The validity of the multifractal formalism. Results and examples

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By obtaining a new sufficient condition for a valid multifractal formalism, we improve in this paper a result developped by Olsen (1995, *Adv. Math.* **116,** 82-196). In particular, we describe a large class of measures satisfying the multifractal formalism and for which the construction of Gibbs measures is not possible. Some of these measures are not unidimensional but have a nontrivial multifractal spectrum, giving a negative answer to a question asked by S.J. Taylor (1995, *J. Fourier Anal. Appl.,* special issue). We also describe a necessary condition of validity for the formalism which is very close to the sufficient one. This necessary condition allows us to describe a measure μ for which the multifractal packing dimension function $B_\mu(q)$ is a nontrivial real analytic function but the multifractal formalism is nowhere satisfied. This example gives also a solution to a problem posed by Taylor (cited above).

Key Words: multifractal formalism; multifractal spectrum; Hausdorff dimension; packing dimension

1. INTRODUCTION AND THEORETICAL RESULTS

In this paper, we want to describe results about the validity of the multifractal formalism of measures. There has recently been a great interest for this subject and positive results have been written in various situations (see for example [2, 3, 5, 6, 7, 10, 13, 16]). The setting used in this article was originally developed by Olsen in [13] and does not require any dynamical context.

Let us briefly recall the notations and the main results proved by Olsen.

In the sequel, $\mathcal{P}(\mathbf{R}^d)$ is the set of Borel probability measures on $\mathbf{R}^d$ and $\mu \in \mathcal{P}(\mathbf{R}^d)$. If E is a non empty subset of $\mathbf{R}^d$ and if $q, t \in \mathbf{R}$ and $\delta > 0$, we introduce the quantities

$$\begin{aligned}
\overline{\mathcal{H}}^{q,t}_{\mu,\delta}(E) &= \inf\{\sum_i \mu(B(x_i,r_i))^q (2r_i)^t \ ; \\
&\qquad (B(x_i,r_i))_i \text{ is a centered } \delta\text{-covering of } E\} \\
\overline{\mathcal{H}}^{q,t}_{\mu}(E) &= \sup_{\delta>0} \overline{\mathcal{H}}^{q,t}_{\mu,\delta}(E) \\
\overline{\mathcal{P}}^{q,t}_{\mu,\delta}(E) &= \sup\{\sum_i \mu(B(x_i,r_i))^q (2r_i)^t \ ; \\
&\qquad (B(x_i,r_i))_i \text{ is a centered } \delta\text{-packing of } E\} \\
\overline{\mathcal{P}}^{q,t}_{\mu}(E) &= \inf_{\delta>0} \overline{\mathcal{P}}^{q,t}_{\mu,\delta}(E)
\end{aligned}$$

The function $\overline{\mathcal{H}}^{q,t}_{\mu}$ is σ-subadditive but not increasing and the the function $\overline{\mathcal{P}}^{q,t}_{\mu}$ is increasing but not σ-subadditive. That is the reason why Olsen introduces the following modifications of $\overline{\mathcal{H}}^{q,t}_{\mu}$ and $\overline{\mathcal{P}}^{q,t}_{\mu}$:

$$\mathcal{H}^{q,t}_{\mu}(E) = \sup_{F\subset E} \overline{\mathcal{H}}^{q,t}_{\mu}(F), \qquad \mathcal{P}^{q,t}_{\mu}(E) = \inf_{E\subset \cup_i E_i} \sum_i \overline{\mathcal{P}}^{q,t}_{\mu}(E_i) \ .$$

The functions $\mathcal{H}^{q,t}_{\mu}$ and $\mathcal{P}^{q,t}_{\mu}$ are outer measures (in the Carathéodory sense) for which Borel sets are measurable. They are multifractal extensions of the Hausdorff measures $\mathcal{H}^t$ and the packing measure $\mathcal{P}^t$. In the same way, the quantity $\overline{\mathcal{P}}^{q,t}_{\mu}$ is a multifractal extension of the prepacking measure $\overline{\mathcal{P}}^t$. For more details on the measures $\mathcal{H}^t$, $\mathcal{P}^t$ and the premeasures $\overline{\mathcal{P}}^t$, see, for example, [8].

The measures $\mathcal{H}^{q,t}_{\mu}$, $\mathcal{P}^{q,t}_{\mu}$ and the pre-measures $\overline{\mathcal{P}}^{q,t}_{\mu}$ assign in the usual way a dimension to each subset E of $\mathbf{R}^d$. They are respectively denoted $\dim^q_{\mu}(E)$, $\mathrm{Dim}^q_{\mu}(E)$, and $\Delta^q_{\mu}(E)$ and characterized by :

$$\begin{aligned}
\overline{\mathcal{P}}^{q,t}_{\mu}(E) &= \begin{cases} \infty & \text{for } t < \Delta^q_{\mu}(E) \\ 0 & \text{for } \Delta^q_{\mu}(E) < t \end{cases} \\
\mathcal{P}^{q,t}_{\mu}(E) &= \begin{cases} \infty & \text{for } t < \mathrm{Dim}^q_{\mu}(E) \\ 0 & \text{for } \mathrm{Dim}^q_{\mu}(E) < t \end{cases} \\
\mathcal{H}^{q,t}_{\mu}(E) &= \begin{cases} \infty & \text{for } t < \dim^q_{\mu}(E) \\ 0 & \text{for } \dim^q_{\mu}(E) < t \end{cases}
\end{aligned}$$

The number $\dim_\mu^q(E)$ is a multifractal extension of the Hausdorff dimension $\dim(E)$ of E whereas the numbers $\mathrm{Dim}_\mu^q(E)$ and $\Delta_\mu^q(E)$ are multifractal extensions of the packing dimension $\mathrm{Dim}(E)$ and the prepacking dimension $\Delta(E)$ of E respectively. More precisely we have the equalities

$$\dim(E) = \dim_\mu^0(E),\ \mathrm{Dim}(E) = \mathrm{Dim}_\mu^0(E) \text{ and } \Delta(E) = \Delta_\mu^0(E)\ .$$

We can also remark that $\dim_\mu^q(E) \leq \mathrm{Dim}_\mu^q(E) \leq \Delta_\mu^q(E)$.

Then, we are able to define the multifractal dimension functions b_μ and B_μ by

$$b_\mu(q) = \dim_\mu^q(\operatorname{supp}\mu),\ \ B_\mu(q) = \mathrm{Dim}_\mu^q(\operatorname{supp}\mu) \ \text{ and } \ \Lambda_\mu(q) = \Delta_\mu^q(\operatorname{supp}\mu)\ .$$

These functions satisfy the following properties :

Proposition 1.1 ([13]). *Let $\mu \in \mathcal{P}(\mathbf{R}^d)$. Then,*

(i) $b_\mu \leq B_\mu \leq \Lambda_\mu$ *and* $b_\mu(1) = B_\mu(1) = \Lambda_\mu(1) = 0$
(ii) $b_\mu(0) = \dim(\operatorname{supp}\mu)$
(iii) $B_\mu(0) = \mathrm{Dim}(\operatorname{supp}\mu)$ *and* $\Lambda_\mu(0) = \Delta(\operatorname{supp}\mu)$
(iv) b_μ *is decreasing and* B_μ *and* Λ_μ *are convex and decreasing.*

The functions b_μ and B_μ are related to the multifractal spectrum of the measure μ. More precisely, if $f^*(x) = \inf_y(xy + f(y))$ denotes the Legendre transform of the function f, and if

$$X(\alpha) = \left\{ x \in \operatorname{supp}\mu\ ;\ \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\},$$

Olsen proved the following statement.

