c. \tag{6.2}$$

Indeed, if $\nu\in F_\delta$ we can find $\beta\in F$ such that $\|\nu-\beta\|_{var}\le\delta$. For any $i = 1\dots p$, since $F\subset N(\nu_i,\delta_i)^c$, there is some j such that

$$|\beta(f_{i,j}) - \nu_i(f_{i,j})| \ge \delta_i.$$

With (6.1) and the fact that $\|f_{i,j}\|_\infty\le 1$, we obtain

$$\begin{array}{rcl}|\nu(f_{i,j}) - \nu_i(f_{i,j})| & \ge & |\beta(f_{i,j}) - \nu_i(f_{i,j})| - |\beta(f_{i,j}) - \nu(f_{i,j})| \\ & \ge & \delta_i - \delta \\ & \ge & \delta_i/2\end{array}$$

for $\delta \leq \min_{i=1\dots p} \delta_i/2$. So if $\nu \in F_\delta$, then $\nu \in N(\nu_i, \delta_i/2)^c$, for any $i = 1\dots p$ and (6.2) is satisfied.

We obtain for $C > 0$ as in Lemma 6.1 and by the upper bound in Lemma 5.1

$$\begin{aligned}
& \limsup_{T\to\infty} \frac{1}{T} \log \sup_{\nu\in\mathcal{M}_{\lambda_0,L}} \mathbb{P}_\nu\left(L_t \in F\right) \\
\leq\ & \limsup_{T\to\infty} \frac{1}{T} \log \sup_{\tilde{\nu}\in\mathcal{M}^*_{\lambda_0,\exp(\lambda_0 C)L}} \mathbb{P}_{\tilde{\nu}}\left(L_t \in F_\delta\right) \\
\leq\ & \limsup_{T\to\infty} \frac{1}{T} \log \sup_{\tilde{\nu}\in\mathcal{M}^*_{\lambda_0,\exp(\lambda_0 C)L}} \mathbb{P}_{\tilde{\nu}}\left(L_t \in \cap_{i=1\dots p} N(\nu_i,\delta_i/2)^c\right) \\
\leq\ & - \inf_{\nu\ \in\ \cap_{i=1}^{i=p} N(\nu_i,\delta_i/2)^c} J(\nu) \\
\leq\ & -b
\end{aligned}$$

since the closed subset $\cap_{i=1}^{i=p} N(\nu_i, \delta_i/2)^c$ is contained in $[J \leq b]^c$. Noting that $0 < b < a$ is arbitrary, we obtain the upper bound (iii) in Theorem 1.1, which finishes the proof of Theorem 1.1. □

7 Extension to unbounded functionals

Let us now precise how the strong τ topology in Theorem 1.1, and the exponential estimate (4.3) imply Proposition 1.3 and more generally Corollary 1.2. In the sequel we suppose that our assumption (1.6) is satisfied for some $\frac{1}{4} < \alpha < \frac{1}{2}$, and that

$$0 < \lambda_0 < \frac{\lambda_1}{2\|Q\|}$$

is a fixed real number.

Proof of Corollary 1.2. For a measurable function $f : D\left(A^{\frac{1}{2}}\right) \to \mathbb{B}$, bounded on balls, let us consider $f_n : H \to \mathbb{B}$,

$$f_n(x) := \begin{cases} f(x) & \text{if } x \in D\left(A^{\frac{1}{2}}\right),\ |A^{\frac{1}{2}}x| \leq n \\ 0 & \text{otherwise} \end{cases} \tag{7.1}$$

which is far from being continuous, but is measurable and bounded on H. Since $\nu \to \nu(f_n)$ is continuous from $M_1(H)$ to $\mathbb{B}$ by [9, Lemma 3.3.8], $L_T(f_n)$ satisfies the LDP on $\mathbb{B}$ by Theorem 1.1 and a standard contraction principle.

Now, by the approximation lemma in large deviations (see [9, Lemma 2.1.4]), it remains to prove for any $L > 0$,

$$\limsup_{n\to\infty} \sup_{\beta\ :\ J(\beta)\leq L} \|\beta(f_n) - \beta(f)\|_{\mathbb{B}} = 0 \tag{7.2}$$

and for any $\delta > 0$

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \log \sup_{\nu\in\mathcal{M}_{\lambda_0,L}} \mathbb{P}_\nu\left(\|L_T(f - f_n)\|_{\mathbb{B}} > \delta\right) = -\infty. \tag{7.3}$$

Thanks to our condition (1.9) on f, we can construct a sequence $(\varepsilon(n))_n$ decreasing to 0 such that, once $|A^{\frac{1}{2}}x| \geq n$, we have

$$\|f(x)\|_{\mathbb{B}} \leq \varepsilon(n)|A^{\frac{1}{2}}x|^2.$$

Denoting by 1_Γ the characteristic function of the set Γ, we have for any β satisfying $J(\beta) < L$,

$$\begin{aligned}
\|\beta(f_n) - \beta(f)\|_{\mathbb{B}} &= \left\|\beta\left(f1_{\{|A^{\frac{1}{2}}x|\geq n\}}\right)\right\|_{\mathbb{B}} \\
&\leq \beta\left(\varepsilon(n)|A^{\frac{1}{2}}x|^2 1_{\{|A^{\frac{1}{2}}x|\geq n\}}\right) \\
&\leq \frac{\varepsilon(n)}{\lambda_0}\beta\left(\lambda_0|A^{\frac{1}{2}}x|^2\right) \\
&\leq \frac{\varepsilon(n)}{\lambda_0}\left(L + \lambda_0\left(tr(Q) + \frac{\lambda_1|f|^2}{\lambda_1 - 2\|Q\|\lambda_0}\right)\right)
\end{aligned}$$

by using (4.5). Thus (7.2) follows. Let us also evaluate

$$\begin{aligned}
\mathbb{P}_\nu\left(\|L_T(f - f_n)\|_{\mathbb{B}} > \delta\right) &= \mathbb{P}_\nu\left(\left\|\frac{1}{T}\int_0^T f(X_s) - f_n(X_s)ds\right\|_{\mathbb{B}} > \delta\right) \\
&\leq \mathbb{P}_\nu\left(\frac{1}{T}\int_0^T \varepsilon(n)|A^{\frac{1}{2}}X(s)|^2 1_{\{|A^{\frac{1}{2}}X(s)|\geq n\}}ds > \delta\right) \\
&\leq \mathbb{P}_\nu\left(\int_0^T \lambda_0|A^{\frac{1}{2}}X(s)|^2 1_{\{|A^{\frac{1}{2}}X(s)|\geq n\}}ds > \frac{\lambda_0 T\delta}{\varepsilon(n)}\right) \\
&\leq \exp\left(-\frac{\lambda_0 T\delta}{\varepsilon(n)}\right)\mathbb{E}^\nu \exp\left(\lambda_0\int_0^T |A^{\frac{1}{2}}X(s)|^2 ds\right)
\end{aligned}$$

so that (7.3) is consequence of (4.3). ☐

Proof of Proposition 1.3. It is a particular case of Corollary 1.2, since the choice $f(x) = x$ on $\mathbb{B} := D\left(A^{\frac{1}{2}}\right)$ is allowed. ☐

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