

# ON MEASURES DRIVEN BY MARKOV CHAINS

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ABSTRACT. We study measures on  $[0, 1]$  which are driven by a finite Markov chain and which generalize the famous Bernoulli products. We propose a hands-on approach to determine the structure function  $\tau$  and to prove that the multifractal formalism is satisfied. Formulas for the dimension of the measures and for the Hausdorff dimension of their supports are also provided. Finally, we identify the measures with maximal dimension.

## 1. INTRODUCTION

Multifractal measures on  $\mathbb{R}^d$  are measures  $m$  for which the level sets

$$E_\alpha = \left\{ x \in \mathbb{R}^d ; \lim_{r \rightarrow 0} \frac{\log m(B(x, r))}{\log r} = \alpha \right\}$$

are non empty for at least two values of the real  $\alpha$ . On the other hand, if there exists a unique value of  $\alpha$  for which the the level set  $E_\alpha$  is non empty, we say that the measure  $m$  is monofractal. In practice, it is impossible to completely describe the sets  $E_\alpha$ , but we can try to calculate their Hausdorff dimensions. To this end, Frisch and Parisi ([8]) were the first to use the Legendre transform of a structure function  $\tau$ . A mathematically rigorous approach was given by Brown, Michon and Peyrière in [2] and by Olsen in [10]. There are many situations in which the Legendre transform formula, now called multifractal formalism, is satisfied. For a comprehensive account see e.g. [3], [4], [6], [7], [15] or [9].

A fundamental model for multifractal measures is given by so called Bernoulli products, see for example Chapter 10 of [5]. Roughly speaking, if  $I_{\varepsilon_1 \dots \varepsilon_n}$  are the  $n^{\text{th}}$   $\ell$ -adic intervals of the  $n^{\text{th}}$  generation and  $(X_n)$  is an i.i.d. sequence of random variables, then Bernoulli product  $m$  can be defined by

$$m(I_{\varepsilon_1 \dots \varepsilon_n}) = \mathbb{P}[X_1 = \varepsilon_1, \dots, X_n = \varepsilon_n].$$

The purpose of this paper is to provide an explicit analysis of a natural generalization of this model. Instead of an i.i.d. sequence, we consider an irreducible homogeneous Markov chain. Consequently, the measure  $m$  satisfies the recurrence relation

$$m(I_{\varepsilon_1 \dots \varepsilon_{n+1}}) = p_{\varepsilon_n \varepsilon_{n+1}} m(I_{\varepsilon_1 \dots \varepsilon_n}),$$

where  $P = (p_{ij})$  is the transition matrix of  $X_n$  (see the next section for full details). Such a model has been already studied by Edgar and Mauldin ([4]). In the case where invariance properties are added, it also falls within the scope of thermodynamic formalism. Nevertheless, we want to study this model in the light of modern multifractal approach, without using invariance properties. This approach

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allows to easily compute the dimension of the measures and their supports. Also, the interrelations between the invariant measures and those which are not can be described and the problem of measures with maximal dimension can be discussed. Moreover this formulation remains open to further extensions.

The article is organized as follows. In Section 3, we identify a formula for the structure function  $\tau$  of such a measure and we compute its dimension. Section 4 contains a construction of auxiliary (Gibbs) measures. While it involves a nontrivial rescaling, we insist on the fact that our results only require some fundamental results of multifractal analysis and the use of Perron-Frobenius theorem. Our approach is to apply the methods developed in [2] and to avoid the classical method of the variational principle for pressure and the theory of equilibrium states. We are then able to prove that the multifractal formalism is satisfied and we give a formula for the Hausdorff dimension of the (closed) support of the measures. Finally, in Section 5 we discuss ergodic properties and prove that for a given support  $K$ , the measure  $m$  with maximal dimension is essentially unique. now again, we do not need any invariance property for the measure  $m$ .

## 2. PRELIMINARIES

Set  $\mathcal{S} = \{0, 1, \dots, \ell - 1\}$ . Let  $\mathcal{W}_n$  be set of words of length  $n$  over the alphabet  $\mathcal{S}$ . A concatenation of two words  $\varepsilon = \varepsilon_1 \cdots \varepsilon_n \in \mathcal{W}_n$  and  $\delta = \delta_1 \cdots \delta_k \in \mathcal{W}_k$  will be denoted by  $\varepsilon\delta = \varepsilon_1 \cdots \varepsilon_n \delta_1 \cdots \delta_k$ . Let  $\mathcal{F}_0 = \{[0, 1)\}$  and for  $n \geq 1$ ,  $\mathcal{F}_n$  be the set of  $\ell$ -adic intervals of order  $n$ , that is the family of intervals of the form

$$I_\varepsilon = I_{\varepsilon_1 \cdots \varepsilon_n} = \left[ \sum_{i=1}^n \frac{\varepsilon_i}{\ell^n}, \sum_{i=1}^n \frac{\varepsilon_i}{\ell^n} + \frac{1}{\ell^n} \right)$$

where  $\varepsilon = \varepsilon_1 \cdots \varepsilon_n \in \mathcal{W}_n$ . If  $I = I_\varepsilon \in \mathcal{F}_n$  and  $J = I_\delta \in \mathcal{F}_k$ , we will write  $IJ = I_{\varepsilon\delta}$ . Note that  $IJ \subset I$ . Finally, we will denote by  $I_n(x)$  the unique interval  $I \in \mathcal{F}_n$  such that  $x \in I$  and  $|I|$  the length of the interval  $I$ .

Consider a discrete Markov chain  $X = (X_k)_{k \geq 1}$  on  $\mathcal{S}$  with an initial distribution  $p_i = \mathbb{P}(X_1 = i)$ ,  $i \in \mathcal{S}$ , and the transition probabilities  $P = (p_{ij})_{i,j=0}^{\ell-1}$  where  $p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i)$ . In order to exclude degenerated cases, we will suppose that  $p_{ij} \neq 1$  for any  $(i, j) \in \mathcal{S} \times \mathcal{S}$ . Note that for the sake of coherence with the definition of  $\mathcal{F}_n$ , the entries of matrices and vectors will be indexed by numbers starting from 0.

If necessary, we will assume that  $X$  is irreducible, that is for all  $i, j \in \mathcal{S}$  there exists  $k \geq 2$  such that  $\mathbb{P}(X_k = j | X_1 = i) > 0$ .

For a matrix  $P = (p_{ij})_{i,j \in \mathcal{S}}$  we denote by  $P^n$  the usual matrix power. This should not be confused with  $P_q$ ,  $q \in \mathbb{R}$ , which stands for the matrix whose entries are given by  $(p_{ij}^q)_{i,j \in \mathcal{S}}$ . In this context, by convention we will set  $0^q = 0$  for any  $q \in \mathbb{R}$ .