Theorem 1.1 ([13]). *Let $\mu \in \mathcal{P}(\mathbf{R}^d)$. Define $\underline{a} = \sup_{q>0} -\frac{b(q)}{q}$ and $\overline{a} = \inf_{q<0} -\frac{b(q)}{q}$. Then,*

$$\dim(X(\alpha)) \leq b_\mu^*(\alpha) \quad \textit{and} \quad \mathrm{Dim}(X(\alpha)) \leq B_\mu^*(\alpha) \quad \textit{for all } \alpha \in (\underline{a}, \overline{a})\ .$$

It is more difficult to obtain a lower bound for the dimension of the set $X(\alpha)$. In general, such a minoration is related to the existence of an auxiliary measure (also called Gibbs measure) which is supported by the set to be analysed. In his paper, Olsen also gives a result in such a way and supposes (among other things) the existence of a Gibbs measure at state q for the measure μ i.e. the existence of a measure ν_q on $\operatorname{supp}\mu$ and constants $C > 0$, $\lambda > 0$ such that for every $x \in \operatorname{supp}\mu$ and every $0 < r < \lambda$,

$$\frac{1}{C}\mu(B(x,r))^q(2r)^{B_\mu(q)} \leq \nu_q(B(x,r)) \leq C\mu(B(x,r))^q(2r)^{B_\mu(q)} \qquad (1)$$

to conclude that

$$\dim(X(-B'_\mu(q))) = \mathrm{Dim}(X(-B'_\mu(q))) = b^*_\mu(-B'_\mu(q)) = B^*_\mu(-B'_\mu(q)) \ .$$

In general, one needs some degree of similarity to prove the existence of Gibbs measures. For example, in dynamic contexts, the existence of such measures are often natural. Our purpose is to improve Olsen's result and to propose a new sufficient condition that gives the lower bound (Theorem 1.2). We also observe that this sufficient condition is very close to being a necessary and sufficient condition (Theorem 1.3).

The second part of the paper is devoted to the description of examples and conterexamples that illustrate the theoretical results. We explain the difference between our positive result and Olsen's result in describing a class of measures which do not satisfy Olsen's hypothesis but satisfy the multifractal formalism. We also describe a measure μ for which the function B_μ is of class C^1 (and even real analytic) but the multifractal formalism is nowhere valid. This measure has another interest. It gives a solution to a problem due to S.J. Taylor ([17], page 567).

Let us now explain our results. For simplicity, we will write in the sequel $b = b_\mu$, $B = B_\mu$ and $\Lambda = \Lambda_\mu$. If $x \in \mathbf{R}^d$, define the local dimensions of the measure μ at point x by

$$\overline{\alpha}_\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \quad \text{and} \quad \underline{\alpha}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \ .$$

Then, if $\alpha \geq 0$, let us introduce the fractal sets

$$\begin{aligned} \overline{X}^\alpha &= \{x \in \mathrm{supp}\,\mu \ ; \ \overline{\alpha}_\mu(x) \leq \alpha\} \\ \underline{X}_\alpha &= \{x \in \mathrm{supp}\,\mu \ ; \ \underline{\alpha}_\mu(x) \geq \alpha\} \end{aligned}$$

and $X(\alpha) = \underline{X}_\alpha \cap \overline{X}_\alpha$.

THEOREM 1.2. *Let $\mu \in \mathcal{P}(\mathbf{R}^d)$ and $q \in \mathbf{R}$ such that $\mathcal{H}^{q,B(q)}_\mu(\mathrm{supp}\,\mu) > 0$. Then*

$$\dim\left(\underline{X}_{-B'_+(q)} \cap \overline{X}^{-B'_-(q)}\right) \geq \begin{cases} -B'_+(q)q + B(q) & if \quad q \geq 0 \\ -B'_-(q)q + B(q) & if \quad q \leq 0 \ . \end{cases}$$

The following result proves that the condition $\mathcal{H}^{q,B(q)}_\mu(\mathrm{supp}\,\mu) > 0$ is very close to being a necessary and sufficient condition for the validity of the multifractal formalism.

THEOREM 1.3. *Let $\mu \in \mathcal{P}(\mathbf{R}^d)$ and $q \in \mathbf{R}$. Suppose that*

$$\begin{aligned} &\dim\left(\underline{X}_{-B'_+(q)} \cap \overline{X}^{-B'_-(q)}\right) \geq -B'_-(q)q + B(q) \quad \textit{and} \quad q \geq 0 \\ \textit{or} \quad &\dim\left(\underline{X}_{-B'_+(q)} \cap \overline{X}^{-B'_-(q)}\right) \geq -B'_+(q)q + B(q) \quad \textit{and} \quad q \leq 0 \ . \end{aligned}$$

Then, $b(q) = B(q)$. In other words, $\mathcal{H}_\mu^{q,t}(\operatorname{supp}\mu) > 0$, for every $t < B(q)$.

In the case where $B'(q)$ exists, Theorems 1.2 and 1.3 take a simpler form. More precisely, using Theorem 1.1 and the relations $b^* \leq B^*$ and $B^*(-B'(q)) = -B'(q)q + B(q)$, we obtain :

COROLLARY 1.1. (A sufficient and a necessary condition for a valid multifractal formalism). *Let $\mu \in \mathcal{P}(\mathbf{R}^d)$ and $q \in \mathbf{R}$. Suppose that $\alpha = -B'(q)$ exists.*

(i) *If $\mathcal{H}_\mu^{q,B(q)}(\operatorname{supp}\mu) > 0$, then*

$$\dim(X(\alpha)) = \operatorname{Dim}(X(\alpha)) = B^*(\alpha) = b^*(\alpha) \ .$$

(ii) *If $\dim(X(\alpha)) \geq B^*(\alpha)$, then $b(q) = B(q)$. That is, $\mathcal{H}_\mu^{q,t}(\operatorname{supp}\mu) > 0$, for every $t < B(q)$.*

Remarks. 1. The hypothesis $\mathcal{H}_\mu^{q,B(q)}(\operatorname{supp}\mu) > 0$ implies that $b(q) = B(q)$ which is also known as the Taylor regularity condition (see [17] or [15]). Nevertheless, we don't know if the weaker condition $b(q) = B(q)$ is sufficient to obtain the conclusion of Theorem 1.2.

2. In [2] and [3], the first and second authors obtained a similar result in the case $q < 0$. Using the hypothesis $\mathcal{H}_\mu^{q,B(q)}(\operatorname{supp}\mu) > 0$ and Frostman's technique, they constructed an auxiliary Radon measure ν_q satisfying

$$\nu_q(B(x,r)) \leq C\mu(B(x,r))^q(2r)^{B(q)} \ . \tag{2}$$

In the case $q > 0$, such a construction is only possible for doubling measures (see [3]). In fact, the knowledge of the auxilliary measure ν_q is unnecessary to obtain the minoration of the Hausdorff dimension of $\underline{X}_{-B'_+(q)} \cap \overline{X}^{-B'_-(q)}$. The outer measure $\mathcal{H}_\mu^{q,B(q)}$ makes the work.

3. It is clear that the existence of a nontrivial measure ν_q satisfying (2) implies the condition $\mathcal{H}_\mu^{q,B(q)}(\operatorname{supp}\mu) > 0$. This is in particular the case if there exists a Gibbs measure (that is a measure satisfying (1)) for the state q. In Section 2.1, we will see that the existence of a measure ν_q satisfying (2) is strictly weaker than the existence of a Gibbs measure.

Proof of Theorem 1.2. Let $E_q = \underline{X}_{-B'_+(q)} \cap \overline{X}^{-B'_-(q)}$. It is well known that for all $\delta > 0$

$$\begin{cases} \mathcal{H}^{-B'_-(q)q+t-\delta}(E_q) \geq \mathcal{H}^{q,B(q)}_\mu(E_q) & \text{if} \quad q \geq 0 \\ \mathcal{H}^{-B'_+(q)q+t-\delta}(E_q) \geq \mathcal{H}^{q,B(q)}_\mu(E_q) & \text{if} \quad q \leq 0 \end{cases}$$

(see [13], Proposition 7.2). Theorem 1.2 is then an easy consequence of the following lemma.

