MULTIFRACTAL ANALYSIS OF THE DIVERGENCE OF FOURIER SERIES: THE EXTREME CASES

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Abstract. We study the size, in terms of the Hausdorff dimension, of the subsets of $\mathbb{T}$ such that the Fourier series of a generic function in $L^1(\mathbb{T})$, $L^p(\mathbb{T})$ or in $\mathcal{C}(\mathbb{T})$ may behave badly. Genericity is related to the Baire category theorem or to the notion of prevalence. This paper is a continuation of [2].

1. Introduction

This paper, which can be seen as a continuation of [2], deals with the divergence of Fourier series of functions in $L^p(\mathbb{T})$, $p \geq 1$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, or in $\mathcal{C}(\mathbb{T})$, from the multifractal point of view. More precisely, let f be in $L^p(\mathbb{T})$, or in $\mathcal{C}(\mathbb{T})$, and let $(S_n f)_{n\geq 0}$ be the sequence of partial sums of its Fourier series. We are interested in the size of the sets of the real numbers x such that $(S_n f(x))_{n\geq 0}$ diverges with a prescribed growth.
We will measure the size of subsets of $\mathbb{T}$ using the Hausdorff dimension. Let us recall the relevant definitions (we refer to [5] and to [8] for more on this subject). If $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing continuous function satisfying $\phi(0) = 0$ (ϕ is called a *dimension function* or a *gauge function*), the ϕ-Hausdorff outer measure of a set $E \subset \mathbb{R}^d$ is

$$\mathcal{H}^\phi(E) = \lim_{\varepsilon\to 0} \inf_{r\in R_\varepsilon(E)} \sum_{B\in r} \phi(|B|),$$

where $R_\varepsilon(E)$ is the set of (countable) coverings of E with balls B of diameter $|B| \leq \varepsilon$. When $\phi(x) = \phi_s(x) = x^s$, we write for short $\mathcal{H}^s$ instead of $\mathcal{H}^{\phi_s}$. The Hausdorff dimension of a set E is defined by

$$\dim_{\mathcal{H}}(E) := \sup\{s > 0; \mathcal{H}^s(E) > 0\} = \inf\{s > 0;\ \mathcal{H}^s(E) = 0\}.$$

The first result studying the Hausdorff dimension of the divergence sets of Fourier series is due to J-M. Aubry [1].

Theorem 1.1. *Let $f \in L^p(\mathbb{T})$, $1 < p < +\infty$. If $\beta \geq 0$, define*

$$\mathcal{E}(\beta, f) = \left\{x \in \mathbb{T};\ \limsup_{n\to+\infty} n^{-\beta}|S_n f(x)| > 0\right\}.$$

Then $\dim_{\mathcal{H}}\left(\mathcal{E}(\beta, f)\right) \leq 1 - \beta p$. Conversely, given a set E such that $\dim_{\mathcal{H}}(E) < 1 - \beta p$, there exists a function $f \in L^p(\mathbb{T})$ such that, for any $x \in E$, $\limsup_{n\to+\infty} n^{-\beta}|S_n f(x)| = +\infty$.

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This result motivated us to introduce in [2] the notion of divergence index. For a given function $f \in L^p(\mathbb{T})$ and a given point $x_0 \in \mathbb{T}$, we can define $\beta(x_0)$ as the infimum of the nonnegative real numbers β such that $|S_n f(x_0)| = O(n^\beta)$. The real number $\beta(x_0)$ will be called the *divergence index* of the Fourier series of f at point x_0. As a consequence of Nikolsky inequality and F. Riesz theorem, it is easy to see that, for any function $f \in L^p(\mathbb{T})$ $(1 \leq p < +\infty)$ and any point $x_0 \in \mathbb{T}$, $0 \leq \beta(x_0) \leq 1/p$ (see [9] for the Nikolsky inequality and [11], Vol. I, page 266 for the Riesz theorem). Moreover, when $p > 1$, Carleson's theorem implies that $\beta(x_0) = 0$ almost surely. In [2], we gave precise estimates on the size of the level sets of the function β. These are defined as

$$\begin{aligned} E(\beta, f) &= \{x \in \mathbb{T};\ \beta(x) = \beta\} \\ &= \left\{x \in \mathbb{T};\ \limsup_{n\to+\infty} \frac{\log |S_n f(x)|}{\log n} = \beta\right\}. \end{aligned}$$

Theorem 1.2 ([2])**.** *Let $1 < p < +\infty$. For quasi-all functions $f \in L^p(\mathbb{T})$, for any $\beta \in [0, 1/p]$, $\dim_{\mathcal{H}}\big(E(\beta, f)\big) = 1 - \beta p$.*

The terminology "quasi-all" used here is relative to the Baire category theorem. It means that this property is true for a residual set of functions in $L^p(\mathbb{T})$ and we can say that the behaviour of the Fourier series of any function of this residual set is multifractal.

In the case of continuous functions, the situation breaks down dramatically. If $(D_n)_{n\geq 0}$ denotes the Dirichlet kernel, we can first observe that, when $f \in \mathcal{C}(\mathbb{T})$,

$$\|S_n f\|_\infty \leq \|D_n\|_1 \|f\|_\infty \leq C\|f\|_\infty \log n.$$

This motivated us in [2] to introduce the following level sets:

$$\begin{aligned} \mathcal{F}(\beta, f) &= \left\{x \in \mathbb{T};\ \limsup_{n\to+\infty} (\log n)^{-\beta} |S_n f(x)| > 0\right\} \\ F(\beta, f) &= \left\{x \in \mathbb{T};\ \limsup_{n\to+\infty} \frac{\log |S_n f(x)|}{\log\log n} = \beta\right\}. \end{aligned}$$

Whereas, on $L^p(\mathbb{T})$, $1 < p < +\infty$, the divergence index takes its biggest value ($\beta(x) = 1/p$) on small sets, this is far from being the case on $\mathcal{C}(\mathbb{T})$, as the following very surprizing result indicates.

Theorem 1.3 ([2])**.** *For quasi-all functions $f \in \mathcal{C}(\mathbb{T})$, for any $\beta \in [0, 1]$, $F(\beta, f)$ is non-empty and has Hausdorff dimension 1.*

However, several questions were left open in [2].

Question 1: what happens on $L^1(\mathbb{T})$?
In view of the differences between $L^p(\mathbb{T})$, $p \in (1, +\infty)$, and $\mathcal{C}(\mathbb{T})$, it seems *a priori* not clear what situation should be expected on $L^1(\mathbb{T})$. Moreover, Carleson's theorem is false on $L^1(\mathbb{T})$ and Kolmogorov's theorem ensures that there exist functions in $L^1(\mathbb{T})$ with everywhere divergent Fourier series.
The proof of Theorem 1.2 proceeds in two steps. In a first time, we build a residual set of functions in $L^p(\mathbb{T})$ such that, if f lies in this residual set and if $0 \leq \beta \leq 1/p$,

$\dim_{\mathcal{H}}(E(\beta,f)) \geq 1-\beta p$. In a second time, we use Theorem 1.1 to conclude that necessarily $\dim_{\mathcal{H}}(E(\beta,f)) = 1-\beta p$. The first step works as well in $L^1(\mathbb{T})$ and the trouble comes from Aubry's result, which uses the Carleson-Hunt maximal inequality. In Section 2, we succeed to overcome this difficulty by proving a (very weak!) version of Carleson's maximal inequality in $L^1(\mathbb{T})$ which is sufficient to prove the analogue of Theorem 1.1. Thus, we will show that

Theorem 1.4. *For quasi-all functions $f \in L^1(\mathbb{T})$, for any $\beta \in [0,1]$,*

$$\dim_{\mathcal{H}}\big(E(\beta,f)\big) = 1-\beta.$$

Question 2: what about the size of the set of functions satisfying the conclusion of Theorem 1.2 and Theorem 1.4?

Theorem 1.2 and Theorem 1.4 say that, in $L^p(\mathbb{T})$ ($p \geq 1$), the set of functions for which the Fourier series has a multifractal behaviour is big in a topological sense. One can ask if it remains big for other points of view. We deal here with an infinite-dimensional version of the notion of "almost-everywhere". This notion, called *prevalence*, has been introduced by J. Christensen in [4] and has been widely studied since then. In multifractal analysis, some properties which are true on a dense G_δ-set are also prevalent (see for instance [7] or [6]), whereas some are not (see for instance [10] or [7]). This motivated us to examine Theorem 1.2 and Theorem 1.4 under this point of view.

