Generic boundary behaviour for harmonic functions in the ball

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Introduction

Sets of divergence

Generic behavior

Dimension of measures

Selfsimilar measures

Harmonic measure

The structure function

Quasi Bernoulli measures

Merci !
The beginning of the story

- If \( f \in L^p(\mathbb{T}) \), \( p > 1 \), \( S_n f(x) = \sum_{k=-n}^{n} \hat{f}(k)e^{inx} \) is almost surely convergent (Carleson Theorem) but there are possible divergence points.
- For a given \( \beta \), what is the size of the set of points \( x \) for which \( |S_n f(x)| \gg n^\beta \) i.o. ? (Aubry 2006)
- What is the behaviour of \( S_n f \) for a generic function \( f \in L^p \)?
- Let \( \beta(x) \) be the supremum of the beta such that \( |S_n f(x)| \gg n^\beta \) i.o. and \( E(\beta, f) = \{ x \in \mathbb{T} ; \beta(x) = \beta \} \).
  If \( f \) is a generic function in \( L^p(\mathbb{T}) \),
  \[
  \text{for any } \beta \in [0, 1/p], \quad \dim_H(E(\beta, f)) = 1 - \beta p. \]
  (Bayart, H., 2011)
- Always true when \( p = 1 \) (Bayart, H., 2012)
- What about \( P_r \ast f(x) = \sum_{k=-\infty}^{+\infty} r|k|\hat{f}(k)e^{ikx} \) when \( r \to 1 ? \)
  \( h(re^{ix}) = P_r \ast f(x) \) is harmonic in the unit disk.
  \( r \to 1 \) corresponds to the radial convergence in the disk.
Harmonic functions in the ball $B_{d+1}$

The Poisson kernel:

$$P(x, \xi) = \frac{1 - \|x\|^2}{\|x - \xi\|^{d+1}}.$$ 

- Bounded harmonic functions

$$h(x) = P[f](x) = \int_{S_d} P(x, \xi) f(\xi) d\sigma(\xi) \quad \text{with} \quad f \in L^\infty(S_d)$$

- Nonnegative harmonic functions

$$h(x) = P[\mu](x) = \int_{S_d} P(x, \xi) d\mu(\xi) \quad \text{with} \quad \mu \in \mathcal{M}^+(S_d)$$

- Harmonic functions with $L^1$ data

$$h(x) = P[f](x) = \int_{S_d} P(x, \xi) f(\xi) d\sigma(\xi) \quad \text{with} \quad f \in L^1(S_d)$$
Fatou’s Theorem

- Fatou (1906) : if $f \in L^\infty(\mathbb{T})$, then
  \[ P_r \ast f(x) \to f(x) \text{ almost surely.} \]

- Generalizations (Hardy-Littlewood, Wiener, Bochner . . .)
  \[ P[\mu](ry) \to \frac{d\mu}{d\sigma}(y) \text{ } d\sigma\text{-almost surely when } r \to 1. \]

- Hunt and Wheeden (1970) : If $h$ is a nonnegative harmonic function in a Lipschitz domain $U \subset \mathbb{R}^n$, then $h$ has a non tangential limit at almost every point of the boundary $\partial U$. 
An elementary upper bound for $|P[f](ry)|$ when $r \to 1$:

$$P(x, \xi) = \frac{1 - \|x\|^2}{\|x - \xi\|^{d+1}} \leq \frac{1 - \|x\|^2}{(1 - \|x\|)^{d+1}} \leq \frac{2}{(1 - \|x\|)^d}$$

$$|P[f](ry)| = \left| \int_{S_d} \frac{1 - \|ry\|^2}{\|ry - \xi\|^{d+1}} f(\xi) d\sigma(\xi) \right| \leq \frac{2\|f\|_1}{(1 - r)^d}$$

**Question**

Let $0 < \beta \leq d$. What can we say about the size of the set of points $y$ such that $|P[f](ry)| \approx (1 - r)^{-\beta}$ when $r \to 1$?
Hausdorff dimension of exceptional sets

\[ 0 < \beta < d \]

\[ \mathcal{E}(\beta, f) = \left\{ y \in S_d; \limsup_{r \to 1} \frac{|P[f](ry)|}{(1 - r)^{-\beta}} = +\infty \right\} \]

Theorem (Bayart, H.)