The Hausdorff dimension (box dimension, respectively) of a set  $E$  will be denoted by  $\dim_H E$  ( $\dim_B E$ ). Recall the definitions of the lower and upper dimension of a probability measure  $\mu$  on  $\mathbb{R}^d$ :

$$\begin{aligned} \dim_* \mu &= \inf\{\dim_H(E) : \mu(E) > 0\} = \sup\{s > 0 : \mu \ll \mathcal{H}^s\} \\ \dim^* \mu &= \inf\{\dim_H(E) : \mu(E) = 1\} = \inf\{s > 0 : \mu \perp \mathcal{H}^s\} \end{aligned}$$

where  $\mathcal{H}^s$  denotes the Hausdorff measure in dimension  $s$ . If these two quantities agree, the common value is called the dimension of  $\mu$  and denoted by  $\dim \mu$ . In

this case the measure is said to be *unidimensional*. For more details, we refer the reader to e.g. [9].

By  $c$  or  $C$  we will denote a generic positive constant whose exact value is not important and may change from line to line. For functions or expressions  $f$  and  $g$ , depending on a variable  $x$ , say, we will write  $f \asymp g$  if there exists a constant  $c$  which does not depend on  $x$  and such that  $c^{-1}g(x) \leq f(x) \leq cg(x)$  for any admissible  $x$ .

### 3. THE MEASURES AND THE STRUCTURE FUNCTION $\tau$

Let  $(X_n)_{n \geq 1}$  be a Markov chain on  $\mathcal{S}$  with initial distribution  $(p_i)_{i \in \mathcal{S}}$  and transition matrix  $P = (p_{ij})$ . Define the measure  $m$  as follows:

$$m(I_{\varepsilon_1 \dots \varepsilon_n}) = \mathbb{P}(X_1 = \varepsilon_1, \dots, X_n = \varepsilon_n).$$

Then  $m(I_i) = p_i$ ,  $i \in \mathcal{S}$ , and by the Markov property,

$$\begin{aligned} m(I_{\varepsilon_1 \dots \varepsilon_n}) &= \mathbb{P}(X_n = \varepsilon_n \mid X_{n-1} = \varepsilon_{n-1} \cdots X_1 = \varepsilon_1) \mathbb{P}(X_{n-1} = \varepsilon_{n-1} \cdots X_1 = \varepsilon_1) \\ &= p_{\varepsilon_{n-1} \varepsilon_n} m(I_{\varepsilon_1 \dots \varepsilon_{n-1}}). \end{aligned}$$

Iterating, we get

$$(3.1) \quad m(I_{\varepsilon_1 \dots \varepsilon_n}) = p_{\varepsilon_1} p_{\varepsilon_1 \varepsilon_2} \cdots p_{\varepsilon_{n-1} \varepsilon_n}.$$

In other words, a finite trajectory of  $X$  selects an interval and assigns it a mass equal to the probability of the trajectory. The measure is well defined since for any given  $\ell$ -adic interval  $I$  there is exactly one path to it and its subintervals can be reached only through  $I$ .

Because of the additivity property  $m(I_{\varepsilon_1 \dots \varepsilon_n}) = \sum_{k=0}^{n-1} m(I_{\varepsilon_1 \dots \varepsilon_{n-1} k})$  and the property  $\lim_{n \rightarrow +\infty} m(I_{\varepsilon_1 \dots \varepsilon_n}) = 0$ , it is well known that formula (3.1) defines a probability Borel measure whose support is contained in  $[0, 1]$  (see for example [16]). Moreover the measure  $m$  is such that  $m(\{x\}) = 0$  for any point  $x$ .

The construction proposed here can be viewed as a generalization of a classical Bernoulli measure. Our goal is to give a hands-on approach to the multifractal analysis of such measures.

**Example 3.1.** (1) Natural measure on the triadic Cantor set. Let

$$P = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

and set the initial distribution  $(p_0, p_1, p_2) = (1/2, 0, 1/2)$ . Then  $\mathcal{C} = \text{supp } m$  is the ternary Cantor set and  $m$  is the normalized  $(\log_3 2)$ -Hausdorff measure on  $\mathcal{C}$ .

(2) Bernoulli measures. Let  $p = (p_0, \dots, p_{\ell-1})$  be a probability vector and suppose that  $(X_i)$  are i.i.d. random variables with  $\mathbb{P}(X_1 = j) = p_j$ . By independence  $p_{ij} = p_j$  so that

$$P = \begin{pmatrix} p_0 & p_1 & \cdots & p_{\ell-1} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ p_0 & p_1 & \cdots & p_{\ell-1} \end{pmatrix}.$$

Hence, by (3.1) we get  $m(I_{\varepsilon_1 \dots \varepsilon_n}) = p_{\varepsilon_1} \cdots p_{\varepsilon_n}$ , and the measure  $m$  is the classical Bernoulli product.

(3) Let  $\ell = 2$ ,  $p \neq 1/2$  and

$$P = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}.$$

The associated measure  $m$  was introduced by Tukia in [17]. It is a doubling measure with dimension  $\dim(m) = -(p \log_2 p + (1-p) \log_2 (1-p)) < 1$ . The associated repartition function  $f(x) = m([0, x])$  is a singular quasisymmetric function.

(4) Random walk on  $\mathbb{Z}_\ell$ . Let  $p_{ij} = 1/2$  if  $|i - j| = 1$  or  $(i, j) = (0, \ell - 1)$  or  $(i, j) = (\ell - 1, 0)$ . If, for example,  $\ell = 3$ ,

$$P = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

As we will see later, the associated measure is monofractal with dimension  $\frac{\log 2}{\log \ell}$  (see the definition in the introduction).

Define as usual the structure function  $\tau(q)$  by

$$(3.2) \quad \tau(q) = \limsup_{n \rightarrow \infty} \frac{1}{n \log \ell} \log \left( \sum_{I \in \mathcal{F}_n} m(I)^q \right),$$

with the usual convention  $0^q = 0$  for any  $q \in \mathbb{R}$ .