Lemma 1.1. $\mathcal{H}^{q,B(q)}_\mu(\operatorname{supp}\mu \setminus E_q) = 0.$

The hypothesis of Theorem 1.2 implies that $b(q) = B(q)$. So, Lemma 1.1 is nothing but Theorem 2.2 in [15]. In fact, it was in some sense already announced by the first and second authors in [2] and [3], the measure $\mathcal{H}^{q,B(q)}_\mu$ being replaced by an auxiliary measure ν (see [2], page 255). That is why we propose the following short proof of Lemma 1.1. Let us introduce $F_\alpha = \operatorname{supp}\mu \setminus \underline{X}_\alpha$ and $G_\beta = \operatorname{supp}\mu \setminus \overline{X}^\beta$. We only have to prove that $\mathcal{H}^{q,B(q)}_\mu(F_\alpha) = 0$ for every $\alpha < -B'_+(q)$ and $\mathcal{H}^{q,B(q)}_\mu(G_\beta) = 0$ for every $\beta > -B'_-(q)$. Let us sketch the proof for the set F_α. If $\alpha < -B'_+(q)$, we can choose $t > 0$ such that $B(q+t) < B(q) - \alpha t$. It follows that $\mathcal{P}^{q+t,B(q)-\alpha t}_\mu(\operatorname{supp}\mu) = 0$. Let $\delta > 0$. For every $x \in F_\alpha$, we can find $r_x < \delta$ such that $\mu(B(x,r_x)) > (2r_x)^\alpha$. The family $(B(x,r_x))_{x\in F_\alpha}$ is then a centered δ-covering of F_α. Using Besicovitch's Covering Theorem, we can construct ζ finite or countable subfamilies $(B(x_{1j},r_{1j}))_j, \cdots, (B(x_{\zeta j},r_{\zeta j}))_j$ such that each $(B(x_{ij},r_{ij}))_j$ is a δ-packing of F_α and $F_\alpha \subset \bigcup_{i=1}^\zeta \bigcup_j B(x_{ij},r_{ij})$. Observing that

$$\mu(B(x_{ij},r_{ij}))^q(2r_{ij})^{B(q)} \leq \mu(B(x_{ij},r_{ij}))^{q+t}(2r_{ij})^{B(q)-\alpha t}$$

we successively obtain

$$\begin{cases} \overline{\mathcal{H}}^{q,B(q)}_{\mu,\delta}(F_\alpha) \leq \zeta \overline{\mathcal{P}}^{q+t,B(q)-\alpha t}_{\mu,\delta}(F_\alpha) \\ \overline{\mathcal{H}}^{q,B(q)}_{\mu}(F_\alpha) \leq \zeta \overline{\mathcal{P}}^{q+t,B(q)-\alpha t}_{\mu}(F_\alpha)\ . \end{cases}$$

In fact, in the last inequality, we can replace F_α by an arbitrary subset of F_α. Then, by standard arguments we can finally conclude that

$$\mathcal{H}^{q,B(q)}_\mu(F_\alpha) \leq \zeta \mathcal{P}^{q+t,B(q)-\alpha t}_\mu(F_\alpha) = 0$$

and the proof is finished. ∎

Proof of Theorem 1.3. It is an easy consequence of the following result.

Proposition 1.2. *Let $\mu \in \mathcal{P}(\mathbf{R}^d)$, $\alpha > 0$ and q such that $\alpha q + B(q) \geq 0$. Suppose that*

$$\dim\left(\overline{X}^{\alpha}\right) \geq \alpha q + B(q) \text{ and } q \geq 0 \quad \text{or} \quad \dim\left(\underline{X}_{\alpha}\right) \geq \alpha q + B(q) \text{ and } q \leq 0 \ .$$

Then, $b(q) = B(q)$. In other words, $\mathcal{H}_{\mu}^{q,t}(\operatorname{supp}\mu) > 0$, for every $t < B(q)$.

Let us sketch the proof when $q \geq 0$. Suppose that $\dim\left(\overline{X}^{\alpha}\right) \geq \alpha q + B(q)$. Let $t < B(q)$ and choose β such that $\alpha < \beta$ and $\beta q + t < \alpha q + B(q)$. If $p \in \mathbf{N}^*$, let

$$E_p = \left\{ x \in \overline{X}^{\alpha} \ ; \ \mu(B(x,r)) \geq (2r)^{\beta}, \ \forall 0 < r < 1/p \right\} \ .$$

Observe that $\bigcup_{p \in N^*} E_p = \overline{X}^{\alpha}$. We can then find an integer $p > 0$ such that $\dim E_p > \beta q + t$. If $0 < \delta < 1/p$ and if $(B(x_i, r_i))_i$ is a centered δ-covering of E_p, we have

$$\sum_i \mu(B(x_i, r_i))^q (2r_i)^t \geq \sum_i (2r_i)^{\beta q + t} \ .$$

Then, we easily get $\mathcal{H}_{\mu}^{q,t}(\operatorname{supp}\mu) \geq \mathcal{H}_{\mu}^{q,t}(E_p) > 0$. ▮

Remark. We proposed this proof of Proposition 1.2 in order to be selfcontained. In fact, Proposition 1.2 is also an immediate consequence of Proposition 2.4 in [13] which says that

$$\begin{cases} \dim\left(\overline{X}^{\alpha}\right) \leq \alpha q + b(q) & \text{if} \quad q \geq 0 \quad \text{and} \quad \alpha q + b(q) \geq 0 \\ \dim\left(\underline{X}_{\alpha}\right) \leq \alpha q + b(q) & \text{if} \quad q \leq 0 \quad \text{and} \quad \alpha q + b(q) \geq 0 \ . \end{cases}$$

2. EXAMPLES

2.1. The validity of the multifractal formalism does not imply the existence of Gibbs measures

Theorem 1.2 and the remarks following Theorem 1.2 make sense if we can construct measures with valid multifractal formalism but for which it is not possible to construct Gibbs measures. That is what we do in the following results.

PROPOSITION 2.1. *Let $q > 0$. The set*

$$\mathcal{A}_q = \{\mu \in \mathcal{P}(\mathbf{R}^d)\ ;\ \mathcal{H}_\mu^{q,B_\mu(q)}(\operatorname{supp}\mu) > 0\}$$

is a convex subset of $\mathcal{P}(\mathbf{R}^d)$.

THEOREM 2.1. *Let μ_1, $\mu_2 \in \mathcal{P}(\mathbf{R}^d)$ such that $\operatorname{supp}\mu_1 = \operatorname{supp}\mu_2$ and $q > 0$. Suppose that there exists a Gibbs measure at state q for the measures μ_1 and μ_2. Let $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ $(0 < \alpha < 1)$. In general, there is no Gibbs measure at state q for the measure μ. More precisely, if one exists, then*

$$B_{\mu_1}(q) = B_{\mu_2}(q) \quad or \quad \exists c > 0\ ;\ \mu_1 \le c\mu_2 \quad or \quad \exists c > 0\ ;\ \mu_2 \le c\mu_1\ .$$

Nevertheless, $\mu \in \mathcal{A}_q$ and the conclusion of Theorem 1.2 is valid.

As a result of Theorem 2.1, we easily obtain the following corollary.

COROLLARY 2.1. *Suppose that μ_1 and μ_2 are mutually singular with the same support, have Gibbs measures at state $q > 0$, and are such that $B_{\mu_1}(q) \neq B_{\mu_2}(q)$. Then $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ $(0 < \alpha < 1)$ verifies the conclusion of Theorem 1.2 without having a Gibbs measure at state q.*

Let us begin with the proof of Proposition 2.1. Let $\mu_1, \mu_2 \in \mathcal{A}_q$ and $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$. The convexity of the set $\mathcal{A}_q$ is an easy consequence of the following lemma.