- For any \( f \in L^1(S_d) \), \( \dim_\mathcal{H}(\mathcal{E}(\beta, f)) \leq d - \beta \).
- If \( E \subset S_d \) is such that \( \dim_\mathcal{H}(E) < d - \beta \), there exists \( f \in L^1(S_d) \) such that \( E \subset \mathcal{E}(\beta, f) \).

The first part was already obtained by Armitage (1981) in the context of the half upper space.
A more precise result

Let $\tau$ be a nonnegative nonincreasing function such that

$$\lim_{s \to 0^+} \tau(s) = +\infty, \quad \tau(s) \ll s^{-d} \quad \text{and} \quad \tau(s) \approx \tau(2s).$$

Define

$$\mathcal{E}(\tau, f) = \left\{ y \in S_d; \limsup_{r \to 1} \frac{|P[f](ry)|}{\tau(1 - r)} = +\infty \right\}. $$

Let $\phi$ be the gauge function defined by $\phi(s) = \tau(s)s^d$.

Theorem (Bayart, H.)

- For any $f \in L^1(S_d)$, $\mathcal{H}^\phi(\mathcal{E}(\tau, f)) = 0$.
- If $E \subset S_d$ is such that $\mathcal{H}^\phi(E) = 0$, there exists $f \in L^1(S_d)$ such that $E \subset \mathcal{E}(\tau, f)$. 
The Hardy-Littlewood maximal inequality

\[ P[\mu](x) = \int_{S_d} P(x, \xi) \, d\mu(\xi) \]

\[ \sup_{r \in (0,1)} |P[\mu](ry)| \leq \sup_{\delta > 0} \frac{|\mu|(\kappa(y, \delta))}{\sigma(\kappa(y, \delta))} \]

where \( \kappa(y, \delta) = \{ \xi \in S_d; \|\xi - y\| < \delta \} \).
\( \kappa(y, \delta) \) is called a cap.

Lemma (a quantitative improvement - Bayart, H.)

Let \( 0 < r < 1 \). There exists \( \delta \geq 1 - r \) such that

\[ |P[\mu](ry)| \leq C \frac{|\mu|(\kappa(y, \delta))}{\sigma(\kappa(y, \delta))}, \]

where \( C \) is a constant independent of \( \mu, r \) and \( y \).
Dimension of $\mathcal{E}(\beta, \mu) :$ the upper bound

$$\tau(s) = s^{-\beta}.$$ 

$$\mathcal{E}(\beta, \mu) = \left\{ y \in S_d; \limsup_{r \to 1} \frac{|P[\mu](ry)|}{(1 - r)^{-\beta}} = +\infty \right\}$$

$$\mathcal{E}_M = \left\{ y \in S_d; \limsup_{r \to 1} \frac{|P[\mu](ry)|}{(1 - r)^{-\beta}} > M \right\}.$$ 

Let $y \in \mathcal{E}_M.$

Using the previous lemma, we can find $r_y$ as close to 1 as we want and a cap $\kappa_y = \kappa(y, \delta_y)$ with $\delta_y \geq 1 - r_y$

$$M(1 - r_y)^{-\beta} < |P[\mu](r_yy)| \leq C \frac{|\mu|(\kappa_y)}{\sigma(\kappa_y)}.$$ 

$\delta_y$ goes to 0 when $r_y$ goes to 1.
Dimension of $\mathcal{E}(\beta, \mu)$: the upper bound

$$(1 - r_y)^{-\beta} \sigma(\kappa_y) < \frac{C}{M} |\mu|_y(\kappa_y).$$

By the Vitali covering lemma, we can find a family of disjoint caps $(\kappa_{y_i})_{j \in \mathbb{N}}$ such that $\mathcal{E}_M \subset \bigcup_i 5\kappa_{y_i}$.

$$\sum_i (1 - r_{y_i})^{-\beta} \sigma(\kappa_{y_i}) \leq \frac{C}{M} \|\mu\|$$

$$\sum_i \delta_{y_i}^{d-\beta} \leq \frac{C}{M} \|\mu\|$$

$$\mathcal{H}^{d-\beta}(\mathcal{E}_M) \leq \frac{C}{M} \|\mu\|$$

$$\mathcal{H}^{d-\beta}(\mathcal{E}(\beta, \mu)) = 0$$
Lower bound for the dimension: an elementary lemma