Definition 1.5. Let E be a complete metric vector space. A Borel set $A \subset E$ is called *Haar-null* if there exists a compactly supported probability measure μ such that, for any $x \in E$, $\mu(x+A) = 0$. If this property holds, the measure μ is said to be *transverse* to A. A subset of E is called *Haar-null* if it is contained in a Haar-null Borel set. The complement of a Haar-null set is called a *prevalent* set.

The following results enumerate important properties of prevalence and show that this notion supplies a natural generalization of "almost every" in infinite-dimensional spaces:

- If A is Haar-null, then $x+A$ is Haar-null for every $x \in E$.
- If $\dim(E) < +\infty$, A is Haar-null if and only if it is negligible with respect to the Lebesgue measure.
- Prevalent sets are dense.
- The intersection of a countable collection of prevalent sets is prevalent.
- If $\dim(E) = +\infty$, compacts subsets of E are Haar-null.

In Section 3, we will prove the following result.

Theorem 1.6. *Let $1 \leq p < +\infty$. The set of functions $f \in L^p(\mathbb{T})$ such that, for any $\beta \in [0,1/p]$, $\dim_{\mathcal{H}}(E(\beta,f)) = 1-\beta p$, is prevalent.*

Thus, almost every function in $L^p(\mathbb{T})$ is multifractal with respect to the summation of its Fourier series.

Question 3: can we say more on $\mathcal{C}(\mathbb{T})$?

Theorem 1.3 implies that there exists a residual subset $A \subset \mathcal{C}(\mathbb{T})$ such that, if $f \in A$ and if $\beta < 1$, one can find a set $E \subset \mathbb{T}$ with Hausdorff dimension 1 such that

$$\limsup_{n\to+\infty} \frac{|S_n f(x)|}{(\log n)^\beta} = +\infty \text{ for any } x \in E. \tag{1}$$

On the other hand, we know that, for any fixed $f \in \mathcal{C}(\mathbb{T})$, $\|S_n f\|_\infty$ is negligible compared to $\log n$ and that, conversely, given any sequence $(\delta_n)_{n\geq 2}$ of positive real numbers going to zero, we can find $f \in \mathcal{C}(\mathbb{T})$ such that

$$\limsup_{n\to+\infty} \frac{|S_n f(0)|}{\delta_n \log n} = +\infty.$$

These statements can be found for example in [11], Vol I, page 298. It seems then natural to ask whereas this property can be ensured in a set with Hausdorff dimension equal to 1 (relation (1) means that this is true when $\delta_n = (\log n)^{\beta-1}$, $0 < \beta < 1$). This is indeed true.

Theorem 1.7. *Let $(\delta_n)_{n\geq 2}$ be a sequence of positive real numbers going to zero. For quasi-all functions $f \in \mathcal{C}(\mathbb{T})$, there exists $E \subset \mathbb{T}$ with Hausdorff dimension 1 such that, for any $x \in E$,*

$$\limsup_{n\to+\infty} \frac{|S_n f(x)|}{\delta_n \log n} = +\infty.$$

The same result also holds in a prevalent subset of $\mathcal{C}(\mathbb{T})$.

Theorem 1.8. *Let $(\delta_n)_{n\geq 2}$ be a sequence of positive real numbers going to zero. For almost every function $f \in \mathcal{C}(\mathbb{T})$, there exists $E \subset \mathbb{T}$ with Hausdorff dimension 1 such that, for any $x \in E$,*

$$\limsup_{n\to+\infty} \frac{|S_n f(x)|}{\delta_n \log n} = +\infty.$$

The proof of Theorems 1.7 and 1.8 are proposed in Section 4.

2. Multifractal analysis of the divergence of Fourier series in $L^1(\mathbb{T})$

We first recall some basic facts on Fourier series and Fourier transforms.
A function $f \in L^1(\mathbb{T})$ is identified to a 1-periodic function on $\mathbb{R}$. Its Fourier coefficients are defined by

$$\hat{f}(k) = \langle f, e_k\rangle = \int_{\mathbb{T}} f(t)\bar{e_k}(t)dt = \int_{\mathbb{T}} f(t)e^{-2i\pi kt}dt.$$

The partial sums of its Fourier series are given by

$$S_n f \ : \ t \mapsto \sum_{k=-n}^{n} \langle f, e_k\rangle e_k(t).$$

We can also write $S_n f = D_n \star f$ where

$$D_n(t) = \sum_{k=-n}^{n} e_k(t) = \frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)}$$

is the Dirichlet kernel. The Riesz theorem says that the projections $(S_n)_{n\geq 1}$ are uniformly bounded on $L^p(\mathbb{T})$ when $p > 1$ (see [11], Vol. I, page 266). This is not yet the case on

$L^1(\mathbb{T})$. However, it is well known that the L^1 norm of the Dirichlet kernel is estimated by $\log n$. It follows that there exists some absolute constant $C>0$ such that, for any $n\geq 2$ and any $f\in L^1(\mathbb{T})$,

$$\|S_nf\|_1\leq C\log n\|f\|_1.$$

Let us also recall the definition of $\sigma_n f$, the n-th Féjer sum of f, namely

$$\sigma_n(f)=\frac{1}{n}\big(S_0f+\cdots+S_{n-1}f\big).$$

We write $\mathcal{E}_n(\mathbb{T}):=S_n(L^1(\mathbb{T}))$ the set of trigonometric polynomials of degree less than n. The classical Nikolsky inequality (see for example [9]) says that if $P\in\mathcal{E}_n(\mathbb{T})$ and $1\leq p\leq q\leq\infty$, then

$$\|P\|_q\leq n^{\frac{1}{p}-\frac{1}{q}}\|P\|_p. \tag{2}$$

Let also finally recall that if $f\in L^p(\mathbb{T})$ with $p\in[1,+\infty)$ and if q is the conjugate exponent (that is $\frac{1}{p}+\frac{1}{q}=1$), then

$$\|S_nf\|_\infty=\|D_n\star f\|_\infty\leq\|D_n\|_q\times\|f\|_p\leq C_pn^{1/p}\|f\|_p\ . \tag{3}$$

In the context of non periodic functions, the Fourier transform of a function $f\in L^1(\mathbb{R})$ is the continuous function

$$\hat{f}:\xi\in\mathbb{R}\mapsto\int_{\mathbb{R}}f(x)\bar{e_\xi}(x)dx,$$

where $e_\xi(t)=e^{2\pi i\xi t}$.

Our first lemma will be helpful to control a function which is locally a Dirichlet kernel.

Lemma 2.1. *There exists a constant $A>0$ such that, for any $N\geq 2$, for any measurable function $n:\mathbb{T}\to\{1,\dots,N\}$, for any $t\in\mathbb{T}$,*

$$\int_{\mathbb{T}}|D_{n(x)}(x-t)|dx\leq A\log N.$$

Proof. It is obvious from the above expression of D_n that, if $k\leq N$ and if $u\in[-1/2,1/2]$,

$$|D_k(u)|\leq\left\{\begin{array}{c}CN\\ \frac{C}{|u|}\end{array}\right.$$

for some absolute constant $C>0$. We then split the integral into two parts:

$$\int_{|x-t|\leq 1/N}|D_{n(x)}(x-t)|dx\leq 2CN\frac{1}{N}$$

and

$$\int_{1/N<|x-t|\leq 1/2}|D_{n(x)}(x-t)|dx\leq C\int_{1/N<|x-t|\leq 1/2}\frac{dx}{|x-t|}\leq 2C\log N.$$

□

Writing $S_{n(x)}f(x)=(f\star D_{n(x)})(x)$ and using Fubini's theorem, it is straightforward to deduce the following inequality on partial sums of Fourier series of L^1-functions.

Lemma 2.2. *There exists a constant $A>0$ such that, for any $N\geq 2$, for any measurable function $n:\mathbb{T}\to\{1,\dots,N\}$, for any $f\in L^1(\mathbb{T})$, then*

$$\int_{\mathbb{T}}|S_{n(x)}f(x)|dx\leq A\log N\,\|f\|_1.$$

We are now ready to prove the following weak version of the maximal inequality of Carleson and Hunt, on $L^1(\mathbb{T})$.