LEMMA 2.1. *For every $q > 0$, $B_\mu(q) = \sup(B_{\mu_1}(q), B_{\mu_2}(q))$.*

Suppose that Lemma 2.1 is true. If, for example, $B_\mu(q) = B_{\mu_1}(q)$, we obtain

$$\mathcal{H}_\mu^{q,B_\mu(q)}(\operatorname{supp}\mu) = \mathcal{H}_\mu^{q,B_{\mu_1}(q)}(\operatorname{supp}\mu) \ge \alpha^q \mathcal{H}_{\mu_1}^{q,B_{\mu_1}(q)}(\operatorname{supp}\mu_1)$$

and we can conclude that $\mu \in \mathcal{A}_q$.

Let us now prove Lemma 2.1. If $(B(x_i,r_i))_i$ is a centered δ-packing of $E \subset \mathbf{R}^d$ and $t \in \mathbf{R}$, we have

$$\sum_i \mu(B(x_i,r_i))^q (2r_i)^t \ge \alpha^q \sum_i \mu_1(B(x_i,r_i))^q (2r_i)^t\ .$$

Taking the supremum over the centered δ-packing of E and the limit when $\delta \to 0$, we get

$$\overline{\mathcal{P}}_\mu^{q,t}(E) \ge \alpha^q \overline{\mathcal{P}}_{\mu_1}^{q,t}(E)\ .$$

Using countable coverings of $\operatorname{supp}\mu$, we can conclude that

$$\mathcal{P}_{\mu}^{q,t}(\operatorname{supp}\mu) \geq \alpha^q \mathcal{P}_{\mu_1}^{q,t}(\operatorname{supp}\mu) \geq \alpha^q \mathcal{P}_{\mu_1}^{q,t}(\operatorname{supp}\mu_1)$$

which gives the inequality $B_\mu(q) \geq B_{\mu_1}(q)$. In the same way, we can prove that $B_\mu(q) \geq B_{\mu_2}(q)$.

On the other hand, observe that for every $a, b > 0$, $(a+b)^q \leq C(a^q + b^q)$, with $C = \sup(1, 2^{q-1})$. If $(B(x_i, r_i))_i$ is a centered δ-packing of $E \subset \mathbf{R}^d$ and $t \in \mathbf{R}$, we have

$$\begin{aligned} \textstyle\sum_i \mu(B(x_i,r_i))^q (2r_i)^t \leq C\alpha^q \ & \textstyle\sum_i \mu_1(B(x_i,r_i))^q (2r_i)^t \\ & +C(1-\alpha)^q \textstyle\sum_i \mu_2(B(x_i,r_i))^q (2r_i)^t \ . \end{aligned}$$

Taking the supremum over the centered δ-packing of E and the limit when $\delta \to 0$, we get

$$\overline{\mathcal{P}}_{\mu}^{q,t}(E) \leq C\alpha^q \overline{\mathcal{P}}_{\mu_1}^{q,t}(E) + C(1-\alpha)^q \overline{\mathcal{P}}_{\mu_2}^{q,t}(E) \ . \tag{3}$$

Let $t > \sup(B_{\mu_1}(q), B_{\mu_2}(q))$. If $k \in \{1, 2\}$, we have

$$\mathcal{P}_{\mu_k}^{q,t}(\operatorname{supp}\mu) = \mathcal{P}_{\mu_k}^{q,t}(\operatorname{supp}\mu_k) = 0 \ .$$

We may then choose countable coverings $(E_i)_i$ and $(F_j)_j$ of $\operatorname{supp}\mu$ such that

$$\sum_i \overline{\mathcal{P}}_{\mu_1}^{q,t}(E_i) \leq 1 \quad \text{and} \quad \sum_j \overline{\mathcal{P}}_{\mu_2}^{q,t}(F_j) \leq 1 \ .$$

Hence, for each $(i,j) \in \mathbf{N}^2$, we have

$$\begin{aligned} \mathcal{P}_{\mu}^{q,t}(E_i \cap F_j) &\leq \overline{\mathcal{P}}_{\mu}^{q,t}(E_i \cap F_j) \\ &\leq C\alpha^q \overline{\mathcal{P}}_{\mu_1}^{q,t}(E_i \cap F_j) + C(1-\alpha)^q \overline{\mathcal{P}}_{\mu_2}^{q,t}(E_i \cap F_j) \\ &\leq C\alpha^q + C(1-\alpha)^q \ . \end{aligned}$$

This implies that

$$\operatorname{Dim}_{\mu}^{q}(E_i \cap F_j) \leq t \ .$$

Observing that $\operatorname{supp}\mu \subset \bigcup_{i,j} E_i \cap F_j$, we conclude that

$$B_\mu(q) = \operatorname{Dim}_{\mu}^{q}(\operatorname{supp}\mu) \leq \sup_{i,j} \left(\operatorname{Dim}_{\mu}^{q}(E_i \cap F_j)\right) \leq t$$

and the proof of the lemma is finished. ▮

Remark. When $q < 0$, the function B_μ is more complicated to compute. The only elementary relation is $B_\mu(q) \le \inf(B_{\mu_1}(q), B_{\mu_2}(q))$.

We can now prove Theorem 2.1. It is easy to check that $\operatorname{supp}\mu = \operatorname{supp}\mu_1 = \operatorname{supp}\mu_2$. Suppose that there exists a Gibbs measure at state q for the measure μ. Then, there exist probability measures ν, ν_1, ν_2 on $\operatorname{supp}\mu$ and constants $C > 0$, $\lambda > 0$ such that for every $x \in \operatorname{supp}\mu$ and for every $0 < r < \lambda$,

$$\begin{cases} \frac{1}{C}\mu(B(x,r))^q(2r)^{B_\mu(q)} \le \nu(B(x,r)) \le C\mu(B(x,r))^q(2r)^{B_\mu(q)} \\ \frac{1}{C}\mu_1(B(x,r))^q(2r)^{B_{\mu_1}(q)} \le \nu_1(B(x,r)) \le C\mu_1(B(x,r))^q(2r)^{B_{\mu_1}(q)} \\ \frac{1}{C}\mu_2(B(x,r))^q(2r)^{B_{\mu_2}(q)} \le \nu_2(B(x,r)) \le C\mu_2(B(x,r))^q(2r)^{B_{\mu_2}(q)}. \end{cases}$$

Finally, suppose that $B_{\mu_1}(q) \ne B_{\mu_2}(q)$. Without loss of generality, we will sketch the proof in the case where $B_{\mu_1}(q) > B_{\mu_2}(q)$ and prove that $\mu_2 \le c\mu_1$. According to Lemma 2.1, we know that $B_\mu(q) = B_{\mu_1}(q)$. Let $x \in \operatorname{supp}\mu$, $0 < r < \lambda$ and $\varepsilon > 0$ such that $r + \varepsilon < \lambda$. For simplicity, denote $B = B(x,r)$. Let $(B(x_i,\varepsilon))_{i\in\{1,\cdots,n\}}$ be a finite centered covering of $B \cap \operatorname{supp}\mu$. Using Besicovitch's Covering Theorem, we can construct ζ subfamilies $(B_{1j})_j, \cdots, (B_{\zeta j})_j$ constituted of disjoint balls and such that $B \cap \operatorname{supp}\mu \subset \bigcup_{i=1}^{\zeta}\bigcup_j B_{ij}$. The measure ν being supported by $\operatorname{supp}\mu$, we have

$$\begin{aligned} \mu(B)^q(2r)^{B_{\mu_1}(q)} &\le C\nu(B) \\ &\le C\sum_{i=1}^{\zeta}\sum_j \nu(B_{ij}) \\ &\le C^2\sum_{i=1}^{\zeta}\sum_j \mu(B_{ij})^q(2\varepsilon)^{B_{\mu_1}(q)}\ . \end{aligned}$$