If \( r > 1/2 \),

\[
\int_{\kappa(N,1-r)} P(rN,\xi) \, d\sigma(\xi) \geq C
\]
Lower bound for the dimension : an elementary lemma

If $r > 1/2$, \[ \int_{\kappa(N,1-r)} P(rN,\xi) d\sigma(\xi) \geq C \]
Lower bound for the dimension: the construction

Let $E$ be such that $\mathcal{H}^{d-\beta}(E) = 0$. Let $\mathcal{R}_j$ be a $2^{-j}$-covering of $E$ by caps such that

$$\sum_{\kappa \in \mathcal{R}_j} |\kappa|^{d-\beta} \leq 2^{-j}.$$

Define

$$C_n = \left\{ \kappa \in \bigcup_j \mathcal{R}_j; \ 2^{-(n+1)} < |\kappa| \leq 2^{-n} \right\}.$$

$E \subset \limsup_n E_n$ where $E_n = \bigcup_{\kappa \in C_n} \kappa$.

$$\sum_{n \geq 1} \sum_{\kappa \in C_n} |\kappa|^{d-\beta} \leq \sum_{j \geq 1} \sum_{\kappa \in \mathcal{R}_j} |\kappa|^{d-\beta} \leq 1.$$

Choose $(\omega_n)_{n \geq 1}$ tending to infinity such that

$$\sum_{n \geq 1} \omega_n \sum_{\kappa \in C_n} |\kappa|^{d-\beta} < +\infty.$$
Lower bound for the dimension: the function $f$

$$f = \sum_{n \geq 1} \omega_n 2^{-n\beta} \sum_{\kappa \in C_n} \mathbb{1}_{4\kappa}$$

Let $y \in E_n = \bigcup_{\kappa \in C_n} \kappa$. Let $\kappa_0 \in C_n$ such that $y \in \kappa_0$ and $r = 1 - 2^{-n}$.

$$P[f](ry) \geq \omega_n 2^{-n\beta} \int_{4\kappa_0} P(ry, \xi) \, d\sigma(\xi)$$

$$\geq \omega_n 2^{-n\beta} \int_{\kappa(y, 2^{-n})} P(ry, \xi) \, d\sigma(\xi)$$

$$\geq C\omega_n (1 - r)^{-\beta}.$$
The divergence index

Let $f \in L^1(S_d)$ and $y_0 \in S_d$.

$$\beta(y_0) = \sup (\beta \ ; \ y_0 \in E(\beta, f))$$

$$= \inf \left( \beta \ ; \ |P[f](ry_0)| = O((1 - r)^{-\beta}) \right)$$

$$= \limsup_{r \to 1} \frac{\log |P[f](ry_0)|}{-\log(1 - r)} .$$

Level sets :

$$E(\beta, f) = \{ y \in S_d ; \ \beta(y) = \beta \} .$$

The family $(E(\beta, f))_\beta$ is a nonincreasing family of sets and the sets $(E(\beta, f))_\beta$ are disjoints.

Spectrum of singularities :

$$\beta \mapsto \dim_H (E(\beta, f)) .$$
Multifractal behavior of $P[f]$

Of course,

$$\dim_{\mathcal{H}}(E(\beta, f)) \leq d - \beta .$$

Theorem (Bayart, H.)

For quasi-all functions $f \in L^1(S_d)$,

$$\forall \beta \in [0, d], \quad \dim_{\mathcal{H}}(E(\beta, f)) = d - \beta .$$

- Roughly speaking, for any $\beta$, $|P[f](ry)| \approx (1 - r)^{-\beta}$ in a set with dimension $d - \beta$.
- “quasi-all” is related to the Baire category theorem.
- For such $f$ we also have $\dim_{\mathcal{H}}(E(\beta, f)) = d - \beta$. 
The analogue of dyadic numbers in the sphere $S_d$

There exists a sequence $(\mathcal{R}_n)_{n \geq 1}$ of finite subsets of $S_d$ satisfying

- $\mathcal{R}_n \subset \mathcal{R}_{n+1}$;
- $\bigcup_{x \in \mathcal{R}_n} \kappa(x, 2^{-n}) = S_d$;
- $\text{card} (\mathcal{R}_n) \leq C2^{nd}$;
- For any $x, y$ in $\mathcal{R}_n$, $x \neq y$, then $|x - y| \geq 2^{-n}$.