Corollary 2.3. *Let $\alpha>0$. There exists $C:=C_\alpha>0$ such that, for any $f\in L^1(\mathbb{T})$,*

$$\int_{\mathbb{T}}\sup_{n\geq 2}\frac{|S_nf(x)|}{(\log n)^{1+\alpha}}dx\leq C\,\|f\|_1.$$

Proof. Using the monotone convergence theorem, we first observe that it is sufficient to prove that, for any $N\geq 2$,

$$\int_{\mathbb{T}}\sup_{2\leq n\leq N}\frac{|S_nf(x)|}{(\log n)^{1+\alpha}}dx\leq C\,\|f\|_1 \tag{4}$$

where, of course, C does not depend on N. Now, we take a measurable function $n:\mathbb{T}\to\mathbb{N}\backslash\{0,1\}$ not necessarily bounded, and observe that (4) will be proved if we are able to show that

$$\int_{\mathbb{T}}\frac{|S_{n(x)}f(x)|}{(\log n(x))^{1+\alpha}}dx\leq C\,\|f\|_1$$

for some constant C independent of the function n. If $k\geq 0$, let

$$A_k=\{x\in\mathbb{T};\ 2^{2^k}\leq n(x)<2^{2^{k+1}}\}.$$

Lemma 2.2 ensures that

$$\begin{aligned}\int_{\mathbb{T}}\frac{|S_{n(x)}f(x)|}{(\log n(x))^{1+\alpha}}dx &= \sum_{k\geq 0}\int_{A_k}\frac{|S_{n(x)}f(x)|}{(\log n(x))^{1+\alpha}}dx\\ &\leq \sum_{k\geq 0}\frac{1}{(2^k\log 2)^{1+\alpha}}\int_{A_k}|S_{n(x)}f(x)|dx\\ &\leq \sum_{k\geq 0}C\frac{2^{k+1}\log 2}{2^{k(1+\alpha)}(\log 2)^{1+\alpha}}\,\|f\|_1\\ &= C_\alpha\,\|f\|_1.\end{aligned}$$

□

We then need the existence of a periodic function which is well-localized and with rapidly decreasing Fourier coefficients.

Lemma 2.4. *Let $\gamma\in(0,1)$. There exist $A,B>0$ such that, given any interval I with length less than 1, there exist a 1-periodic C^∞-function w_I with support in $I+\mathbb{Z}$ satisfying, $w_I(0)=1$, $0\leq w_I\leq 1$ and for any $\lambda\geq 2$,*

$$\sum_{|n|\geq\lambda|I|^{-1}}|\hat{w}_I(n)|\leq Ae^{-B\lambda^\gamma}.$$

Proof. We introduce a C^∞-function $w\ :\mathbb{R}\to\mathbb{R}$ with support in $[-1,1]$, satisfying $0\leq w\leq 1$, $w(0)=1$ and for which there exist two strictly positive constants D and E such that

$$\forall \xi\in\mathbb{R},\quad |\hat{w}(\xi)|\leq De^{-E|\xi|^\gamma}.$$

It is a classical result in Fourier analysis that such a function does exist (see e.g. [1, Lemma 6]). We then define the 1-periodic function w_I by

$$w_I(x)=\sum_{k\in\mathbb{Z}} w\left(\frac{x-k}{|I|}\right)\ .$$

A classical calculation says that the Fourier coefficients of the periodic function w_I are given by

$$\hat{w_I}(n)=|I|\hat{w}(n|I|)\ .$$

Then

$$\begin{aligned}\sum_{|n|\geq\lambda|I|^{-1}}|\hat{w_I}(n)| &\leq 2|I|\sum_{n\geq\lambda|I|^{-1}} De^{-En^\gamma|I|^\gamma}\\ &\leq 2D\int_{\lambda-1}^{+\infty} e^{-Et^\gamma}dt.\end{aligned}$$

Observe that

$$\int_X^{+\infty} e^{-Et^\gamma}dt=\frac{1}{\gamma}\int_{X^\gamma}^{+\infty} e^{-Eu}u^{(1/\gamma)-1}du\leq Ce^{-(E/2)X^\gamma}.$$

This easily yields the result. □

The following lemma is inspired by Aubry's paper. It means that, as soon as a trigonometric polynomial is large at some point $a\in\mathbb{T}$, it is also large in small intervals around a, with a rather good control of the L^p-norm.

Lemma 2.5. *Let $p\geq 1$ and $\varepsilon>0$. There exists $\delta>0$ such that, if n is large enough, if $P\in\mathcal{E}_n(\mathbb{T})$ and if $a\in\mathbb{T}$ is such that $|P(a)|\geq\|P\|_p$, then, for any interval I with center a and with length $|I|\leq\frac{1}{n}$,*

$$\|P\|_{L^p(I)}\geq\delta|P(a)|\times|I|^{1/p}\times(\log n)^{-(1+\varepsilon)/p}\ .$$

Remarks:
- Such a point a does exist because P is continuous.
- In the context of the L^∞-norm, if a is a point such that $|P(a)|\geq\|P\|_\infty$, Bernstein's inequality says that if $x\in I$, then

$$|P(x)|\geq|P(a)|-n\|P\|_\infty\,|x-a|\geq\frac{1}{2}|P(a)|\ .$$

In Lemma 2.5, we try to do the same in the context of the L^p-norm.
- In fact, we will only need the lemma in the case $p=1$, but we give the general case for completeness.

Proof of Lemma 2.5. Without loss of generality, we may assume that $a=0$. The idea is to localize P around $0+\mathbb{Z}$, and to use Nikolsky inequality to estimate the L^p-norm knowing the L^∞-norm. Let $\gamma\in(0,1)$ such that $\gamma(1+\varepsilon)>1$. Let w_I be given by Lemma 2.4 for this value of γ. We decompose Pw_I as f_1+f_2 with $f_1=S_NPw_I$ and $N=[|I|^{-1}(\log n)^{1+\varepsilon}]$, the integer part of $|I|^{-1}(\log n)^{1+\varepsilon}$. On the one hand, if $p\geq 1$ we get

$$\begin{aligned}\|f_1\|_\infty &\leq C_p|I|^{-1/p}(\log n)^{(1+\varepsilon)/p}\|Pw_I\|_p \text{ (Inequality (3))}\\ &\leq C_p|I|^{-1/p}(\log n)^{(1+\varepsilon)/p}\|P\|_{L^p(I)} \ (0\leq w_I\leq 1).\end{aligned}$$

On the other hand, we may write

$$f_2(x)=\sum_{|k|>N}\widehat{Pw_I}(k)e^{2i\pi kx}$$

and we need to estimate the Fourier coefficients of Pw_I in order to obtain an upper bound for $\|f_2\|_\infty$. We know that

$$\widehat{Pw_I}(k)=\hat{P}\star\hat{w}_I(k)=\sum_{j=-n}^{n}\hat{P}(j)\hat{w}_I(k-j),$$

so that

$$\|f_2\|_\infty\leq\sum_{j=-n}^{n}|\hat{P}(j)|\sum_{|k|>N}|\hat{w}_I(k-j)|\ .$$

Now, if n is large enough and $|j|\leq n$, we have since $|I|\leq\frac{1}{n}$

$$\begin{aligned}\sum_{|k|>N}|\hat{w}_I(k-j)| &\leq \sum_{|k|>|I|^{-1}(\log n)^{1+\varepsilon}/2}\hat{w}_I(k)\\ &\leq Ae^{-B(\log n)^{1+\varepsilon}/2}\\ &\leq Cn^{-2}.\end{aligned}$$

This implies

$$\begin{aligned}\|f_2\|_\infty &\leq Cn^{-2}\sum_{j=-n}^{n}|\hat{P}(j)|\\ &\leq Cn^{-2}(2n+1)\|P\|_1\\ &\leq Cn^{-2}(2n+1)\|P\|_p\\ &\leq \frac{1}{2}\|P\|_p\end{aligned}$$

provided n is large enough. If we recall that $|P(0)|\geq\|P\|_p$ and $P(0)=f_1(0)+f_2(0)$, we get

$$\|f_1\|_\infty\geq|P(0)|-\|f_2\|_\infty\geq\frac{1}{2}|P(0)|$$

and the result follows from the above estimates of $\|f_1\|_\infty$. □

We can now conclude by proving the following proposition (Proposition 2.6) and its corollary on the Hausdorff dimension of $E(\beta,f)$ (Corollary 2.7). Recall that it is all that we need to obtain Theorem 1.4 since the construction done in [2] is always true when $p=1$ and shows that there exists a residual set of functions $f \in L^1(\mathbb{T})$ with $\dim_{\mathcal{H}}(E(\beta, f)) \geq 1 - \beta$ for any $\beta \in [0,1]$.