Remember that for every $a, b \ge 0$, $(a+b)^q \le \sup(1, 2^{q-1})(a^q + b^q)$. We can then find a constant C_1 depending only on q and α such that

$$\begin{aligned} \sum_{i=1}^{\zeta}\sum_j \mu(B_{ij})^q(2\varepsilon)^{B_{\mu_1}(q)} \le C_1 \Big(&\sum_{i=1}^{\zeta}\sum_j \mu_1(B_{ij})^q(2\varepsilon)^{B_{\mu_1}(q)} \\ &+ \sum_{i=1}^{\zeta}\sum_j \mu_2(B_{ij})^q(2\varepsilon)^{B_{\mu_1}(q)}\Big)\ . \end{aligned}$$

On the other hand, for $k = 1, 2$, and $i \in \{1, \cdots, \zeta\}$,

$$\sum_j \mu_k(B_{ij})^q(2\varepsilon)^{B_{\mu_k}(q)} \ \le\ C\sum_j \nu_k(B_{ij})$$

$$\begin{aligned} &\leq C\,\nu_k(B(x,r+\varepsilon)) \\ &\leq C^2\mu_k(B(x,r+\varepsilon))^q(2(r+\varepsilon))^{B_{\mu_k}(q)} \ . \end{aligned}$$

If $C_2 = C_1\zeta C^4$, we can conclude that

$$\begin{aligned} \mu(B(x,r))^q(2r)^{B_{\mu_1}(q)} \leq\ & C_2\left(\mu_1(B(x,r+\varepsilon))^q(2(r+\varepsilon))^{B_{\mu_1}(q)}\right. \\ & \left.+\mu_2(B(x,r+\varepsilon))^q(2(r+\varepsilon))^{B_{\mu_2}(q)}(2\varepsilon)^{B_{\mu_1}(q)-B_{\mu_2}(q)}\right) . \end{aligned}$$

Taking the limit when $\varepsilon \to 0$, we deduce that

$$\mu(B(x,r)) \leq C_2^{1/q}\mu_1(B(x,r)) \ .$$

Finally, if $c = (C_2^{1/q} - \alpha)/(1-\alpha)$, we obtain that for every $x \in \operatorname{supp}\mu$ and every $r < \lambda$,

$$\mu_2(B(x,r)) \leq c\mu_1(B(x,r))$$

which says that $\mu_2 \leq c\mu_1$.

2.2. Nonexact dimensional measures with non trivial multifractal spectra

Let $\mathcal{A}$ be the set of measures $\mu \in \mathcal{P}(\mathbf{R}^d)$ satisfying the following properties :

(H1) B_μ is strictly convex and of class C^1

(H2) $\mathcal{H}_\mu^{q,B_\mu(q)}(\operatorname{supp}\mu) > 0$ for all $q \in \mathbf{R}$.

The hypothesis (H2) is satisfied if there exists a Gibbs measure at each state q and it is well known that the set $\mathcal{A}$ is nonempty. In fact, there are measures μ in $\mathcal{A}$ for which the function B_μ is real analytic.

Let $\mu \in \mathcal{A}$, $\delta = -B'_\mu(1)$ and $I = (\underline{a}, \overline{a}) = (-B'_\mu(+\infty), -B'_\mu(-\infty))$. Using [15] or [10] and Corollary 1.1, we can conclude that

(P1) $\underline{\alpha}_\mu(x) = \overline{\alpha}_\mu(x) = \delta$ for μ-almost every x.

(P2) $\dim(X(\alpha)) = \operatorname{Dim}(X(\alpha)) = B^*(\alpha)$ for all α in a nontrivial interval I.

When property (P1) is satisfied, we say that the measure μ is dimension regular with exact dimension δ. In particular, it is supported by a set E with Hausdorff dimension δ and every set F such that $\dim(F) < \delta$ is μ-negligible (see for example [9], [15] or [10]). When property (P2) is satisfied, we say that the multifractal analysis of the measure μ is nontrivial. In [17], Taylor asked the following question.

Question ([17]). Does the existence of a nontrivial multifractal spectrum for the measure μ imply that μ is dimension regular with exact dimension ?

As suggested by the anonymous referee, it is easy to construct a measure μ which gives a negative answer to this question. For example, if $\mu_1 \in \mathcal{A}$ with support $[0,1]$ and if μ_2 is the restriction of the Lebesgue measure to $[2,3]$, then, it is easy to see that $\mu = \frac{\mu_1+\mu_2}{2}$ makes the work. In fact, Proposition 2.1, Lemma 2.1 and Corollary 1.1 allow us to construct a lot of non-exact dimensional measures but with non-trivial multifractal spectra.

THEOREM 2.2. *Let μ_1 and μ_2 be two Borel probability measures on $\mathbf{R}^d$ satisfying* (P1) *and* (P2). *Suppose that $B'_{\mu_1}(1) \neq B'_{\mu_2}(1)$ and let $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$, $0 < \alpha < 1$. Then, μ is a non exact dimensional measure with non trivial multifractal spectrum.*

Proof. According to Lemma 2.1, we have $B_\mu(q) = \sup(B_{\mu_1}(q), B_{\mu_2}(q))$ for every $q > 0$. Suppose for example that $B'_{\mu_1}(1) < B'_{\mu_2}(1)$. There exists an interval $J = [a,b]$ with $0 < a < 1 < b$ such that $B_\mu(q) = B_{\mu_1}(q)$ if $a \le q \le 1$ and $B_\mu(q) = B_{\mu_2}(q)$ if $1 \le q \le b$. It follows that the function B_μ is of class C^1 on $(a,1)\cup(1,b)$. Then, using Proposition 2.1 and Corollary 1.1, we conclude that property (P2) is satisfied. More precisely, the multifractal formalism is satisfied in the non-trivial interval $I_1 = (-B'_{\mu_1}(1), -B'_{\mu_1}(a))$ and in the nontrivial interval $I_2 = (-B'_{\mu_2}(b), -B'_{\mu_2}(1))$.

On the other hand, let $\delta_1 = -B'_{\mu_1}(1)$ and $\delta_2 = -B'_{\mu_2}(1)$. The measures μ_1 and μ_2 being in $\mathcal{A}$, we can find Borel sets E_1 and E_2 of full measure μ_1 and μ_2 respectively and such that

$$\lim_{r\to 0} \frac{\log \mu_k(B(x,r))}{\log r} = \delta_k \quad \text{for all } x \in E_k, \quad k = 1,2\ . \tag{4}$$

In particular, $\dim(E_1) = \delta_1$ and $\dim(E_2) = \delta_2 < \delta_1$. We also know that every Borel set E with Hausdorff dimension $\dim(E) < \delta_1$ is μ_1-negligible. In particular $\mu_1(E_2) = 0$ and the measures μ_1 and μ_2 are singular. We can then suppose that E_1 and E_2 are disjoint. It follows that for every Borel set E we have

$$\mu(E\cap E_1) = \alpha\mu_1(E\cap E_1) \quad \text{and} \quad \mu(E\cap E_2) = (1-\alpha)\mu_2(E\cap E_2)\ . \tag{5}$$

Using a theorem related to the differentiation of measures (see for example [4]), we also know that

$$\begin{cases} \lim\limits_{r\to 0} \dfrac{\mu(B(x,r)\cap E_1)}{\mu(B(x,r))} = 1 \text{ for } \mu\text{-a.a. } x \in E_1 \\ \lim\limits_{r\to 0} \dfrac{\mu_1(B(x,r)\cap E_1)}{\mu_1(B(x,r))} = 1 \text{ for } \mu_1\text{-a.a. } x \in E_1 \ . \end{cases}$$

We can then construct a set $F_1 \subset E_1$ such that $\mu(F_1) = \alpha$, $\mu_1(F_1) = 1$ and for every $x \in F_1$,

$$\lim_{r\to 0} \frac{\mu(B(x,r)\cap E_1)}{\mu(B(x,r))} = \lim_{r\to 0} \frac{\mu_1(B(x,r)\cap E_1)}{\mu_1(B(x,r))} = 1 \ . \tag{6}$$

It follows from (4), (5) and (6) that

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \delta_1 \quad \text{for all } x \in F_1 \ .$$

In the same way, we can construct a set F_2 such that

$$\mu(F_2) = 1-\alpha > 0 \quad \text{and} \quad \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \delta_2 \quad \text{for all } x \in F_2 \ ,$$

and the measure μ is not of exact dimension. ▮

2.3. Measures for which the multifractal formalism is nowhere valid

In the following examples, we prove that the multifractal formalism can be nowhere valid even if the function B is regular. In particular, we construct in Theorem 2.4 a measure m on $[0,1)$ for which B is real analytic but the multifractal formalism is nowhere valid.