**Theorem 3.2.** *Let  $m$  be a probability measure driven by a Markov chain with transition matrix  $P$ . Suppose that the matrix  $P$  is irreducible. For  $q \in \mathbb{R}$ , let  $\lambda_q$  be the spectral radius of  $P_q$ . Then  $\tau(q) = \log_\ell(\lambda_q)$  and the limit does exist in (3.2).*

*Proof.* Let  $\mathcal{W}_{n,k}$  be the subset of  $\mathcal{W}_n$  consisting of words that end with  $k \in \mathcal{S}$ . Set

$$s_{n,k} = \sum_{\varepsilon \in \mathcal{W}_{n,k}} m(I_\varepsilon)^q$$

and let  $S_n$  be the (line) vector  $(s_{n,0}, \dots, s_{n,\ell-1})$ . In particular,  $\mathcal{W}_{1,k} = \{k\}$  and  $S_1 = (p_0^q, \dots, p_{\ell-1}^q)$ . We claim that

$$(3.3) \quad S_n P_q = S_{n+1}, \quad n \geq 1.$$

Indeed, the  $j^{\text{th}}$  coordinate of  $S_n P_q$  is given by

$$\sum_{k \in \mathcal{S}} s_{n,k} p_{kj}^q = \sum_{k \in \mathcal{S}} \sum_{\varepsilon \in \mathcal{W}_{n,k}} p_{kj}^q m(I_\varepsilon)^q$$

Using the Markov property, we have  $p_{kj} m(I_\varepsilon) = m(I_{\varepsilon j})$  when  $\varepsilon \in \mathcal{W}_{n,k}$ . It follows that

$$\sum_{k \in \mathcal{S}} s_{n,k} p_{kj}^q = \sum_{k \in \mathcal{S}} \sum_{\varepsilon \in \mathcal{W}_{n,k}} m(I_{\varepsilon j})^q = \sum_{\varepsilon \in \mathcal{W}_{n+1,j}} m(I_\varepsilon)^q = s_{n+1,j}$$

which is the  $j^{\text{th}}$  coordinate of  $S_{n+1}$ . So that (3.3) follows. Iterating, we obtain

$$(3.4) \quad S_n = S_1 (P_q)^{n-1}.$$

Now, observe that

$$\sum_{\varepsilon \in \mathcal{W}_n} m(I_\varepsilon)^q = \sum_{k \in \mathcal{S}} s_{n,k} = \|S_n\|_1 = \|S_1 P_q^{n-1}\|_1$$

where  $\|\cdot\|_1$  is the  $\ell^1$  norm in  $\mathbb{R}^\ell$ . It follows that

$$\begin{aligned}\tau(q) &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log_\ell \left( \sum_{\varepsilon \in W_n} m(I_\varepsilon)^q \right) \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log_\ell \|S_1 P_q^{n-1}\|_1 \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log_\ell \|S_1 P_q^n\|_1\end{aligned}$$

Let us now introduce the following notation. If  $a = (a_0, \dots, a_{\ell-1})$  and  $b = (b_0, \dots, b_{\ell-1})$  are two vectors in  $\mathbb{R}^\ell$ , we will write  $a \prec b$  when  $a_i \leq b_i$  for any value of  $i$ . Observe in particular that  $S_1 \succ 0$  and is not identically equal to 0. Let  $\lambda_q$  be the spectral radius of the matrix  $P_q$ , which is also the spectral radius of the transposed matrix  $P_q^t$ . The matrix  $P_q$  being positive and irreducible, Perron-Frobenius Theorem ensures the existence of an eigenvector  $\nu_q$  with strictly positive entries, satisfying  $\nu_q P_q = \lambda_q \nu_q$ . Therefore, there exists a constant  $C > 0$  such that  $S_1 \prec C \nu_q$ . It follows that

$$\|S_1 P_q^n\|_1 \leq C \|\nu_q P_q^n\|_1 = C \|\nu_q\|_1 \lambda_q^n.$$

On the other hand, using the irreducibility property of the matrix  $P_q$ , we can find an integer  $k$  such that the matrix  $I + P_q + \dots + P_q^k$  has strictly positive entries. It follows that the line vector  $S_1 + S_1 P_q + \dots + S_1 P_q^k$  has strictly positive entries and we can find a constant  $C > 0$  such that

$$\nu_q \prec C (S_1 + S_1 P_q + \dots + S_1 P_q^k).$$

So

$$\begin{aligned}\|\nu_q P_q^n\|_1 &\leq C \|S_1 P_q^n + S_1 P_q^{n+1} + \dots + S_1 P_q^{n+k}\|_1 \\ &= C \| (S_1 P_q^n) (I + P_q + \dots + P_q^k) \|_1 \\ &\leq C' \|S_1 P_q^n\|_1.\end{aligned}$$

Finally,  $\|S_1 P_q^n\|_1 \asymp \lambda_q^n$ . Taking the logarithm, we can conclude that  $\tau(q) = \log_\ell(\lambda_q)$  and that the limit exists.  $\square$

*Remark 3.3.* Let  $q \in \mathbb{R}$  and  $U_n(q) = \sum_{J \in \mathcal{F}_n} m(J)^q$ . As a consequence of the argument developed in Theorem 3.2, we can easily see that there exists a constant  $C > 0$  (depending on  $q$  but not on  $m$  and  $n$ ) such that

$$\frac{1}{C} U_m(q) U_n(q) \leq U_{m+n}(q) \leq C U_m(q) U_n(q).$$

Such a sub-multiplicativity and super-multiplicativity property was already observed in the context of quasi-Bernoulli measures ([2]) and in the case of weak quasi-Bernoulli measures ([15]). It will be the key of the proof of Theorem 5.5.

**Corollary 3.4.** *The function  $\tau$  is analytic on  $\mathbb{R}$ .*

*Proof.* This can be seen as a consequence of the Kato-Rellich theorem (see for example [14]). But in this finite dimensional context, there is an elementary proof. Let  $F(q, x) = \det(P_q - xI)$  be the characteristic polynomial of  $P_q$  and let  $q_0 \in \mathbb{R}$ . Observing that  $F(q_0, \lambda_{q_0}) = 0$  and  $\frac{\partial F}{\partial x}(q_0, \lambda_{q_0}) \neq 0$  (the eigenvalue  $\lambda_{q_0}$  is simple),

the map  $q \mapsto \lambda_q$  is given around  $q_0$  by the implicit functions theorem. Moreover,  $F$  being analytic in  $q$  and  $x$ , it is well known that the implicit function is analytic.  $\square$

The existence of  $\tau'(1)$  ensures that the measure  $m$  is unidimensional (see e.g. [9], Theorem 3.1).

**Corollary 3.5.** *The measure  $m$  is unidimensional with dimension  $\dim(m) = -\tau'(1)$ .*

Let us now describe some examples. Let  $h_\ell$  be the usual entropy function

$$h_\ell(p) = - \sum_{i=0}^{\ell-1} p_i \log_\ell p_i, \quad p = (p_0, \dots, p_{\ell-1}) \text{ with } \sum_{i=0}^{\ell-1} p_i = 1.$$

In particular, set  $h(x) = h_2(x, 1-x) = -(x \log_2(x) + (1-x) \log_2(1-x))$ .