Proposition 2.6. *Let $f \in L^1(\mathbb{T})$ and $\tau : (0,+\infty) \to (0,+\infty)$ be an increasing function. Define*

$$E(\tau, f) := \left\{ x \in \mathbb{T};\ \limsup_{n\to+\infty} \frac{|S_n f(x)|}{\tau(n)} = +\infty \right\}.$$

If $\nu > 2$ and if ϕ is a dimension function satisfying $c_1 s \leq \phi(s) \leq c_2 \frac{s\tau(s^{-1})}{\log(s^{-1})^\nu}$, then

$$\mathcal{H}^\phi(E(\tau, f)) = 0.$$

Proof. Let $M > 0$ and $\varepsilon = \nu - 2$. Define

$$E_M(\tau, f) = \left\{ x \in \mathbb{T};\ \limsup_{n\to+\infty} \frac{|S_n f(x)|}{\tau(n)} > M \right\}.$$

If $x \in E_M(\tau, f)$, one can find n_x as large as we want such that $|S_{n_x} f(x)| \geq M\tau(n_x)$. Set $I_x = \left[x - \frac{1}{2n_x}, x + \frac{1}{2n_x}\right]$ and observe that $\|S_{n_x} f\|_1 \leq C(\log n_x)$. The hypothesis on the function τ implies that $\tau(n) >> \log n$. It follows that $\|S_{n_x} f\|_1 \leq |S_{n_x} f(x)|$ if n_x is large enough. We can then apply Lemma 2.5 and we get

$$\|S_{n_x} f\|_{L^1(I_x)} \geq \delta \frac{M\tau(n_x)}{n_x(\log n_x)^{1+\varepsilon/2}}.$$

$(I_x)_{x\in E_M(\tau,f)}$ is a covering of $E_M(\tau, f)$. We can extract a Vitali's covering, namely a countable family of disjoint intervals $(I_i)_{i\in\mathbb{N}}$, of length $1/n_i$, such that $E_M(\tau, f) \subset \bigcup_{i\in\mathbb{N}} 5B_i$. Then, Corollary 2.3 implies

$$\begin{aligned}
C\|f\|_1 &\geq \int_{\mathbb{T}} \sup_{n\geq 2} \frac{|S_n f(x)|}{(\log n)^{1+\varepsilon/2}} dx \\
&\geq \sum_i \int_{I_i} \frac{|S_{n_i} f(x)|}{(\log n_i)^{1+\varepsilon/2}} dx \\
&\geq \delta M \sum_i \frac{|I_i|\tau(1/|I_i|)}{\big(\log(1/|I_i|)\big)^{2+\varepsilon}}.
\end{aligned}$$

This yields $\sum_i \phi(5|I_i|) \leq \frac{C\|f\|_1}{\delta M}$ (we recall that τ is increasing), with C another absolute constant and $M > 0$ as large as we want. Hence, $\mathcal{H}^\phi(E_M(\tau, f)) \leq \frac{C\|f\|_1}{\delta M}$ (the length of the intervals of the covering can be arbitrarily small). This in turn implies $\mathcal{H}^\phi(E(\tau, f)) = 0$, since $E(\tau, f) = \bigcap_{M>0} E_M(\tau, f)$. □

By applying the previous proposition to $\tau(s) = s^\beta$ and $\phi(s) = s^{1-\beta}/\log(s^{-1})^3$, we get:

Corollary 2.7. *For any $f \in L^1(\mathbb{T})$ and any $\beta \in [0,1]$, $\dim_{\mathcal{H}}(E(\beta, f)) \leq 1 - \beta$.*

3. Prevalence of multifractal behaviour

3.1. **Strategy.** In all this part, p is a fixed real number such that $1 \leq p < +\infty$. To prove that a set $A \subset E$ is Haar-null, the Lebesgue measure on the unit ball of a finite-dimensional subspace V can often play the role of the transverse measure. Precisely, if there exists a finite-dimensional subspace V of E such that, for any $x \in E$, $V \cap (x + A)$

has full Lebesgue-measure, then A is prevalent. Such a finite-dimensional subspace V is called a *probe* for A. Of course, it is the same to prove that for any $x \in E$, $(x+V)\cap A$ has full Lebesgue-measure.

We shall use this property to prove prevalence. More precisely, we shall first prove that, for a fixed $\beta\in[0,1/p]$, the set of functions f in $L^p(\mathbb{T})$ satisfying $\dim_{\mathcal{H}}\big(E(\beta,f)\big)=1-\beta p$ is prevalent. Then we will conclude because a countable intersection of prevalent sets is prevalent.

3.2. **The construction of saturating functions with disjoint spectra.** In this subsection, $\alpha>1$ is fixed. For $j\geq 1$, we define $J=[j/\alpha]+1$, which is smaller than $j-2$ if j is large enough, say $j\geq j_\alpha$. For $0\leq K\leq 2^J-1$, we define the dyadic intervals

$$I_{K,j}:=\left[\frac{K}{2^J}-\frac{1}{2^j};\frac{K}{2^J}+\frac{1}{2^j}\right].$$

We also define

$$\mathbf{I}_j:=\bigcup_{K=0}^{2^J-1}I_{K,j}\quad\text{ and }\quad \mathbf{I}'_j:=\bigcup_{K=0}^{2^J-1}2I_{K,j}.$$

The condition $j\geq j_\alpha$ ensures that the $2I_{K,j}$ do not overlap. We finally introduce D_α the set of real numbers in $[0,1]$ which are α-approximable by dyadics. Namely, $x\in[0,1]$ belongs to D_α if there exist two sequences of integers $(k_n)_{n\geq 0}$ and $(j_n)_{n\geq 0}$ such that

$$\left|x-\frac{k_n}{2^{j_n}}\right|\leq\frac{1}{2^{\alpha j_n}}.$$

It is easy to check that D_α is contained in $\limsup_{j\to+\infty}\mathbf{I}_j$. Indeed, let $x\in D_\alpha$. One may find J as large as we want and K such that $|x-K/2^J|\leq 1/2^{\alpha J}$. Let j be an integer such that $J-1=[j/\alpha]$ (such an integer exists because $\alpha\geq 1$). We get

$$\left|x-\frac{K}{2^J}\right|\leq\frac{1}{2^j}.$$

Finally, $x\in\mathbf{I}_j$. Furthermore, it is well-known that $\dim_{\mathcal{H}}(D_\alpha)=1/\alpha$ and even that $\mathcal{H}^{1/\alpha}(D_\alpha)=+\infty$ (see for instance [3] and the mass transference principle). It follows that

$$\dim_{\mathcal{H}}\left(\limsup_{j\to+\infty}\mathbf{I}_j\right)\geq\frac{1}{\alpha}.$$

We are going to build finite families of functions which behave badly on each $\mathbf{I}_j$, and which have disjoint spectra. The starting point is a modification of the basic construction of [2].

Lemma 3.1. *Let $j\geq j_\alpha$ and $J=[j/\alpha]+1$. There exists a trigonometric polynomial P_j with spectrum contained in $(0,2^{j+1}-1]$ such that*

- $\|P_j\|_p\leq 1$
- $|P_j(x)|\geq C2^{-(J-j)/p}$ *for any $x\in\mathbf{I}_j$*

where the constant C is independant of j.

Proof. Let χ_j be a continuous piecewise linear function equal to 1 on $\mathbf{I}_j$, equal to 0 outside $\mathbf{I}'_j$ and satisfying $0 \leq \chi_j \leq 1$ and $\|\chi'_j\|_\infty \leq 2^j$. P_j is defined by

$$P_j := 2^{-(J-j+2)/p} e_{2^j} \sigma_{2^j} \chi_j.$$

The L^p-norm of P_j is clearly less than or equal to 1 (observe that the measure of $\mathbf{I}'_J$ is 2^{J-j+2}). Applying Lemma 1.7 of [2] to $1-\chi_j$, we find that $\sigma_{2^j}\chi_j(x) \geq 1/4$ for any $x \in \mathbf{I}_j$. This gives the second assertion of the lemma. □

We now collapse these polynomials to get as many saturating functions as necessary, with disjoint spectra.

