

Multifractal Analysis of the Divergence of Fourier Series

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Divergence of Fourier series : historic results

- Du Bois Raymond (1872) $\exists f \in C(\mathbb{T}) ; S_n f(0)$ diverges
- Kolmogorov (1923) $\exists f \in L^1(\mathbb{T}) ; S_n f(x)$ diverges a.s.
- Kolmogorov (1926) $\exists f \in L^1(\mathbb{T}) ; S_n f(x)$ diverges surely
- Kahane Katznelson (1966) If $m(E) = 0$, $\exists f \in C(\mathbb{T})$ s.t. $\forall x \in E$, $S_n f(x)$ diverges
- Carleson (1966) If $f \in L^2(\mathbb{T})$, $S_n f(x)$ converges a.s.
- Carleson Hunt (1967) Always true if $f \in L^p(\mathbb{T})$, $p > 1$

Natural questions

Question

Let x be a divergence point for $S_n f$. How quick can be the divergence of $S_n f(x)$?

Nikolsky Inequality : If $f \in L^p(\mathbb{T})$,

$$\|S_n f\|_\infty \leq C n^{1/p} \|S_n f\|_p \leq C n^{1/p} \|f\|_p .$$

Question

Let $\beta \in [0, 1/p]$. What is the size of the set of points x such that $|S_n f(x)| \approx n^\beta$ when $n \rightarrow +\infty$?

What about $f \in C(\mathbb{T})$?

If $f \in C(\mathbb{T})$,

$$|S_n f(x)| \leq \|D_n\|_1 \|f\|_\infty \leq C \log n .$$

Question

Let $\beta \in [0, 1]$. What is the size of the set of points x such that $|S_n f(x)| \approx (\log n)^\beta$ when $n \rightarrow +\infty$?

Aubry's result

Define

$$\mathcal{E}(\beta, f) = \left\{ x \in \mathbb{T}; \limsup_{n \rightarrow +\infty} n^{-\beta} |S_n f(x)| > 0 \right\}.$$

Theorem (J.M. Aubry, 2006)

Suppose $p > 1$ and $f \in L^p(\mathbb{T})$. Then

$$\dim_{\mathcal{H}}(\mathcal{E}(\beta, f)) \leq 1 - \beta p.$$

Carleson Hunt maximal inequality :

$$\|S^* f\|_p \leq C \|f\|_p$$

where $S^* f(x) = \sup_n |S_n f(x)|$.

Aubry's result

For a fixed $\beta \in [0, 1/p]$, Aubry's result is optimal :

Theorem (J.M. Aubry, 2006)

Let E such that $\dim_{\mathcal{H}}(E) < 1 - \beta p$. There exists $f \in L^p(\mathbb{T})$ such that,

$$\forall x \in E, \quad \limsup_{n \rightarrow +\infty} n^{-\beta} |S_n f(x)| = +\infty .$$

Question

Can we construct a function $f \in L^p(\mathbb{T})$ for which lower bounds of $\dim_{\mathcal{H}}(\mathcal{E}(\beta, f))$ are valid **for all $\beta \in [0, 1/p]$** ?

The divergence index

Let $f \in L^p(\mathbb{T})$ and $x_0 \in \mathbb{T}$.

$$\begin{aligned}\beta(x_0) &= \inf \left(\beta ; |S_n f(x_0)| = O(n^\beta) \right) \\ &= \limsup_{n \rightarrow +\infty} \frac{\log |S_n f(x_0)|}{\log n} .\end{aligned}$$

Level sets :

$$E(\beta, f) = \{x \in \mathbb{T}; \beta(x) = \beta\} .$$

Multifractal analysis :

$$\beta \mapsto \dim_{\mathcal{H}} (E(\beta, f)) .$$

Multifractal behavior of $S_n f$

Of course, $E(\beta, f) \subset \bigcap_{\gamma < \beta} \mathcal{E}(\gamma, f)$, so that

$$\dim_{\mathcal{H}}(E(\beta, f)) \leq 1 - \beta p .$$

Theorem (Bayart, H., 2010)

For quasi-all functions $f \in L^p(\mathbb{T})$,

$$\forall \beta \in [0, 1/p], \quad \dim_{\mathcal{H}}(E(\beta, f)) = 1 - \beta p .$$

- Roughly speaking, $|S_n f(x)| \approx n^\beta$ in a set with dimension $1 - \beta p$.
- “quasi-all” is related to the Baire category theorem.
- For such f we also have $\dim_{\mathcal{H}}(\mathcal{E}(\beta, f)) = 1 - \beta p$.