**Example 3.6.** If  $m$  is the Bernoulli measure from Example 3.1 (2), then by Theorem 3.2 we get the well known formula for  $\tau$

$$\tau(q) = \log_\ell (p_0^q + \dots + p_{\ell-1}^q).$$

Furthermore, the dimension of the measure  $m$  is  $\dim(m) = -\tau'(1) = h_\ell(p)$ .

**Example 3.7.** Actually, if  $\ell = 2$ , we can obtain an explicit formula for any given Markov chain. Suppose that  $a, b \in (0, 1)$  and

$$P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}.$$

Then

$$(3.5) \quad \dim(m) = \frac{b}{a+b} h(a) + \frac{a}{a+b} h(b).$$

Indeed, by Theorem 3.2 we get

$$\tau(q) = -1 + \log_2 \left( (1-a)^q + (1-b)^q + \sqrt{((1-a)^q - (1-b)^q)^2 + 4a^q b^q} \right).$$

Note that if  $q = 1$ , then  $\sqrt{((1-a)^q - (1-b)^q)^2 + 4a^q b^q}$  simplifies to  $a + b$ . Thus, we obtain

$$\begin{aligned} \tau'(1) = \frac{1}{2 \log 2} & \left( (1-a) \log(1-a) + (1-b) \log(1-b) + \right. \\ & \left. + \frac{(b-a)((1-a) \log(1-a) - (1-b) \log(1-b)) + 2ab \log ab}{a+b} \right) \end{aligned}$$

Rearranging, we get (3.5). Note that the coefficients in (3.5) come from the stationary distribution of  $X$ :

$$\pi = \left( \frac{b}{a+b}, \frac{a}{a+b} \right).$$

This observation will be generalized in Theorem 3.9 below.

**Example 3.8.** Let  $(a_0, \dots, a_{\ell-1})$  be a probability vector (with possibly some entries that are equal to 0) and  $P$  be an  $\ell \times \ell$  irreducible stochastic matrix with entries  $a_0, \dots, a_{\ell-1}$  in every row, but in an arbitrary order. Then  $\tau(q) = \log_\ell (a_0^q + \dots + a_{\ell-1}^q)$  and  $\dim(m) = h_\ell(a_0, \dots, a_{\ell-1})$ . In particular, if  $\kappa$  is the number of  $a_i$ 's that are not equal to 0 and if each nonzero  $a_i$  is equal to  $1/\kappa$ , we get  $\dim(m) = \frac{\log \kappa}{\log \ell}$  which is the maximal possible value and is also the dimension of the support of the measure  $m$ . Such a remark will be generalized below.

*Proof.* Set  $A_q = a_0^q + \dots + a_{\ell-1}^q$ . We have

$$\begin{aligned} \sum_{\varepsilon \in \mathcal{W}_{n+1}} m(I_\varepsilon)^q &= \sum_{k \in \mathcal{S}} \sum_{\varepsilon \in \mathcal{W}_{n,k}} \sum_{j \in \mathcal{S}} p_{kj}^q m(I_\varepsilon)^q \\ &= A_q \sum_{k \in \mathcal{S}} \sum_{\varepsilon \in \mathcal{W}_{n,k}} m(I_\varepsilon)^q \\ &= A_q \sum_{\varepsilon \in \mathcal{W}_n} m(I_\varepsilon)^q. \end{aligned}$$

Iterating, we get

$$\sum_{\varepsilon \in \mathcal{W}_n} m(I_\varepsilon)^q = A_q^{n-1} \sum_{\varepsilon \in \mathcal{W}_1} m(I_\varepsilon)^q.$$

Consequently,

$$\tau(q) = \log_\ell (a_0^q + \dots + a_{\ell-1}^q).$$

It follows that  $\tau'(1) = -h_\ell(a_0, \dots, a_{\ell-1})$ .

If there are only  $\kappa$  nonzero values in the probability vector  $(a_0, \dots, a_{\ell-1})$ , say e.g.  $a_0, \dots, a_{\kappa-1}$ , the formula turns to  $\dim m = -\sum_{j=0}^{\kappa-1} a_j \log_\ell a_j$  which is maximal when  $a_j = 1/\kappa$  for any  $j$ .  $\square$

Let us finish this part with a general formula for the dimension of the measure  $m$ . That is the purpose of the following theorem.

**Theorem 3.9.** *Denote by  $L_k$  the  $k^{\text{th}}$  line of the matrix  $P$ . Let  $H = (h_\ell(L_0), \dots, h_\ell(L_{\ell-1}))$  be the vector of entropies of the lines of  $P$ . Then*

$$\dim(m) = \langle \pi | H \rangle,$$

where  $\pi$  is the stationary distribution of the Markov chain  $X$  and  $\langle | \rangle$  is the canonical scalar product.

*Proof.* Let  $T_n = \sum_{I \in \mathcal{F}_n} m(I) \log_\ell m(I)$ , and  $H_n = \frac{-1}{n} T_n$  the entropy related to the partition  $\mathcal{F}_n$ . We know that  $\tau'(1)$  exists. It follows that  $-\tau'(1) = \lim_{n \rightarrow +\infty} H_n$  (see for example [9], Theorem 3.1). Further,

$$\begin{aligned} T_n &= \sum_{k \in \mathcal{S}} \sum_{\varepsilon \in \mathcal{W}_{n-1,k}} \sum_{j \in \mathcal{S}} m(I_\varepsilon) p_{kj} \log_\ell (m(I_\varepsilon) p_{kj}) \\ &= - \sum_{k \in \mathcal{S}} h_\ell(L_k) \sum_{\varepsilon \in \mathcal{W}_{n-1,k}} m(I_\varepsilon) + \sum_{k \in \mathcal{S}} \sum_{\varepsilon \in \mathcal{W}_{n-1,k}} m(I_\varepsilon) \log_\ell (m(I_\varepsilon)) \end{aligned}$$

Denote, as before,  $s_{n-1,k} = \sum_{\varepsilon \in \mathcal{W}_{n-1,k}} m(I_\varepsilon)$ . It follows that

$$T_n = - \sum_{k \in \mathcal{S}} h_\ell(L_k) s_{n-1,k} + T_{n-1}.$$

Iterating, we get

$$\tau'(1) = \lim_{n \rightarrow +\infty} \sum_{k \in \mathcal{S}} h_\ell(L_k) \frac{1}{n} (s_{n-1,k} + s_{n-2,k} + \dots + s_{1,k})$$