Lemma 3.2. *Let $s \geq 1$. There exist functions $g_1, \dots, g_s$ in $L^p(\mathbb{T})$, two sequences of integers $(n_{j,r})_{j \geq j_\alpha, 1 \leq r \leq s}$, $(m_{j,r})_{j \geq j_\alpha, 1 \leq r \leq s}$ and a constant $C > 0$ satisfying*

- $1 \leq m_{j,r} < n_{j,r} \leq C2^j$ *for any j and any r;*
- *for any $j \geq j_\alpha$, any $x \in \mathbf{I}_j$, any $r \in \{1, \dots, s\}$,*
$$\left|S_{n_{j,r}} g_r(x) - S_{m_{j,r}} g_r(x)\right| \geq \frac{C}{j^2} 2^{(j-J)/p}$$
- *for any $r \in \{1, \dots, s\}$, the spectrum of g_r is included in $\bigcup_{j \geq j_\alpha} (m_{j,r}, n_{j,r}] =: G_r$, where the intervals $(m_{j,r}, n_{j,r}]$ are disjoint.*
- *if $r_1 \neq r_2$, $G_{r_1} \cap G_{r_2} = \emptyset$.*

Proof. For $r \in \{1, \dots, s\}$, we set

$$g_r := \sum_{j \geq j_\alpha} \frac{1}{j^2} e_{(s+r)2^{j+1}} P_j.$$

Define

$$\begin{aligned} m_{j,r} &:= (s+r)2^{j+1} \\ n_{j,r} &:= (s+r)2^{j+1} + (2^{j+1} - 1) \end{aligned}$$

so that each g_r belongs to L^p with spectrum included in $\bigcup_{j \geq j_\alpha} (m_{j,r}, n_{j,r}]$. Moreover, the intervals $(m_{j,r}, n_{j,r}]$ are disjoint, so that

$$\left|S_{n_{j,r}} g_r - S_{m_{j,r}} g_r\right| = \frac{1}{j^2} |P_j|.$$

Let us also remark that, for any $j \geq j_\alpha$ and any $r < s$, $n_{j,r} < m_{j,r+1}$ and $n_{j,s} < m_{j+1,1}$ so that the spectra $G_1, \cdots, G_s$ are disjoint. This ends up the proof. □

It is easy to show that, if $x \in \limsup_j \mathbf{I}_j$, $r \in \{1, \dots, s\}$ and $\beta < \frac{1}{p}\left(1 - \frac{1}{\alpha}\right)$, then

$$\limsup_{n \to +\infty} \frac{|S_n g_r(x)|}{n^\beta} = +\infty.$$

In some sense, the functions g_r have the worst possible behaviour on $\mathbf{I}_j$ if we keep in mind that they have to belong to $L^p(\mathbb{T})$. We now show that this property remains true almost everywhere (in the sense of the Lebesgue measure) on any affine subspace $f + \operatorname{span}(g_1, \dots, g_s)$ provided s is large enough. This is the main step towards the proof of Theorem 1.6.

3.3. **Prevalence of divergence for a fixed divergence index.** We keep the notations of the previous subsection.

Proposition 3.3. *Let* $0 < \beta < \frac{1}{p}\left(1-\frac{1}{\alpha}\right)$. *There exists* $s \geq 1$ *such that, for every* $f \in L^p(\mathbb{T})$, *for almost every* $c = (c_1, \dots, c_s)$ *in* $\mathbb{R}^s$, *the function* $g = f + c_1 g_1 + \cdots + c_s g_s$ *satisfies for every* $x \in D_\alpha$

$$\limsup_{n\to+\infty} \frac{|S_n g(x)|}{n^\beta} = +\infty.$$

Proof. We set $\varepsilon = \frac{1}{p}\left(1-\frac{1}{\alpha}\right) - \beta$. Let $s > 4/\varepsilon$ and let f be an arbitrary function in $L^p(\mathbb{T})$. For such a value of s, we will prove the conclusion of the proposition for every $x \in \limsup_j \mathbf{I}_j$ (recall that $D_\alpha \subset \limsup_j \mathbf{I}_j$).

Let $M > 0$ and let us introduce

$$S_M := \left\{ g \in L^p(\mathbb{T});\ \exists x \in \limsup_{j\to+\infty} \mathbf{I}_j \text{ s.t. } \forall n \geq 1,\ |S_n g(x)| \leq M n^\beta \right\}.$$

It is enough to show that for every $R > 0$, the set of $c \in \mathbb{R}^s$ satisfying $\|c\|_\infty \leq R$ and such that $f + c_1 g_1 + \cdots + c_s g_s$ belongs to S_M has Lebesgue measure 0. In the sequel, we will fix such values of M and R.

If $j \geq 1$, we split each interval $I_{K,j}$ into 2^j subintervals. Each of them has size 2^{-2j+1}, and we get 2^{J+j} intervals $O_{l,j}$ with $\bigcup_{l=1}^{2^{J+j}} O_{l,j} = \mathbf{I}_j$. For $j \geq 1$, $l \in \{1, \dots, 2^{J+j}\}$, we set

$$S_M^{(l,j)} := \left\{ g \in L^p(\mathbb{T});\ \exists x \in O_{l,j} \text{ s.t. } \forall n \geq 1,\ |S_n g(x)| \leq M n^\beta \right\}.$$

Clearly,

$$S_M \subset \limsup_{j\to+\infty} \bigcup_{l=1}^{2^{J+j}} S_M^{(l,j)}$$

and we shall first control the size of the $c \in \mathbb{R}^s$ with $\|c\|_\infty \leq R$ such that

$$f + c_1 g_1 + \cdots + c_s g_s \in S_M^{(l,j)}.$$

We denote by λ_s the Lebesgue measure on $\mathbb{R}^s$ and we fix $j \geq j_\alpha$, l in $\{1, \dots, 2^{J+j}\}$ and c, c^0 in $\mathbb{R}^s$ such that $\|c\|_\infty \leq R$, $\|c^0\|_\infty \leq R$ and

$$\begin{cases} f + c_1 g_1 + \cdots + c_s g_s & \in & S_M^{(l,j)} \\ f + c_1^0 g_1 + \cdots + c_s^0 g_s & \in & S_M^{(l,j)}. \end{cases}$$

The goal is to find an upper bound for $\|c - c_0\|_\infty$.

Let $r \in \{1, \dots, s\}$ and let us apply the definition of $S_M^{(l,j)}$ with $n = n_{j,r}$ and $n = m_{j,r}$. The spectra $(G_l)_{l\neq r}$ being disjoint from G_r, we can find $x \in O_{l,j}$ such that

$$\left| S_{n_{j,r}} f(x) - S_{m_{j,r}} f(x) + c_r\big(S_{n_{j,r}} g_r(x) - S_{m_{j,r}} g_r(x)\big) \right| \leq M n_{j,r}^\beta + M m_{j,r}^\beta \leq 2CM2^{\beta j}.$$

In the same way, we can find $y \in O_{l,j}$ such that

$$\left| S_{n_{j,r}} f(y) - S_{m_{j,r}} f(y) + c_r^0\big(S_{n_{j,r}} g_r(y) - S_{m_{j,r}} g_r(y)\big) \right| \leq 2CM2^{\beta j}.$$

Using the triangle inequality, we get

$$\begin{aligned} \left|c_r\big(S_{n_{j,r}}g_r(x)-S_{m_{j,r}}g_r(x)\big)-c_r^0\big(S_{n_{j,r}}g_r(y)-S_{m_{j,r}}g_r(y)\big)\right| &\leq \\ 4CM2^{\beta j}+|S_{n_{j,r}}f(x)-S_{n_{j,r}}f(y)|+|S_{m_{j,r}}f(x)-S_{m_{j,r}}f(y)|. \end{aligned} \tag{5}$$

Now, by combining the norm of the Riesz projection, Nikolsky's inequality and Bernstein's inequality, we know that

$$\|(S_nf)'\|_\infty \leq C(\log n)n^{1+1/p}\|f\|_p$$

(the factor $\log n$ disappears when $p>1$). This yields

$$\begin{aligned} |S_{n_{j,r}}f(x)-S_{n_{j,r}}f(y)| &\leq C\log(n_{j,r})n_{j,r}^{1+1/p}|x-y|\|f\|_p \\ &\leq Cj2^{j(1+1/p)}2^{-2j+1}\|f\|_p \\ &\ll 2^{\beta j}. \end{aligned}$$