Dyadic approximation

The real number x is said to be α -approximable by dyadics if $\left| x - \frac{k}{2^j} \right| \leq \frac{1}{2^{\alpha j}}$ for infinitely many j .

$$\begin{aligned} D_\alpha &= \{x \in [0, 1] ; x \text{ is } \alpha\text{-approximable}\} \\ &= \limsup_{j \rightarrow +\infty} \bigcup_{k=0}^{2^j-1} \left[\frac{k}{2^j} - \frac{1}{2^{\alpha j}}, \frac{k}{2^j} + \frac{1}{2^{\alpha j}} \right]. \end{aligned}$$

Well known : $\dim_{\mathcal{H}}(D_\alpha) = 1/\alpha$ (in fact $\mathcal{H}^{1/\alpha}(D_\alpha) > 0$).

The saturating function

- $I_{k,j} = \left[\frac{k}{2^j} - \frac{1}{2^j}, \frac{k}{2^j} + \frac{1}{2^j} \right]$
- $\frac{k}{2^j} = \frac{K}{2^J}$ with $K \notin 2\mathbb{Z}$
- $\mathbf{I}_{J,j} = \bigcup_{\frac{k}{2^j} = \frac{K}{2^J}} I_{k,j}$
- $\chi_{J,j} = 1$ on $\mathbf{I}_{J,j}$, $\chi_{J,j} = 0$ outside “ $2\mathbf{I}_{J,j}$ ” and is smooth
- $e_n(x) = e^{inx}$

$$f(x) = \sum_{j \geq 1} \frac{1}{j^3} \sum_{J=1}^j e_{(j+J)2^{j+1}} 2^{(j-J)/p} \sigma_{2^j} \chi_{J,j} .$$

$$x \in D_\alpha \Rightarrow \limsup \frac{\log S_n f(x)}{\log n} \geq \frac{1}{p} \left(1 - \frac{1}{\alpha} \right)$$

$$\beta = \frac{1}{p} \left(1 - \frac{1}{\alpha} \right) \Rightarrow \frac{1}{\alpha} = 1 - \beta p$$

What about $p = 1$?

Theorem (Bayart, H., 2011)

For quasi-all functions $f \in L^1(\mathbb{T})$,

$$\forall \beta \in [0, 1], \quad \dim_{\mathcal{H}}(E(\beta, f)) = 1 - \beta .$$

Lower bound : Same proof as before

Upper bound :

- Carleson Hunt maximal inequality is false in this context
- We can prove the following sufficient estimation :

$$\int_{\mathbb{T}} \sup_n \left| \frac{S_n f(x)}{(\log n)^{1+\varepsilon}} \right| dx \leq C \|f\|_1 .$$

The continuous case : divergence index

Remember that $|S_n f(x)| \leq C \log n$.

Aubry's type sets :

$$\mathcal{F}(\beta, f) = \left\{ x \in \mathbb{T}; \limsup_{n \rightarrow +\infty} (\log n)^{-\beta} |S_n f(x)| > 0 \right\} .$$

Adapted divergence index :

$$\begin{aligned} \beta(x_0) &= \inf \left(\beta ; |S_n f(x_0)| = O((\log n)^\beta) \right) \\ &= \limsup_{n \rightarrow +\infty} \frac{\log |S_n f(x_0)|}{\log \log n} . \end{aligned}$$

Level sets :

$$F(\beta, f) = \{x \in \mathbb{T}; \beta(x) = \beta\} .$$

Surprising results

Theorem (Bayart, H., 2010)

For quasi-all functions $f \in \mathcal{C}(\mathbb{T})$,

$$\dim_{\mathcal{H}}(F(1, f)) = 1 .$$

Theorem (Bayart, H., 2010)

For quasi-all functions $f \in \mathcal{C}(\mathbb{T})$,

$$\forall \beta \in [0, 1] \quad \dim_{\mathcal{H}}(F(\beta, f)) = 1 .$$

Roughly speaking, for all β , $S_n f(x) \approx (\log n)^\beta$ in a set with dimension 1.