If $\alpha > 1$, let $N_{n,\alpha} = \lfloor n/\alpha \rfloor + 1$ and

$$D_{n,\alpha} = \bigcup_{x \in \mathcal{R}_{N_{n,\alpha}}} \kappa(x, 2^{-n}).$$

**Proposition**

$$\mathcal{H}^{d/\alpha} \left( \limsup_{n \to +\infty} D_{n,\alpha} \right) = +\infty.$$

**Proof**: mass transference principle.

**Remark**: we can replace $n$ by a subsequence $n_k$. 
In the way of saturating functions

\[ f_n := \frac{1}{n+1} \sum_{N=1}^{n+1} \sum_{x \in \mathcal{R}_N} 2^{(n-N)d} 1_{\mathcal{K}(x, 2^{2-n})}. \]

Proposition

\[ f_n \in L^1(S_d) \text{ and } \|f_n\|_1 \leq C. \]

Moreover, for any \( \alpha > 1 \), for any \( y \in D_{n, \alpha} \),

\[ P[f_n](r_n y) \geq \frac{C}{n} 2^{(n-N_{n, \alpha})d}, \]

where \( 1 - r_n = 2^{-n} \), \( N_{n, \alpha} = \lceil n/\alpha \rceil + 1 \) and \( C \) is independent of \( n \) and \( \alpha \).

Remark: \( 2^{(n-N_{n, \alpha})d} \approx (1 - r_n)^{-\beta} \) if \( \frac{d}{\alpha} = d - \beta \).
Construction of a dense sequence

Proposition

There exists a dense sequence \((h_n)_{n \geq 1}\) in \(L^1(S_d)\) such that for any \(n \geq 1\), for any \(\alpha > 1\) and any \(y \in D_{n,\alpha}\),

\[
P[h_n](r_n y) \geq \frac{C}{n^2} 2^{(n-N_{n,\alpha})d},
\]

where \(r_n = 1 - 2^{-n}\).

Let \((g_n)_{n \geq 1}\) be a sequence of continuous functions which is dense in \(L^1(S_d)\) ans such that \(\|g_n\|_\infty \leq n\).

\[
h_n = \frac{1}{n} f_n + g_n
\]
The dense $G_δ$ set

The residual set we will consider is the dense $G_δ$-set

$$A = \bigcap_{k \geq 1} \bigcup_{n \geq k} B_{L^1}(h_n, \delta_n).$$

where $\delta_n$ is such that

$$\|f\|_1 \leq \delta_n \Rightarrow \|P[f](r_n \cdot)\|_\infty \leq 1.$$

If $\|f - h_n\|_1 < \delta_n$ and $y \in D_{n,\alpha}$,

$$\frac{\log |P[f](r_n y)|}{-\log(1 - r_n)} \geq \left( d - \frac{N_{n,\alpha} d}{n} \right) + o(1).$$

$$d - \frac{N_{n,\alpha} d}{n} \approx d - \frac{d}{\alpha} := \beta \quad \text{if} \quad \frac{d}{\alpha} = d - \beta.$$
The case of nonnegative harmonic functions

The set $\mathcal{H}^+(B_{d+1})$ of nonnegative harmonic functions in the ball $B_{d+1}$ endowed with the topology of the locally uniform convergence is a closed cone in the space of all continuous functions in the ball: it satisfies Baire’s property.

**Theorem**

For quasi-all nonnegative harmonic functions $h$ in the unit ball $B_{d+1}$, for any $\beta \in [0, d]$, 

$$\dim_{\mathcal{H}} \left( E(\beta, h) \right) = d - \beta$$

where

$$E(\beta, h) = \left\{ y \in S_d ; \limsup_{r \to 1} \frac{\log h(ry)}{-\log(1-r)} = \beta \right\}.$$
Thank you for your attention