Observe now that  $s_{n-1,k}$  is the  $k^{\text{th}}$  component of  $S_{n-1} = S_1 P^{n-2}$  (see (3.4)). It is well known that the Cesaro means converge to the stationary distribution  $\pi$  (even in the periodic case). The theorem follows.  $\square$

**Example 3.10.** Let

$$P = \begin{pmatrix} 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}.$$

An analytic formula for  $\tau(q)$  is complicated. Nevertheless, a numerical evaluation of  $\tau(0)$  (which is also the box dimension of the support of  $m$ ) is possible and Theorem 3.9 allows us to estimate the dimension of the measure  $m$ . We find

$$\dim m \approx 0.58 \quad \text{and} \quad \dim_B(\text{supp } m) \approx 0.60.$$

This example shows that the “uniform” transition densities does not need to imply the maximality of  $\dim m$ . Indeed, we will see in Corollary 4.3 that there always exists a choice of the transition matrix for which the dimension of the measure coincides with the dimension of its support.

#### 4. MULTIFRACTAL ANALYSIS AND DIMENSION OF THE SUPPORT

By the definition of  $\tau$  we have

$$\tau(0) = \limsup_{n \rightarrow \infty} \frac{\log N_n}{n \log \ell},$$

where  $N_n$  is the number of intervals from  $\mathcal{F}_n$  having positive measure. In our context the limit exists, so  $\tau(0) = \dim_B(\text{supp } m)$ . Observe that the support of the measure  $m$  doesn't depend on the specific values of  $p_{ij}$  but only on the configuration of the nonzero entries in the matrix  $P_0$  and in the initial distribution  $p = (p_0, \dots, p_{\ell-1})$ . More precisely, the support of the measure  $m$  is the compact set

$$K = \bigcap_{n \geq 1} \bigcup_{p_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_{n-1} \varepsilon_n} > 0} \overline{I_{\varepsilon_1 \dots \varepsilon_n}}.$$

Indeed, the construction of the support can be viewed as a Cantor-like removal process. Given an interval  $I_{\varepsilon_1 \dots \varepsilon_n}$  of the  $n^{\text{th}}$  generation with  $\varepsilon_n = i$ , its  $j^{\text{th}}$  subinterval will be removed if and only if  $p_{ij} = 0$  (cf. Example 3.1 (1)).

According to Theorem 3.2,  $\overline{\dim}_B(\text{supp } m) = \log_\ell \lambda_0$  where  $\lambda_0$  is the spectral radius of the matrix  $P_0$ , so that the box dimension of  $\text{supp } m$  does not depend on the initial distribution  $p$  and only depends on the configuration of the nonzero entries of the matrix  $P_0$ .

This motivates the following questions. Given a configuration of nonzero entries of  $P_0$  and of the initial distribution  $p$ , which values of  $p_{ij}$  maximize  $\dim m$ ? Is the maximal value of  $\dim m$  equal to the box dimension of the support? Is this maximal measure unique?

In some cases one has an immediate answer. In particular, Example 3.8 says that if each row of the matrix  $P$  has the same number  $\kappa$  of nonzero entries, the maximum of  $\dim m$  is obtained when each nonzero entry of the matrix  $P$  is equal to  $1/\kappa$  and is then equal to  $\frac{\log \kappa}{\log \ell}$ .



The general answer will be a consequence of the following result which says that the measure  $m$  satisfies the multifractal formalism.

**Theorem 4.1.** *Let  $X$  be an irreducible Markov chain with transition matrix  $P$  and let  $m$  be the associated measure. Then  $m$  satisfies the multifractal formalism. More precisely, define*

$$E_\alpha = \left\{ x \in [0, 1] ; \lim_{n \rightarrow \infty} \frac{\log m(I_n(x))}{\log |I_n(x)|} = \alpha \right\}.$$

Then, for any  $-\tau'(+\infty) < \alpha < -\tau'(\infty)$ ,

$$\dim(E_\alpha) = \tau^*(\alpha),$$

where  $\tau^*(\alpha) = \inf_q(\alpha q + \tau(q))$  is the Legendre transform of the function  $\tau$ .

*Remark 4.2.* In the case where  $m$  is  $\sigma$ -invariant, where  $\sigma$  is the shift operator, Theorem 4.1 is related to the work of Pesin and Weiss (see [13] or [12] and Section 5 for a precise definition of the shift  $\sigma$ ).

*Proof.* We will prove the existence of a Gibbs measure at a given state  $q$ , that is an auxiliary measure  $m_q$  such that for any  $\ell$ -adic interval  $I$  one has

$$(4.1) \quad m_q(I) \asymp |I|^{\tau(q)} m(I)^q.$$

Note that in (4.1), the constant may depend on  $q$ . Since the function  $\tau$  is differentiable, it is well known that the existence of such a measure at each state  $q$  implies the validity of the multifractal formalism for  $m$  (see for example [2] or [9]).

Now again, such a Gibbs measure will be obtained with an elementary construction. Note that  $P$  is irreducible if and only if  $P_q$  is. By Perron-Frobenius Theorem, the spectral radius  $\lambda_q$  of  $P_q$  is a simple eigenvalue and there exists a unique probability vector  $\pi_q$  with strictly positive entries and satisfying  $P_q \pi_q = \lambda_q \pi_q$ .

Define  $D_q$  as the  $\ell \times \ell$ -matrix having the coordinates of  $\pi_q$  on the diagonal and zeros elsewhere. Set  $Q_q = \frac{1}{\lambda_q} D_q^{-1} P_q D_q$  and let  $\mathbf{1}$  be the column vector  $\mathbf{1} = (1, \dots, 1)^t \in \mathbb{R}^\ell$ . Then we have

$$Q_q \mathbf{1} = \frac{1}{\lambda_q} D_q^{-1} P_q D_q \mathbf{1} = \frac{1}{\lambda_q} D_q^{-1} P_q \pi_q = D_q^{-1} \pi_q = \mathbf{1}.$$

In other words,  $Q_q$  is a stochastic matrix and thus it can be associated to a Markov chain  $X^{(q)}$ . We may and do assume that this chain has the initial distribution  $q_i = \alpha p_i^q$ ,  $i \in \mathcal{S}$ , where  $(p_0, \dots, p_{\ell-1})$  is the initial distribution of  $X$  and  $\alpha = (p_0^q + \dots + p_{\ell-1}^q)^{-1}$  is the normalization constant. Let  $m_q$  be the measure induced by  $X^{(q)}$ . Remark that if  $Q_q = (q_{ij})$ ,  $D_q = (d_{ij})$ , then we have

$$q_{ij} = \frac{1}{\lambda_q} d_{ii}^{-1} p_{ij}^q d_{jj}, \quad i, j \in \mathcal{S}.$$

Clearly,  $q_{ij} > 0$  if and only if  $p_{ij} > 0$  and  $q_i > 0$  if and only if  $p_i > 0$ . Therefore the measures  $m_q$  and  $m$  have the same support.