The same is true for $|S_{m_{j,r}}f(x)-S_{m_{j,r}}f(y)|$ and we get

$$\left|c_r\big(S_{n_{j,r}}g_r(x)-S_{m_{j,r}}g_r(x)\big)-c_r^0\big(S_{n_{j,r}}g_r(y)-S_{m_{j,r}}g_r(y)\big)\right|\leq\kappa 2^{\beta j} \tag{6}$$

for some constant κ depending on M and $\|f\|_p$ but not on j.
In the same way,

$$\|(S_ng_r)'\|_\infty \leq C(\log n)n^{1+1/p}\|g_r\|_p \leq C(\log n)n^{1+1/p}.$$

It follows that

$$\begin{aligned} \left|c_r^0\left(\big(S_{n_{j,r}}g_r(x)-S_{m_{j,r}}g_r(x)\big)-\big(S_{n_{j,r}}g_r(y)-S_{m_{j,r}}g_r(y)\big)\right)\right| &\leq CRj2^{j(1+1/p)}2^{-2j+1} \\ &\ll 2^{\beta j}. \end{aligned}$$

Combining with (6) we obtain a new constant κ depending on M, $\|f\|_p$ and R but not on j such that

$$\left|(c_r-c_r^0)\big(S_{n_{j,r}}g_r(x)-S_{m_{j,r}}g_r(x)\big)\right|\leq\kappa 2^{\beta j}. \tag{7}$$

Dividing (7) by $|S_{n_{j,r}}g_r(x)-S_{m_{j,r}}g_r(x)|$ (which is not equal to zero), we find

$$\begin{aligned} |c_r-c_r^0| &\leq \kappa 2^{\beta j}|S_{n_{j,r}}g_r(x)-S_{m_{j,r}}g_r(x)|^{-1} \\ &\leq \frac{\kappa}{C}2^{\beta j}j^2 2^{-(j-J)/p} \\ &\leq \frac{\kappa 2^{1/p}}{C}j^2 2^{-\varepsilon j} \\ &\leq 2^{-\varepsilon j/2} \end{aligned}$$

provided j is large enough. Thus, the set of $c\in\mathbb{R}^s$ with $\|c\|_\infty\leq R$ and such that $f+c_1g_1+\cdots+c_sg_s\in S_M^{(l,j)}$ is contained in a ball (for the l^∞-norm) of radius $2^{-\varepsilon j/2}$. Taking the s-dimensional Lebesgue measure, this yields

$$\lambda_s\left(\left\{c\in\mathbb{R}^s;\ \|c\|_\infty\leq R \text{ and } f+c_1g_1+\cdots+c_sg_s\in S_M^{(l,j)}\right\}\right)\leq 2^s 2^{-\varepsilon sj/2}.$$

This in turn gives

$$\lambda_s\left(\left\{c\in\mathbb{R}^s;\ \|c\|_\infty\leq R \text{ and } f+c_1g_1+\cdots+c_sg_s\in\bigcup_{l=1}^{2^{J+j}}S_M^{(l,j)}\right\}\right)\leq 2^s 2^{2j-\varepsilon sj/2}.$$

Thus, since $\varepsilon s/2>2$, this last quantity is the general term of a convergent series. Recall that

$$S_M\subset\limsup_{j\to+\infty}\bigcup_{l=1}^{2^{J+j}}S_M^{(l,j)}.$$

The conclusion of Proposition 3.3 follows from Borel-Cantelli's lemma. □

Corollary 3.4. *Let $\alpha>1$. For almost every function f in $L^p(\mathbb{T})$, for every $x\in D_\alpha$,*

$$\limsup_{n\to+\infty}\frac{\log|S_nf(x)|}{\log n}\geq\frac{1}{p}\left(1-\frac{1}{\alpha}\right).$$

Proof. This follows immediately from Proposition 3.3, if one takes a sequence (β_n) increasing to $\frac{1}{p}\left(1-\frac{1}{\alpha}\right)$ and using the fact that a countable intersection of prevalent sets remains prevalent. □

3.4. **The general case.** We are now able to complete the proof of Theorem 1.6, that is to prove that almost every function $f\in L^p(\mathbb{T})$ in the sense of prevalence has a multifractal behaviour with respect to the summation of its Fourier series. Indeed, let $(\alpha_k)_{k\geq 0}$ be a dense sequence in $(1,+\infty)$. By Corollary 3.4, for almost every function $f\in L^p(\mathbb{T})$, for every $k\in\mathbb{N}$ and every $x\in D_{\alpha_k}$,

$$\limsup_{n\to+\infty}\frac{\log|S_nf(x)|}{\log n}\geq\frac{1}{p}\left(1-\frac{1}{\alpha_k}\right).$$

Now, let $\alpha>1$ and consider a subsequence $(\alpha_{\phi(k)})_{k\geq 0}$ which increases to α. Then $D_\alpha\subset\bigcap_{k\geq 0}D_{\alpha_{\phi(k)}}$ and for any $x\in D_\alpha$,

$$\limsup_{n\to+\infty}\frac{\log|S_nf(x)|}{\log n}\geq\frac{1}{p}\left(1-\frac{1}{\alpha}\right).$$

The conclusion follows now exactly the argument of [2]. For the sake of completeness, we give a complete account. Define

$$\begin{aligned}D_\alpha^1 &= \left\{x\in D_\alpha;\ \limsup_{n\to+\infty}\frac{\log|S_nf(x)|}{\log n}=\frac{1}{p}\left(1-\frac{1}{\alpha}\right)\right\}\\ D_\alpha^2 &= \left\{x\in D_\alpha;\ \limsup_{n\to+\infty}\frac{\log|S_nf(x)|}{\log n}>\frac{1}{p}\left(1-\frac{1}{\alpha}\right)\right\},\end{aligned}$$

so that $\mathcal{H}^{1/\alpha}(D_\alpha^1\cup D_\alpha^2)=\mathcal{H}^{1/\alpha}(D_\alpha)=+\infty$. It suffices to prove that $\mathcal{H}^{1/\alpha}(D_\alpha^2)=0$. Let $(\beta_n)_{n\geq 0}$ be a sequence of real numbers such that

$$\beta_n>\frac{1}{p}\left(1-\frac{1}{\alpha}\right)\quad\text{and}\quad\lim_{n\to+\infty}\beta_n=\frac{1}{p}\left(1-\frac{1}{\alpha}\right).$$

Let us observe that

$$D_\alpha^2\subset\bigcup_{n\geq 0}\mathcal{E}(\beta_n,f).$$

Moreover, Theorem 1.1 for $p>1$ and Corollary 2.7 for $p=1$ imply that $\mathcal{H}^{1/\alpha}(\mathcal{E}(\beta_n,f))=0$ for all n. Hence, $\mathcal{H}^{1/\alpha}(D_\alpha^2)=0$ and $\mathcal{H}^{1/\alpha}(D_\alpha^1)=+\infty$, which proves that

$$\dim_{\mathcal{H}}\left(E\left(\frac{1}{p}\left(1-\frac{1}{\alpha}\right),f\right)\right)\geq\frac{1}{\alpha}.$$

By Theorem 1.1 and Corollary 2.7 again, this inequality is necessarily an equality. Finally, such a function f satisfies the conclusion of Theorem 1.6, setting $1-\beta p=1/\alpha$.

4. Rapid divergence on big sets for Fourier series of continuous functions

This section is devoted to the proof of Theorem 1.7 and Theorem 1.8. We need to construct functions in $\mathcal{C}(\mathbb{T})$ for which the Fourier series behave badly on a set with Hausdorff dimension 1. We will construct these functions by blocks. For $k\geq 1$ and $\omega>1$, we set

$$J_k^\omega:=\bigcup_{j=0}^{k-1}\left[\frac{j}{k}-\frac{1}{2\omega k},\frac{j}{k}+\frac{1}{2\omega k}\right]$$

which will be seen as a subset of $\mathbb{T}=\mathbb{R}/\mathbb{Z}$. The construction makes use of holomorphic functions, so that we will also see $\mathbb{T}$ as the boundary of the unit disk $\mathbb{D}$ and J_k^ω as a part of $\partial\mathbb{D}$.