The precised Hausdorff dimension

Precised gauge functions :

$$\phi_{s,t}(x) = x^s \exp [(\log 1/x)^{1-t}].$$

Definition

We say that E has **precised Hausdorff dimension** (α, β) if α is the Hausdorff dimension of E and

- $\beta = 0$ if $\mathcal{H}^{\phi_{\alpha,t}}(E) = 0$ for every $t \in (0, 1)$;
- $\beta = \sup \{t \in (0, 1); \mathcal{H}^{\phi_{\alpha,t}}(E) > 0\}$ otherwise.

Natural order : lexicographical order. The bigger is the set, the bigger is the precised Hausdorff dimension.

Precised Theorem

Theorem (Bayart, H., 2010)

For quasi-all functions $f \in C(\mathbb{T})$, $\forall \beta \in [0, 1]$, the precised Hausdorff dimension of $F(\beta, f)$ is $(1, 1 - \beta)$.

Lemma

Let $\beta \in (0, 1)$ and $f \in C(\mathbb{T})$.

$$\forall \gamma > 1 - \beta, \quad \mathcal{H}^{\phi_{1,\gamma}}(\mathcal{F}(\beta, f)) = 0.$$

Remarks

- Aubry's type result
- The lemma gives the upper bound for the precised dimension
- Tool : $m(\{x \in \mathbb{T} ; S^*f(x) > y\}) \leq Ae^{-By/\|f\|_\infty}$

Basic idea to construct divergence points

$$P_N(x) = e_N(x) \sum_{j=1}^N \frac{\sin(2\pi jx)}{j},$$

$$\|P_N\|_\infty \leq C \quad \text{but} \quad |S_N(P)(0)| \sim \frac{1}{2} \log N.$$

Similar results are needed in a **big set** of points x .

Construction of polynomials P

Fact (Kahane Katznelson) : Let $Q(z) = \sum_{k=1}^{n-1} a_k z^k$. Set $P(\theta) = e^{in\theta} \times \Im Q(e^{i\theta})$.

$$|S_n P(\theta)| = \frac{1}{2} \left| \sum_{k=1}^{n-1} \bar{a}_k e^{i(n-k)\theta} \right| = \frac{1}{2} |Q(e^{i\theta})| .$$

If Q is such that $|\Im Q|$ is small and $|Q|$ is large on a set $E \subset \partial\mathbb{D}$, then $|P|$ is small and $|S_n P|$ is large on \tilde{E} .

The polynomials Q are constructed by taking the logarithm of the holomorphic function

$$f(z) = \frac{1}{k} \sum_{j=0}^{k-1} \frac{1 + \varepsilon}{1 + \varepsilon - \bar{z}_j z}$$

$$z_j = e^{(2i\pi j)/k} .$$

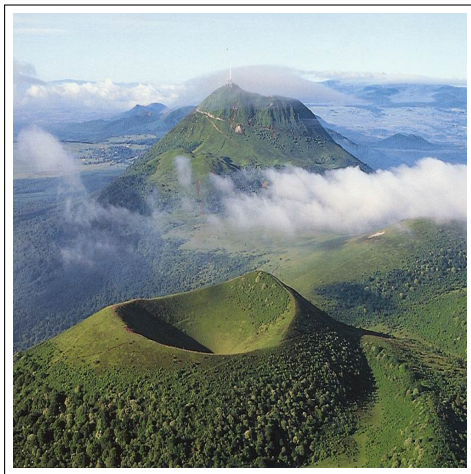
The main lemma

Lemma

Let $\beta \in (0, 1)$. Then there exist k and n as large as we want and a trigonometric polynomial P with spectrum contained in $[0, 2n - 1]$ such that

- $\forall x \in \mathbb{T}, \quad |P(x)| \leq 1$
- $\forall x \in I_k^\beta, \quad \log |S_n P(x)| \geq \beta \log \log n$ for any $x \in I_k^\beta$,

where $I_k^\beta = \bigcup_{j=0}^{k-1} \left[\frac{j}{k} - \frac{1}{2k \exp((\log k)^\beta)}; \frac{j}{k} + \frac{1}{2k \exp((\log k)^\beta)} \right]$.



Merci !