Let  $I = I_{\varepsilon_1 \dots \varepsilon_n} \in \mathcal{F}_n$ . We have

$$\begin{aligned} m_q(I) &= q_{\varepsilon_1} q_{\varepsilon_1 \varepsilon_2} \cdots q_{\varepsilon_{n-1} \varepsilon_n} \\ &= \alpha p_{\varepsilon_1}^q \left( \frac{1}{\lambda_q} d_{\varepsilon_1 \varepsilon_1}^{-1} p_{\varepsilon_1 \varepsilon_2}^q d_{\varepsilon_2 \varepsilon_2} \right) \cdots \left( \frac{1}{\lambda_q} d_{\varepsilon_{n-1} \varepsilon_{n-1}}^{-1} p_{\varepsilon_{n-1} \varepsilon_n}^q d_{\varepsilon_n \varepsilon_n} \right) \\ &= \frac{\alpha}{\lambda_q^{n-1}} d_{\varepsilon_1 \varepsilon_1}^{-1} m(I_{\varepsilon_1 \dots \varepsilon_n})^q d_{\varepsilon_n \varepsilon_n} \end{aligned}$$

Further, since the entries of the the eigenvector  $\pi_q$  are strictly positive, there is a constant  $c$  (possibly depending on  $q$ ) such that

$$c^{-1} \leq \frac{d_{ii}}{d_{jj}} \leq c$$

for any  $i, j \in \mathcal{S}$ . This yields

$$m_q(I) \asymp \frac{1}{\lambda_q^{n-1}} m(I)^q.$$

Now, observe that by Theorem 3.2,  $\ell^{\tau(q)} = \lambda_q$  so that

$$|I|^{\tau(q)} = (\ell^{-n})^{\tau(q)} = \lambda_q^{-n}.$$

It follows that for any  $I \in \mathcal{F}_n$  we have

$$m_q(I) \asymp |I|^{\tau(q)} m(I)^q.$$

Note that in the above estimate the implicit constants may depend on  $q$  but not on  $n$  and  $I$ . Therefore  $m_q$  is the needed Gibbs measure and the theorem follows.  $\square$

**Corollary 4.3.** *Let  $K$  be the support of the measure  $m$ . The measure  $\bar{m} = m_0$  associated to the matrix  $Q_0$  satisfies  $\text{supp } \bar{m} = K$ , is monofractal and strongly equivalent to the Hausdorff measure  $\mathcal{H}^{\tau(0)}$  on  $K$ . In particular,*

$$\dim(\bar{m}) = \dim_H(K) = \dim_B(K)$$

and  $\bar{m}$  is a measure driven by a Markov chain, with support  $K$  and with maximal dimension.

*Proof.* Let  $\bar{m} = m_0$  from the previous proof. Then we have

$$\bar{m}(I) \asymp |I|^{\tau(0)}$$

for any  $\ell$ -adic interval  $I$  such that  $m(I) > 0$ . In particular, for any  $x \in K$ ,  $\bar{m}(I_n(x)) \asymp |I_n(x)|^{\tau(0)}$ . By Billingsley's theorem (see e.g. [5], Propositions 2.2 and 2.3), we conclude that  $\bar{m}$  is equivalent to  $\tau(0)$ -dimensional Hausdorff measure  $\mathcal{H}^{\tau(0)}$  on  $K$ . In particular  $\mathcal{H}^{\tau(0)}(K)$  is positive and finite. It follows that

$$\dim_H(K) = \tau(0) = \dim_B(K).$$

On the other hand, the structure function  $\bar{\tau}$  of the measure  $\bar{m}$  is  $\bar{\tau}(q) = \tau(0)(1-q)$ . It follows that the measure  $\bar{m}$  is monofractal, and that

$$\dim \bar{m} = -\bar{\tau}'(1) = \tau(0) = \dim_B(K).$$

$\square$

**Example 4.4.** Suppose that  $X$  is a random walk on  $\mathbb{Z}_\ell$  (cf. Example 3.1 (3)). Then  $\tau(q) = (1-q) \log_\ell 2$  and

$$\dim m = -\tau'(1) = \log_\ell 2 = \tau(0) = \dim_B(K).$$

**Example 4.5.** Let

$$P = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}.$$

It can be easily seen that  $P$  is irreducible (actually,  $P^2$  has only positive entries). We obtain

$$\tau(q) = -\log_3 2 + \log_3 \left( 2^{-q} + 3^{-q} + \sqrt{4^{-q} + 6^{1-q} + 9^{-q}} \right).$$

Hence

$$\dim_H(\text{supp } m) = \log_3(1 + \sqrt{2}) \approx 0.802 \quad \text{and} \quad \dim m = \frac{1}{7}(3 + 4 \log_3 2) \approx 0.789.$$

As observed in Example 3.10, the transitions are uniform row by row, but the measure  $m$  is not monofractal. The Markov chain inducing the maximal measure  $\bar{m}$  is associated to the following transition matrix :

$$Q_0 = (\sqrt{2} - 1) \times \begin{pmatrix} 1 & \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2} & 0 & 1 \\ \sqrt{2} & 0 & 1 \end{pmatrix}.$$

#### 5. INVARIANCE, ERGODICITY AND APPLICATION TO THE UNIQUENESS OF $\bar{m}$

The goal of this section is to discuss the uniqueness of measure  $\bar{m}$  with maximal dimension given in Corollary 4.3. This is the object of Theorem 5.5.

We need to start with some preliminary results. Let us introduce the shift  $\sigma$  on  $[0, 1)$  defined by  $\sigma(x) = \ell x - E(\ell x)$ , where  $E(y)$  is the integer part of  $y$ . Observe that  $\sigma(I_{\varepsilon_1 \dots \varepsilon_n}) = I_{\varepsilon_2 \dots \varepsilon_n}$  and that

$$\sigma(x) = \bigcap_n I_{\varepsilon_2 \dots \varepsilon_n} \quad \text{if} \quad x = \bigcap_n I_{\varepsilon_1 \dots \varepsilon_n}.$$

That is why  $\sigma$  is called the shift.