Lemma 4.1. *There exist three absolute constants $C_1,C_2,C_3>0$ such that, for any $k\geq 3$, for any $\omega\geq\log k$, one can find a function f which is holomorphic in a neighbourhood of $\mathbb{D}$ and which satisfies :*

$$\forall z\in\overline{\mathbb{D}},\quad \Re e f(z)\ \geq\ \frac{C_1}{\omega k} \tag{8}$$

$$\forall z\in J_k^\omega,\quad |f(z)|\ \geq\ C_2\omega \tag{9}$$

$$\forall z\in\mathbb{T},\quad |f(z)|\ \leq\ C_3\omega \tag{10}$$

$$\forall z\in\mathbb{T},\quad \left|\frac{f'(z)}{f(z)}\right|\ \leq\ \omega k. \tag{11}$$

Proof. We set:

$$\begin{aligned}\varepsilon &= \frac{1}{\omega k}\\ z_j &= e^{\frac{2\pi ij}{k}},\ j=0,\dots,k-1\\ f(z) &= \frac{1}{k}\sum_{j=0}^{k-1}\frac{1+\varepsilon}{1+\varepsilon-\overline{z_j}z}\end{aligned}$$

and we claim that f is the function we are looking for. Indeed, for any $z\in\overline{\mathbb{D}}$ and any $j\in\{0,\dots,k-1\}$,

$$\Re e\left(\frac{1+\varepsilon}{1+\varepsilon-\overline{z_j}z}\right)=\frac{1+\varepsilon}{|1+\varepsilon-\overline{z_j}z|^2}\Re e\big(1+\varepsilon-z_j\overline{z}\big)\geq\frac{1+\varepsilon}{(2+\varepsilon)^2}\times\varepsilon\geq C_1\varepsilon, \tag{12}$$

which proves (8). To prove (9), we may assume that $z=e^{2\pi i\theta}$ with $\theta\in\left[\frac{-\varepsilon}{2};\frac{\varepsilon}{2}\right]$. Then

$$\Re e\left(\frac{1+\varepsilon}{1+\varepsilon-\overline{z_0}z}\right)=\frac{1+\varepsilon}{|1+\varepsilon-z|^2}\Re e\big(1+\varepsilon-z\big)\geq\frac{C_2}{\varepsilon}.$$

Moreover, (12) says that for any j, $\Re e\left(\frac{1+\varepsilon}{1+\varepsilon-\overline{z_j}z}\right)\geq 0$. It follows that

$$\Re e f(z)\geq \frac{C_2}{k\varepsilon}=C_2\omega.$$

Conversely, we want to control $\sup_{z\in\mathbb{T}}|f(z)|$. Pick any $z=e^{2\pi i\theta}\in\mathbb{T}$. By symmetry, we may and shall assume that $|\theta|\leq\frac{1}{2k}$. Then we get

$$\left|\frac{1+\varepsilon}{1+\varepsilon-\overline{z_0}z}\right|\leq\frac{C}{\varepsilon}$$

for some constant $C>0$. Now, for any $j\in\{1,\dots,k/4\}$, we can write

$$\begin{aligned}|1+\varepsilon-\overline{z_j}z| &\geq |\Im m(\overline{z_j}z)|\\ &\geq \sin\left(\frac{2\pi j}{k}-2\pi\theta\right)\\ &\geq \frac{2}{\pi}\times 2\pi\left(\frac{j}{k}-\theta\right)\\ &\geq \frac{4}{k}\left(j-\frac{1}{2}\right).\end{aligned}$$

Taking the sum,

$$\left|\sum_{j=1}^{k/4}\frac{1+\varepsilon}{1+\varepsilon-\overline{z_j}z}\right|\leq\frac{k(1+\varepsilon)}{4}\sum_{j=1}^{k/4}\frac{1}{j-1/2}\leq Ck\log k$$

(the constant C may change from line to line). In the same way, we have

$$\left|\sum_{j=3k/4}^{k-1}\frac{1+\varepsilon}{1+\varepsilon-\overline{z_j}z}\right|\leq Ck\log k.$$

If $j\in[k/4,3k/4]$, we also have $|1+\varepsilon-\overline{z_j}z|\geq C$, so that

$$\left|\sum_{j=k/4}^{3k/4}\frac{1+\varepsilon}{1+\varepsilon-\overline{z_j}z}\right|\leq Ck.$$

Putting this together, we get

$$|f(z)|=\left|\frac{1}{k}\sum_{j=0}^{k-1}\frac{1+\varepsilon}{1+\varepsilon-\overline{z_j}z}\right|\leq C\left(\frac{1}{k\varepsilon}+\log k+1\right)\leq C_3\omega$$

(this is the place where we need that $\omega\geq\log k$). Finally, it remains to prove (11). We observe that

$$\frac{f'(z)}{f(z)}=\frac{\sum_{j=0}^{k-1}\frac{\overline{z_j}}{(1+\varepsilon-\overline{z_j}z)^2}}{\sum_{j=0}^{k-1}\frac{1}{1+\varepsilon-\overline{z_j}z}}.$$

We deduce that

$$\begin{aligned}\left|\frac{f'(z)}{f(z)}\right| &\leq \frac{\sum_{j=0}^{k-1}\frac{1}{|1+\varepsilon-\overline{z_j}z|^2}}{\sum_{j=0}^{k-1}\frac{\Re e(1+\varepsilon-z_j\bar{z})}{|1+\varepsilon-\overline{z_j}z|^2}}\\ &\leq \frac{\sum_{j=0}^{k-1}\frac{1}{|1+\varepsilon-\overline{z_j}z|^2}}{\sum_{j=0}^{k-1}\frac{\varepsilon}{|1+\varepsilon-\overline{z_j}z|^2}}\\ &\leq \frac{1}{\varepsilon}=\omega k.\end{aligned}$$

□

The crucial step is given by the following lemma.

Lemma 4.2. *Let $(\varepsilon_n)_{n\geq 1}$ be a sequence of positive real numbers decreasing to zero. Then, if n is large enough, one can find an integer k_n, a real number $\omega_n>1$ and a trigonometric polynomial P_n with spectrum in $[1,2n-1]$ such that*

- $\|P_n\|_\infty\leq 1$*;*
- *For any $x\in J_{k_n}^{\omega_n}$, $|S_nP_n(x)|\geq \varepsilon_n\log(n)$.*

Moreover, we can choose k_n and ω_n such that (k_n) goes to $+\infty$ and $\omega_n=o(k_n^\alpha)$ for any $\alpha>0$.

Proof. It is clear that the conclusion of the lemma is more difficult to obtain when the sequence (ε_n) is large. Thus, we may assume that

$$\varepsilon_n\geq\frac{\log\log n}{4\pi\log n}.$$

In particular, $\varepsilon_n\log n$ goes to infinity. We define k_n and ω_n by

- $\omega_n=\exp\big(4\pi(\log n)\varepsilon_n\big)$
- k_n is the biggest integer k satisfying

$$2\pi k\omega_n\leq n.$$

Observe that $\omega_n\geq\log n$ and $\omega_n=o(n^\alpha)$ for all $\alpha>0$. Then, the inequalities

$$2\pi k_n\omega_n\leq n\leq 2\pi(k_n+1)\omega_n$$

ensure that

$$k_n\leq n\leq Ck_nn^{1/2}$$

if n is large enough. It follows that (k_n) goes to $+\infty$, that $\omega_n\geq\log k_n$ and that $\omega_n=o(k_n^\alpha)$ for any $\alpha>0$.
Let f_n be the holomorphic function given by Lemma 4.1 for the values $k=k_n$ and $\omega=\omega_n$. We take $h_n(z)=\log(f_n(z))$, which defines a holomorphic function in a neighbourhood of $\overline{\mathbb{D}}$ (see (8) in Lemma 4.1). Moreover, $|\Im m(h_n(z))|\leq\pi/2$ for any $z\in\overline{\mathbb{D}}$ and $h_n(0)=0$. Now, we look at the function h_n on the boundary of the unit disk $\mathbb{D}$, that is we introduce

the function $g_n(x)=h_n(e^{2i\pi x})$ defined on the circle $\mathbb{T}=\mathbb{R}/\mathbb{Z}$. The properties satisfied by f_n translate into

$$\begin{array}{rrcl}\forall x\in J_{k_n}^{\omega_n}, & |g_n(x)| & \geq & \log\omega_n+\log C_2\\ \forall x\in\mathbb{T}, & |g_n(x)| & \leq & \log\omega_n+\log C_3\\ \forall x\in\mathbb{T}, & |g_n'(x)| & \leq & 2\pi k_n\omega_n\leq n.\end{array}$$