Theorem 2.3. *Let $p, \tilde{p} \in (0,1/2)$ with $p \neq \tilde{p}$. There exists a probability measure μ on $[0,1)$ such that for every $q \in \mathbf{R}$,*

$$\begin{cases} B(q) &= \sup\left[\log_2(p^q + (1-p)^q), \log_2(\tilde{p}^q + (1-\tilde{p})^q)\right] \\ b(q) &= \inf\left[\log_2(p^q + (1-p)^q), \log_2(\tilde{p}^q + (1-\tilde{p})^q)\right], \end{cases}$$

where $\log_2$ is the logarithm in base 2. We can deduce that for every $q \neq 0,1$, $B'(q)$ exists and

$$\dim(X(-B'(q))) < B^*(-B'(q)) \ .$$

Using the same ideas as in Theorem 2.3 but in a more complicated situation, we obtain the stronger following result.

Theorem 2.4. *There exists a probability measure μ on $[0,1)$ such that B is real analytic, $B'(\mathbf{R})$ is an interval of positive length and*

$$\overline{X}^{\alpha} = \emptyset \quad \textit{for all } \alpha < -B'(1) \quad \textit{and} \quad \underline{X}_{\alpha} = \emptyset \quad \textit{for all } \alpha > -B'(1)\ .$$

In particular, if $\alpha \in (-B'(+\infty), -B'(-\infty))$ and $\alpha \neq -B'(1)$,

$$\mathrm{Dim}(X(\alpha)) = 0 < B^*(\alpha)\ .$$

Remarks. 1. In [14], Olsen previously proposed an example of a self-affine measure μ on $[0,1]^2$ such that B is real analytic and

$$\mathrm{Dim}\,(X(-B'(q))) = 0 < B^*(-B'(q))$$

for all $q \in \mathbf{R} \setminus \{1\}$. Olsen's example uses the geometry of $\mathbf{R}^2$ and the fact that, in contrast with self-similar sets, self-affine sets in $\mathbf{R}^2$ may be very irregular. This idea cannot be adapted to the dimension one. This gives interest to our example.

2. The relation $b(1) = B(1) = 0$ is always true. If $B'(1)$ exists, the measure μ is unidimensional and satisfies

$$\overline{\alpha}_{\mu}(x) = \underline{\alpha}_{\mu}(x) = -B'(1) \quad d\mu(x) - \text{almost surely}\ .$$

We can deduce that

$$\dim(X(-B'(1))) = \mathrm{Dim}(X(-B'(1))) = -B'(1) = B^*(-B'(1))$$

and we can't hope that $\mathrm{Dim}(X(-B'(1))) < B^*(-B'(1))$. For more details, see [15, 10, 12, 11].

3. The measure proposed in Theorem 2.4 is dimension regular with exact dimension $\delta = -B'(1)$ and verifies $b(q) < B(q)$ for all $q \neq 1$. It gives a positive answer to a problem posed by Taylor ([17], page 567).

Proof of Theorem 2.3. Let $(T_k)_{k\geq 1}$ be a sequence of integers such that

$$T_1 = 1, \quad T_k < T_{k+1} \quad \text{and} \quad \lim_{k\to+\infty} \frac{T_{k+1}}{T_k} = +\infty\ .$$

Then, define the family of parameters p_i :

$$p_i = p \quad \text{if} \quad T_{2n-1} \leq i < T_{2n} \quad \text{and} \quad p_i = \tilde{p} \quad \text{if} \quad T_{2n} \leq i < T_{2n+1}\ .$$

Finally, if $\varepsilon_1, \cdots, \varepsilon_n \in \{0,1\}$, denote by $I_{\varepsilon_1\cdots\varepsilon_n}$ the diadic interval of the n^{th} generation

$$I_{\varepsilon_1\cdots\varepsilon_n} = \left[\sum_{i=1}^{n} \frac{\varepsilon_i}{2^i}, \sum_{i=1}^{n} \frac{\varepsilon_i}{2^i} + \frac{1}{2^n}\right)$$

and $\mathcal{F}_n$ the set of diadic intervals of the n^{th} generation included in $[0,1)$. We consider the measure μ such that $\mu([0,1/2)) = p_1$, $\mu([1/2,1)) = 1-p_1$ and for every $n \geq 1$,

$$\begin{cases} \dfrac{\mu(I_{\varepsilon_1\cdots\varepsilon_{n+1}})}{\mu(I_{\varepsilon_1\cdots\varepsilon_n})} = p_{n+1} & \text{if } \varepsilon_n = \varepsilon_{n+1} \\ \dfrac{\mu(I_{\varepsilon_1\cdots\varepsilon_{n+1}})}{\mu(I_{\varepsilon_1\cdots\varepsilon_n})} = 1-p_{n+1} & \text{if } \varepsilon_n \neq \varepsilon_{n+1} \end{cases}$$

There is another way to describe the measure μ. It is the law of the random variable $Z = \sum_{n=1}^{+\infty} 2^{-n} X_n$ where $(X_n)_{n\geq 1}$ is a Markov chain with transition matrices

$$Q_{n+1} = \begin{pmatrix} p_{n+1} & 1-p_{n+1} \\ 1-p_{n+1} & p_{n+1} \end{pmatrix}$$

and with initial law $\mathbf{P}[X_1 = 0] = p_1$ and $\mathbf{P}[X_1 = 1] = 1-p_1$.

Remark. A more classical measure (used in [1]) is the measure $\tilde{\mu}$ (also called the Bernoulli product) which is the law of the random variable $\tilde{Z} = \sum_{n=1}^{+\infty} 2^{-n} \tilde{X}_n$ where $(\tilde{X}_n)_{n\geq 1}$ is an independent sequence of random variables satisfying $\mathbf{P}[\tilde{X}_n = 0] = p_n$ and $\mathbf{P}[\tilde{X}_n = 1] = 1-p_n$.

The measures μ and $\tilde{\mu}$ are very similar. For every n, the families of numbers

$$[\mu(I_{\varepsilon_1\cdots\varepsilon_n})]_{\varepsilon_1,\cdots,\varepsilon_n} \quad \text{and} \quad [\tilde{\mu}(I_{\varepsilon_1\cdots\varepsilon_n})]_{\varepsilon_1,\cdots,\varepsilon_n}$$

are globally the same but the classifications are different. The only interest in choosing the measure μ with regard to the measure $\tilde{\mu}$ is the following lemma.