**Proposition 5.1.** *Let  $P$  be an irreducible  $\ell \times \ell$  transition matrix,  $\nu$  be the (unique) probability vector such that  $\nu P = \nu$  and  $X$  be the Markov chain with transition  $P$  and initial law  $\nu$ . Set  $m_P$  to be the probability measure driven by the Markov chain  $X$ . Then,  $m_P$  is  $\sigma$ -invariant and ergodic.*

*Proof.*

$$\begin{aligned} m_P(\sigma^{-1}(I_{\varepsilon_1 \dots \varepsilon_n})) &= \sum_{j=0}^{\ell} m_P(I_{j\varepsilon_1 \dots \varepsilon_n}) \\ &= \sum_{j=0}^{\ell} \nu_j p_{j\varepsilon_1} \cdots p_{\varepsilon_{n-1}\varepsilon_n} \\ &= \nu_{\varepsilon_1} p_{\varepsilon_1 \varepsilon_2} \cdots p_{\varepsilon_{n-1}\varepsilon_n} \\ &= m_P(I_{\varepsilon_1 \dots \varepsilon_n}) \end{aligned}$$

So, by the monotone class theorem, the measure  $m_P$  is  $\sigma$ -invariant.

Let  $k$  be an integer such that  $P + P^2 + \dots + P^k$  has strictly positive entries. We claim that there exists a constant  $C > 0$  such that for any  $I, J \in \bigcup_n \mathcal{F}_n$ , we have

$$(5.1) \quad \frac{1}{C} m_P(I) m_P(J) \leq \sum_{j=0}^{k-1} \sum_{K \in \mathcal{F}_j} m_P(IKJ) \leq C m_P(I) m_P(J).$$

Indeed, if  $I = I_{\varepsilon_1 \dots \varepsilon_n}$ ,  $J = I_{\delta_1 \dots \delta_m}$ , and if  $(\pi_{ij})_{i,j}$  denote the coefficients of the matrix  $P + P^2 + \dots + P^k$ , it is easy to check that

$$\sum_{j=0}^{k-1} \sum_{K \in \mathcal{F}_j} m_P(IKJ) = \nu_{\varepsilon_1} p_{\varepsilon_1 \varepsilon_2} \dots p_{\varepsilon_{n-1} \varepsilon_n} \times \pi_{\varepsilon_n \delta_1} p_{\delta_1 \delta_2} \dots p_{\delta_{m-1} \delta_m}$$

and the claim follows.

Inequality (5.1) can be rewritten as

$$(5.2) \quad \sum_{j=0}^{k-1} m_P \left( I \cap \sigma^{-(n+j)}(J) \right) \asymp m_P(I) \times m_P(J)$$

where  $n$  is the generation of  $I$ . If we observe that any open set is a countable union of disjoint intervals in  $\bigcup_n \mathcal{F}_n$ , inequality (5.2) remains true when  $J$  is an open set. Finally, by regularity of the measure  $m_P$ , it is also true for any Borel set  $J$ . In particular, if  $E$  is a  $\sigma$ -invariant Borel set, we get

$$\forall I \in \bigcup_n \mathcal{F}_n, \quad k m_P(I \cap E) \asymp m_P(I) \times m_P(E).$$

Again, it remains true when  $I$  is an arbitrary Borel set. In particular,

$$m_P \left( ([0, 1] \setminus E) \cap E \right) \asymp m_P \left( [0, 1] \setminus E \right) \times m_P(E)$$

which proves that  $m_P(E) = 0$  or  $m_P([0, 1] \setminus E) = 0$ .  $\square$

*Remark 5.2.* Inequality 5.2 is a particular case of the so called weak quasi-Bernoulli property which was introduced by B. Testud in [15].

**Corollary 5.3.** *Let  $P$  and  $\tilde{P}$  be two different irreducible  $\ell \times \ell$  transition matrices. Then  $m_P$  is singular with respect to  $m_{\tilde{P}}$ .*

*Proof.* According to the ergodic theorem, it suffices to show that  $m_P \neq m_{\tilde{P}}$ . Let  $\nu$  and  $\tilde{\nu}$  be the invariant distributions of the stochastic matrix  $P$  and  $\tilde{P}$ . If  $\nu_i \neq \tilde{\nu}_i$  for some  $i$ , then  $m_P(I_i) \neq m_{\tilde{P}}(I_i)$ . If  $\nu = \tilde{\nu}$  and if  $p_{ij} \neq \tilde{p}_{ij}$ , we can write :

$$m_P(I_{ij}) = \nu_i p_{ij} \neq \tilde{\nu}_i \tilde{p}_{ij} = m_{\tilde{P}}(I_{ij}).$$

$\square$

**Corollary 5.4.** *Let  $P, \tilde{P}$  be two irreducible  $\ell \times \ell$  transition matrices,  $p$  and  $\tilde{p}$  to be two probability vectors. Set  $m$  and  $\tilde{m}$  be the associated measures. Suppose that  $\text{supp}(m) = \text{supp}(\tilde{m})$ . Then, there are only two possible cases :*

- (1)  $P = \tilde{P}$  and the measures  $m$  and  $\tilde{m}$  are strongly equivalent (i.e.  $m \asymp \tilde{m}$ )
- (2)  $P \neq \tilde{P}$  and the measures  $m$  and  $\tilde{m}$  are mutually singular.

*Proof.* Let  $\mathcal{A} \subset \mathcal{S}$  be the set of ranges of the nonzero entries of  $p$  (which is also the set of ranges of nonzero entries of  $\tilde{p}$ ). Let  $F = \bigcup_{\varepsilon \in \mathcal{A}} \overline{I_\varepsilon}$ . We claim that  $m$  is strongly equivalent to the measure  $m_P$  restricted to  $F$ . This is an easy consequence of the fact that the invariant probability vector  $\nu$  (satisfying  $\nu P = \nu$ ) has strictly positive

entries. In the same way,  $\tilde{m}$  is strongly equivalent to the measure  $m_{\tilde{P}}$  restricted to  $F$ . Corollary 5.4 is then a consequence of Corollary 5.3.  $\square$

Now, we are able to prove the following theorem on the measure  $\bar{m}$  given by Corollary 4.3.

**Theorem 5.5.** *Let  $P_0 = (p_{ij}^0)$  be an irreducible  $\ell \times \ell$  matrix such that  $p_{ij}^0 \in \{0, 1\}$  for any  $ij$  and let  $p^0 = (p_0^0, \dots, p_{\ell-1}^0)$  be a line vector such that  $p_i^0 \in \{0, 1\}$  for any  $i$ . Suppose that  $p^0 \neq (0, \dots, 0)$  and define the compact set  $K$  by*

$$K = \bigcap_{n \geq 1} \bigcup_{p_{\varepsilon_1}^0 p_{\varepsilon_1 \varepsilon_2}^0 \cdots p_{\varepsilon_{n-1} \varepsilon_n}^0 = 1} \overline{I_{\varepsilon_1 \cdots \varepsilon_n}}.$$

Let  $\delta = \dim_H(K)$  and let  $m$  be a measure with support  $K$ , driven by a Markov chain  $X$  with irreducible transition matrix  $P$ . Then,  $\dim m = \delta$  if and only if

$$P = \frac{1}{\lambda_0} D_0^{-1} P_0 D_0$$

where  $\lambda_0$  is the spectral radius of  $P_0$  and  $D_0$  is the diagonal matrix whose diagonal entries are the coordinates of the (unique) probability vector  $\pi_0$  satisfying  $P_0 \pi_0 = \lambda_0 \pi_0$ . Moreover, the case  $P = \frac{1}{\lambda_0} D_0^{-1} P_0 D_0$  is the only case where the measure  $m$  is monofractal.

*Proof.* Let  $p$  be the initial law of the Markov chain  $X$ . Recall that the support of the measure  $m$  is equal to  $K$ . It follows that the vectors  $p$  and  $p^0$  have the same non-zero entries. Let  $\mathcal{J}$  be the set of  $j \in \{0, \dots, \ell-1\}$  such that  $p_j^0 = 1$ . Remember also that the support of the measure  $m$  only depends on the positions of the non-zero entries of the matrix  $P$  and on the set  $\mathcal{J}$ . It follows the non-zero entries of the matrices  $P$  and  $P_0$  are located at the same places. Let  $Q_0 = \frac{1}{\lambda_0} D_0^{-1} P_0 D_0$ .

Suppose that  $P = Q_0$ . According to Corollary 4.3 and Corollary 5.4, the measure  $m$  is strongly equivalent to  $\bar{m} = m_0$  which satisfies  $\bar{m}(I_n(x)) \asymp |I_n(x)|^\delta$  for any  $x \in K$ . In particular,  $m$  is such that  $\dim m = \delta$  and is monofractal.

Suppose now that  $P \neq Q_0$ . Corollary 5.4 says that  $m$  is singular with respect to  $m_0$  and we have to prove that  $\dim m < \delta$ . Let

$$\tau(q) = \lim_{n \rightarrow +\infty} \frac{1}{n \log \ell} \log \left( \sum_{I \in \mathcal{F}_n} m(I)^q \right).$$

Recall that  $\tau$  is analytic and such that  $\tau(0) = \delta$  and  $\tau(1) = 0$ . Using the convexity of  $\tau$ , it is clear that

$$\tau'(1) = -\delta \iff \forall q \in [0, 1], \tau(q) = \delta(1 - q) \iff \forall q \in \mathbb{R}, \tau(q) = \delta(1 - q).$$

In order to prove that  $\dim m < \delta$ , it is then sufficient to establish that  $\tau(2) > -\delta$ .

Denote by  $(I_j)_{j \in \{0, \dots, \ell-1\}}$  the intervals of the first generation  $\mathcal{F}_1$ . If  $j \in \{0, \dots, \ell-1\}$  and  $n \geq 1$ , let  $\mathcal{F}_n(j)$  be the intervals of  $\mathcal{F}_n$  that are included in  $I_j$ . The measures  $m$  and  $m_0$  being mutually singular, we know that for  $dm$ -almost every  $x \in K$ ,

$$\lim_{n \rightarrow +\infty} \frac{m_0(I_n(x))}{m(I_n(x))} = 0,$$

which can be rewritten as

$$\lim_{n \rightarrow +\infty} \frac{\ell^{-n\delta}}{m(I_n(x))} = 0.$$

Using Egoroff's theorem in each  $I_j$  with  $j \in \mathcal{J}$ , we can find a set  $A \subset K$  such that  $m(A \cap I_j) \geq \frac{1}{2} m(I_j)$  for any  $j \in \mathcal{J}$  and satisfying

$$\forall \varepsilon > 0, \quad \exists n_0 \geq 1; \quad \forall n \geq n_0, \forall x \in A, \quad m(I_n(x)) \geq \frac{1}{\varepsilon} \ell^{-n\delta}.$$

It follows that

$$\sum_{J \in \mathcal{F}_{n_0}(j)} m(J)^2 \geq \sum_{J \in \mathcal{F}_{n_0}(j); J \cap A \neq \emptyset} \frac{1}{\varepsilon} \ell^{-n_0\delta} m(J) \geq \frac{1}{2\varepsilon} \ell^{-n_0\delta} m(I_j),$$

so that

$$\sum_{J \in \mathcal{F}_{n_0}} m(J)^2 = \sum_{j \in \mathcal{J}} \sum_{J \in \mathcal{F}_{n_0}(j)} m(J)^2 \geq \frac{1}{2\varepsilon} \ell^{-n_0\delta}.$$

Using the super multiplicativity property observed in Remark 3.3, we get for any  $k \geq 0$

$$\sum_{J \in \mathcal{F}_{n_0+k}} m(J)^2 \geq \frac{C}{2\varepsilon} \ell^{-n_0\delta} \sum_{J \in \mathcal{F}_k} m(J)^2,$$

where the constant  $C$  does not depend on  $n_0$  and  $k$ . In particular,

$$\sum_{J \in \mathcal{F}_{p n_0}} m(J)^2 \geq \left( \frac{C}{2\varepsilon} \ell^{-n_0\delta} \right)^p.$$

Let  $\varepsilon = \frac{C}{4}$  and the corresponding  $n_0$ . If  $\eta$  is such that  $\ell^{n_0\eta} = 2$ , we can rewrite the last inequality as

$$\sum_{J \in \mathcal{F}_{p n_0}} m(J)^2 \geq \ell^{-p n_0(\delta - \eta)}$$

which gives

$$\tau(2) \geq -(\delta - \eta) > -\delta.$$

Moreover, it is clear that  $\tau'(\mathbb{R})$  is not reduced to a single point. It follows that the measure  $m$  is multifractal.  $\square$

*Remark 5.6.* As was observed by a Referee, in view of Theorem 3.9, the case where the measure  $m$  is  $\sigma$ -invariant, reduces elementarily to the uniqueness of the ergodic measure with maximal entropy on a Markov shift specified by an irreducible incidence matrix (see for example [11]). Now again, we are able to obtain the non-dynamical case, by using elementary arguments.

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