Lemma 2.2. *The measure μ is a doubling measure : there exists $C > 0$ and $r_0 > 0$ such that for every $x \in \operatorname{supp}\mu$ and for every $r \leq r_0$,*

$$\mu(B(x,2r)) \leq C\mu(B(x,r)) \ .$$

Proof. Let I and J be two adjacent diadic intervals of the n^{th} generation. We can find $k < n$ and $\varepsilon_1, \cdots, \varepsilon_k$ such that

$$\begin{aligned} & I = I_{\varepsilon_1\cdots\varepsilon_k 01\cdots 1} \quad \text{and} \quad J = I_{\varepsilon_1\cdots\varepsilon_k 10\cdots 0} \\ \text{or} & \\ & I = I_{\varepsilon_1\cdots\varepsilon_k 10\cdots 0} \quad \text{and} \quad J = I_{\varepsilon_1\cdots\varepsilon_k 01\cdots 1} \ . \end{aligned}$$

The case $k = n-1$ occurs if I and J have the same father. In that case, the words labelling the intervals I and J are $\varepsilon_1\cdots\varepsilon_{n-1}0$ and $\varepsilon_1\cdots\varepsilon_{n-1}1$.

It is easy to check that

$$\frac{\mu(I)}{\mu(J)} = \frac{p_{k+1}}{1-p_{k+1}} \quad \text{or} \quad \frac{\mu(I)}{\mu(J)} = \frac{1-p_{k+1}}{p_{k+1}}$$

and we can conclude that there exists a universal constant $C > 0$ such that

$$\frac{1}{C} \leq \frac{\mu(I)}{\mu(J)} \leq C \ . \tag{7}$$

Now, let $x \in [0,1]$ and $r > 0$ such that $[x-2r, x+2r] \subset [0,1]$. If n is the unique integer such that $2^{-n} \leq r < 2^{-n+1}$ and if I is the unique diadic interval of the n^{th} generation which contains x, observe that $I \subset [x-r, x+r]$. On the other hand, the interval $[x-2r, x+2r]$ is included in the union of 9 contiguous diadic intervals of the n^{th} generation denoted by $I_{-4}, I_{-3}, I_{-2}, I_{-1}, I, I_1, I_2, I_3, I_4$. Using (7), we can conclude that

$$\begin{aligned} \mu([x-2r, x+2r]) &\leq \mu(I)[1 + 2C + 2C^2 + 2C^3 + 2C^4] \\ &\leq \mu([x-r, x+r])[1 + 2C + 2C^2 + 2C^3 + 2C^4] \ . \end{aligned}$$

Remark. In [19], Tukia already studied this kind of measure in the case where the p_n are independent of n.

Now, we can compute the function B. This is the object of the following lemma.

LEMMA 2.3. *Let*

$$\tau_n(q) = \frac{1}{n \log 2} \log \left(\sum_{I \in \mathcal{F}_n} \mu(I)^q \right) \quad \textit{and} \quad \tau(q) = \limsup_{n \to +\infty} \tau_n(q) \ .$$

We have

$$\tau(q) = B(q) = \Lambda(q) = \sup \left[\log_2(p^q + (1-p)^q), \log_2(\tilde{p}^q + (1-\tilde{p})^q) \right] \ .$$

Proof. It is well known that $B(q) \leq \Lambda(q)$ and $\tau(q) \leq \Lambda(q)$. Moreover, according to Lemma 2.2 and the fact that $\operatorname{supp} \mu = [0,1]$, it is sufficient to consider diadic packings in order to compute the functions B and Λ. The calculation of the function τ is classical. As for Bernoulli products, we observe that

$$\mu(I_{\varepsilon_1 \cdots \varepsilon_{n-1} 0})^q + \mu(I_{\varepsilon_1 \cdots \varepsilon_{n-1} 1})^q = [p_n^q + (1-p_n)^q] \mu(I_{\varepsilon_1 \cdots \varepsilon_{n-1}})^q \ .$$

We can easily deduce that

$$\sum_{I\in\mathcal{F}_n} \mu(I)^q = \prod_{k=1}^{n} [p_k^q + (1-p_k)^q] \ .$$

Finally, if N_n is the number of integers $k \le n$ such that $p_k = p$, we have

$$\tau_n(q) = \frac{N_n}{n}\log_2(p^q + (1-p)^q) + (1-\frac{N_n}{n})\log_2(\tilde{p}^q + (1-\tilde{p})^q) \ .$$

Observing that $\liminf_{n\to\infty} \frac{N_n}{n} = 0$ and $\limsup_{n\to\infty} \frac{N_n}{n} = 1$, we can then conclude that $\tau(q) = \sup[\log_2(p^q + (1-p)^q), \log_2(\tilde{p}^q + (1-\tilde{p})^q]$.

To obtain the equalities $\tau(q) = B(q) = \Lambda(q)$, it is now sufficient to prove that $\tau(q) \ge \Lambda(q)$ and $\tau(q) \le B(q)$. It is a consequence of the following computations :

$$\begin{cases} \overline{\mathcal{P}}_\mu^{q,\tau(q)}(\operatorname{supp}\mu) < +\infty \\ \mathcal{P}_\mu^{q,\tau(q)-\varepsilon}(\operatorname{supp}\mu) > 0 \quad \text{for every } \varepsilon > 0 \ . \end{cases} \tag{8}$$

Remember that it is sufficient to use diadic packings in order to prove that $\overline{\mathcal{P}}_\mu^{q,\tau(q)}(\operatorname{supp}\mu) < +\infty$ and $\mathcal{P}_\mu^{q,\tau(q)-\varepsilon}(\operatorname{supp}\mu) > 0$. Property (8) is then an easy consequence of the following lemma.

LEMMA 2.4. *Denote by $|I|$ the length of the interval I. We can construct a subsequence of integers $(n_k)_{k\ge1}$ and a probability measure ν on $[0,1)$ such that*

$$\nu(I) \ge \mu(I)^q |I|^{\tau(q)} \quad \text{if } I \in \bigcup_{n\in\mathbf{N}^*} \mathcal{F}_n$$

and for every $\varepsilon > 0$,

$$\nu(I) \le \mu(I)^q |I|^{\tau(q)-\varepsilon} \quad \text{if } I \in \mathcal{F}_{n_k} \text{ with } k \text{ sufficiently large .}$$

Proof. We will construct the measure ν as a weak limit of the sequence of measures ν_n defined by

$$d\nu_n(t) = \left[\sum_{I\in\mathcal{F}_n} \mu(I)^q 2^{n-n\tau_n(q)} 1\!\!1_I\right] dt,$$

where dt is the Lebesgue measure. The measure ν_n is a probability measure on $[0,1)$ and assigns the mass $\mu(I)^q |I|^{\tau_n(q)}$ to each interval $I \in \mathcal{F}_n$.

Let $I \in \mathcal{F}_n$ and $p \geq 0$. If $I = I_{\varepsilon_1 \cdots \varepsilon_n}$ and $J = I_{\varepsilon_{n+1} \cdots \varepsilon_{n+p}}$, denote by IJ the interval $I_{\varepsilon_1 \cdots \varepsilon_{n+p}}$. We have

$$\nu_{n+p}(I) = \sum_{J \in \mathcal{F}_p} \nu_{n+p}(IJ) = \sum_{J \in \mathcal{F}_p} \mu(IJ)^q 2^{-(n+p)\tau_{n+p}(q)} .$$

On the other hand,

$$\sum_{J \in \mathcal{F}_p} \mu(IJ)^q = \mu(I)^q \prod_{k=n+1}^{n+p} [p_k^q + (1-p_k)^q] = \mu(I)^q \frac{2^{(n+p)\tau_{n+p}(q)}}{2^{n\tau_n(q)}} .$$

We can conclude that

$$\nu_{n+p}(I) = \mu(I)^q 2^{-n\tau_n(q)} = \mu(I)^q |I|^{\tau_n(q)} . \tag{9}$$

Let $(n_k)_{k \geq 1}$ be a subsequence such that $\tau(q) = \lim_{k \to \infty} \tau_{n_k}(q)$ and choose ν as a weak limit of a subsequence of ν_{n_k}. Observing that $\tau_n(q) \leq \tau(q)$, we deduce from (9) that

$$\forall n \geq 1, \quad \forall I \in \mathcal{F}_n, \quad \nu(I) \geq \mu(I)^q |I|^{\tau(q)} .$$

On the other hand, if $\varepsilon > 0$ and if k is sufficently large, we deduce from (9) that

$$\nu_{n_k+p}(I) \leq \mu(I)^q |I|^{\tau(q)-\varepsilon} \quad \text{if } I \in \mathcal{F}_{n_k}$$

and the second property of the measure ν is then proved.

We can now finish the proof of Theorem 2.3. Similar arguments as previous ones allow us to prove that

$$b(q) = \liminf_{n \to \infty} \tau_n(q) = \inf \left[\log_2(p^q + (1-p)^q), \log_2(\tilde{p}^q + (1-\tilde{p})^q)\right] .$$

Remember that $p \neq \tilde{p}$. We conclude that $b(q) < B(q)$ if $q \neq 0, 1$. Then, Theorem 1.3 ensures that $\dim(X(-B'(q)) < B^*(-B'(q))$. ▮

Sketch of proof of Theorem 2.4. From now on, $\mathcal{F}_n$ denotes the set of the 3-adic intervals of the n^{th} generation. Let $a_1, a_2, a_3 \in (0,1)$ such that $a_1 + a_2 + a_3 = 1$. The integers $(T_k)_{k \geq 1}$ are defined as above.

Finally, if $(I_{\varepsilon_1 \cdots \varepsilon_n})_{\varepsilon_1, \cdots, \varepsilon_n \in \{0,1,2\}}$ are the 3^n intervals of $\mathcal{F}_n$, define the measure μ with the following transitions :

- $\mu(I_0) = a_1$, $\mu(I_1) = a_2$ and $\mu(I_2) = a_3$.

- If $T_{2n-1} \leq k+1 < T_{2n}$, then

$$\begin{aligned} \frac{\mu(I_{\varepsilon_1\cdots\varepsilon_{k+1}})}{\mu(I_{\varepsilon_1\cdots\varepsilon_k})} &= a_1 \quad \text{if} \quad \varepsilon_{k+1} = \varepsilon_k \\ &= a_2 \quad \text{if} \quad \varepsilon_{k+1} = \varepsilon_k + 1 \text{ (mod. 3)} \\ &= a_3 \quad \text{if} \quad \varepsilon_{k+1} = \varepsilon_k + 2 \text{ (mod. 3)}. \end{aligned}$$

- If $T_{2n} \leq k+1 < T_{2n+1}$, then

$$\begin{aligned} \frac{\mu(I_{\varepsilon_1\cdots\varepsilon_{k+1}})}{\mu(I_{\varepsilon_1\cdots\varepsilon_k})} &= 1/2 \quad \text{if} \quad \varepsilon_{k+1} = 0 \text{ or } 2 \\ &= 0 \quad \text{if} \quad \varepsilon_{k+1} = 1. \end{aligned}$$

The measure μ is supported by a Cantor set. More precisely, if

$$\mathcal{T} = \{k \in \mathbf{N}^* \; ; \; T_{2n} \leq k < T_{2n+1} \text{ for some } n > 0\}$$

and if $\mathcal{G}_n \subset \mathcal{F}_n$ is the set of intervals $I_{\varepsilon_1\cdots\varepsilon_n}$ of the n^{th} generation such that $\varepsilon_k \neq 1$ for every $k \in \mathcal{T} \cap \{1, \cdots, n\}$, we have

$$\operatorname{supp} \mu = \bigcap_{n \geq 1} \bigcup_{I \in \mathcal{G}_n} \overline{I} \; .$$

If $I \in \bigcup_n \mathcal{G}_n$ (that is if $I \in \bigcup_n \mathcal{F}_n$ with positive mass) and if J is a contiguous 3-adic interval of the same generation, we observe once again that $\mu(J) \leq C\mu(I)$ for some constant C independent of I and J. The difference with the situation proposed in Theorem 2.3 is that $\mu(J)$ may be equal to 0.

Let $r < 1/9$ and $n \in \mathbf{N}^*$ such that $3^{-n} \leq r < 3^{-n+1}$. If $x \in \operatorname{supp} \mu$, there exists $I \in \mathcal{F}_n$ with positive mass such that $x \in \overline{I}$. The choice of the integer n implies that $I \subset [x-r, x+r]$. Let I_1 be the unique interval of $\mathcal{F}_{n-2}$ such that $I \subset I_1$. Of course $\mu(I_1) > 0$ and there exists a constant $K > 0$ depending only on a_1, a_2 and a_3 such that $\mu(I_1) \leq K\mu(I)$. If J_1 and J_2 are the two intervals of $\mathcal{F}_{n-2}$ which are contiguous with I_1, we can conclude that

$$\begin{aligned} \mu([x-2r, x+2r]) &\leq \mu(J_1) + \mu(I_1) + \mu(J_2) \\ &\leq (2C+1)\mu(I_1) \\ &\leq (2C+1)K\mu(I) \\ &\leq (2C+1)K\mu([x-r, x+r]) \; . \end{aligned}$$

This proves that the measure μ is doubling.

Observe that two contiguous 3-adic intervals I and J with positive mass and of the same generation are such that $\frac{1}{C}\mu(I) \leq \mu(J) \leq C\mu(I)$ for some constant C independent of I and J. It is then once again possible to establish that the functions B and b may be calculated only using 3-adic packings and 3-adic coverings of the Cantor set $\operatorname{supp}\mu$. Using the same method as in Theorem 2.3, we can then compute the functions b and B. We obtain

$$\begin{cases} b(q) &= \inf\left[\log_3\left(a_1^q + a_2^q + a_3^q\right), (1-q)\log_3 2\right] \\ B(q) &= \sup\left[\log_3\left(a_1^q + a_2^q + a_3^q\right), (1-q)\log_3 2\right] \end{cases}.$$

Choose a_1, a_2 and a_3 such that

$$a_1 \log_3(a_1) + a_2 \log_3(a_2) + a_3 \log_3(a_3) = -\log_3 2$$

(such a choice is possible in the set of parameters $a_1, a_2, a_3 > 0$ such that $a_1 + a_2 + a_3 = 1$). In that case, the line $y = (1-q)\log_3 2$ is tangent to the curve $y = \log_3\left(a_1^q + a_2^q + a_3^q\right)$ at point $(1, 0)$. Using the strict convexity of the function $q \mapsto \log_3\left(a_1^q + a_2^q + a_3^q\right)$, we conclude that

$$\begin{cases} b(q) &= (1-q)\log_3 2 \\ B(q) &= \log_3\left(a_1^q + a_2^q + a_3^q\right) \end{cases}.$$

In particular, $b(q) < B(q)$ if $q \neq 1$ and Corollary 1.1 ensures that

$$\dim(X(\alpha)) < B^*(\alpha) \quad \text{for all } \alpha \neq -B'(1).$$

In fact, Olsen's result allows us to obtain a stronger conclusion. According to [13], Lemma 4.4,

$$\overline{X}^{\alpha} = \emptyset \quad \text{if } \alpha < \underline{a} \quad \text{and} \quad \underline{X}_{\alpha} = \emptyset \quad \text{if } \alpha > \overline{a},$$

where $\underline{a} = \sup_{q>0} -\frac{b(q)}{q}$ and $\overline{a} = \inf_{q<0} -\frac{b(q)}{q}$. In the particular case of the measure μ we are working with, $\underline{a} = \overline{a} = \log_3 2 = -B'(1)$ and the conclusion of Theorem 2.4 follows. ▌

Final remark. The measure μ is such that $b(0) = \log_3 2$ and $B(0) = 1$. This means that the Cantor set $\operatorname{supp}\mu$ has Hausdorff dimension $\log_3 2$ but packing dimension 1.

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