We apply Lemma 1.7 of [2], which is a precised version of Féjer's theorem, to the function $\theta_x(t)=g_n(t)-g_n(x)$ for $x\in\mathbb{T}$. Since $\|\theta_x\|_\infty\leq 2\log\omega_n+2\log C_3$, $\|\theta_x'\|_\infty\leq n$ and $\theta_x(x)=0$, we get

$$|\sigma_n\theta_x(x)|\leq\frac{1}{2}\log\omega_n+C_4$$

for some absolute constant C_4. If $x\in J_{k_n}^{\omega_n}$ we deduce that

$$|\sigma_n g_n(x)|\geq\frac{1}{2}\log\omega_n-C_5.$$

Finally we set

$$P_n=\frac{2}{\pi}e_n\,\sigma_n(\Im m g_n)=\frac{2}{\pi}e_n\Im m(\sigma_n g_n),$$

so that $\|P_n\|_\infty\leq 1$. Now, recall that g_n is the restriction to the circle of an holomorphic function h_n satisfying $h_n(0)=0$. We can then write $\sigma_n g_n=\sum_{j=1}^{n-1}a_je_j$, so that $2i\Im m\ \sigma_n g_n=-\sum_{j=1}^{n-1}\overline{a_j}e_{-j}+\sum_{j=1}^{n-1}a_je_j$. Thus, the spectrum of P_n is contained in $[1,2n-1]$. Moreover, for any $x\in J_{k_n}^{\omega_n}$, we get

$$\begin{array}{rcl}|S_nP_n(x)| & = & \displaystyle\frac{1}{\pi}\left|\sum_{j=1}^{n-1}\overline{a_j}e_{-j+n}\right|\\ & = & \displaystyle\frac{1}{\pi}|\sigma_n g_n(x)|\\ & \geq & \displaystyle\frac{1}{2\pi}\log\omega_n-C_6\\ & = & 2\varepsilon_n\log n-C_6\\ & \geq & \varepsilon_n\log n\end{array}$$

if n is large enough. □

We are now ready to construct the dense G_δ-set of functions required in Theorem 1.7.

Proof of Theorem 1.7. Let $(\delta_n)_{n\geq 2}$ be a sequence going to 0. We first consider an auxiliary sequence $(\delta_n')_{n\geq 1}$ such that

$$\lim_{n\to+\infty}\delta_n'=0,\quad \lim_{n\to+\infty}\frac{\delta_n'}{\delta_n}=+\infty\quad\text{and}\quad\lim_{n\to+\infty}\delta_n'\log n=+\infty\ .$$

Let $(g_n)_{n\geq 1}$ be a dense sequence in $\mathcal{C}(\mathbb{T})$, such that the spectrum of g_n is contained in $[-n,n]$. We set $\eta_n=\max(\delta_k';\ n\leq k)$. The sequence $(\eta_n)_{n\geq 1}$ decreases to zero. Moreover, we fix a sequence $(\varepsilon_n)_{n\geq 1}$, going to zero, such that ε_n/η_n tends to infinity. Lemma 4.2 gives us an integer N, a sequence $(P_j)_{j\geq N}$ of trigonometric polynomials with spectrum contained in $[1, 2j-1]$, a sequence $(k_j)_{j\geq N}$ of integers going to $+\infty$ and a sequence $(\omega_j)_{j\geq N}$ satisfying $\omega_j > 1$, such that

$$|S_j P_j(x)| \geq \varepsilon_j \log j$$

for any $x \in J_{k_j}^{\omega_j}$. Moreover, we can choose ω_j such that $\omega_j = o(k_j^\alpha)$ for any $\alpha > 0$.
Let us define for $j \geq N$

$$h_j := g_j + \frac{\eta_j}{\varepsilon_j} e_j P_j.$$

The sequence $(h_j)_{j\geq N}$ remains dense in $\mathcal{C}(\mathbb{T})$. Let us also observe that the spectra of g_j and $\frac{\eta_j}{\varepsilon_j} e_j P_j$ are disjoint. It follows that if $x \in J_{k_j}^{\omega_j}$,

$$|S_{2j}h_j(x) - S_j h_j(x)| = \left|\frac{\eta_j}{\varepsilon_j} S_j P_j(x)\right| \geq \eta_j \log j.$$

Thus, for any $x \in J_{k_j}^{\omega_j}$, one may find $n \in \{j, 2j\}$ such that

$$|S_n h_j(x)| \geq \frac{1}{2}\eta_j \log j \geq \frac{1}{2}\delta'_n(\log n - \log 2).$$

Let $r_j > 0$ be small enough so that

$$|S_n h(x)| \geq |S_n h_j(x)| - 1$$

for any $h \in B(h_j, r_j)$ and any $n \in \{j, 2j\}$ (the open balls are related to the norm $\| \|_\infty$). Then, we claim that the following dense G_δ-set of $\mathcal{C}(\mathbb{T})$ fulfills all the requirements:

$$G := \bigcap_{p\geq N} \bigcup_{j\geq p} B(h_j, r_j).$$

Indeed, pick any h in G and any increasing sequence (j_p) such that h belongs to $B(h_{j_p}, r_{j_p})$. Setting $\rho_p = \omega_{j_p}$ and $s_p = k_{j_p}$, it is not hard to show that

$$E := \limsup_{p\to+\infty} E_p, \text{ with } E_p = J_{s_p}^{\rho_p}$$

has Hausdorff dimension 1. Indeed, recall that for any $\alpha > 0$, $\omega_j = o(k_j^\alpha)$. It follows for any $\alpha > 0$ and for p large enough, E_p contains

$$F_p = \bigcup_{j=0}^{s_p-1} \left[\frac{j}{s_p} - \frac{1}{2s_p^{1+\alpha}}; \frac{j}{s_p} + \frac{1}{2s_p^{1+\alpha}}\right],$$

Now, it is well-known that $\limsup_p F_p$ has Hausdorff dimension equal to $1/(1+\alpha)$ (this follows for instance from the mass transference principle of [3]). Finally, $\dim_{\mathcal{H}}(E) \geq \frac{1}{1+\alpha}$. Moreover, for any $x \in E$, the work done before and the fact that $\delta'_n \log n$ goes to $+\infty$ show that

$$|S_n h(x)| \geq \frac{1}{2}\delta'_n(\log n - \log 2) - 1 \geq \frac{1}{4}\delta'_n \log n$$

for infinitely many values of n. We then get

$$\frac{|S_n h(x)|}{\delta_n \log n} \geq \frac{\delta'_n}{4\delta_n}$$

for infinitely many values of n. This achieves the proof of Theorem 1.7. □

We can finally construct the prevalent set of functions required in Theorem 1.8.

Proof of Theorem 1.8. Let $(\delta_n)_{n\geq 2}$ be a sequence going to 0 and denote by A the set of continuous functions $f\in\mathcal{C}(\mathbb{T})$ such that

$$\dim_{\mathcal{H}}\left(\left\{x\in\mathbb{T}\ ;\ \limsup_{n\to+\infty}\frac{|S_nf(x)|}{\delta_n\log n}=+\infty\right\}\right)<1.$$

We have to prove that A is Haar-null in $\mathcal{C}(\mathbb{T})$.
Let f_0 be a fixed function in the complementary of A (such a function does exist by Theorem 1.7) and let g be an arbitrary function in $\mathcal{C}(\mathbb{T})$. Suppose that t_1 and t_2 are two real numbers such that

$$t_1f_0\in(g+A)\quad\text{ and }\quad t_2f_0\in(g+A).$$

We can then find $f_1\in A$ and $f_2\in A$ such that $(t_1-t_2)f_0=f_1-f_2$. It is clear that $f_1-f_2\in A$ (A is a vector subspace of $\mathcal{C}(\mathbb{T})$). It follows that $t_1=t_2$, so that

$$\#\left(\mathrm{span}\,(f_0)\cap(g+A)\right)\leq 1.$$

In particular, the Lebesgue-measure in $\mathrm{span}\,(f_0)$ is transverse to A and A is Haar-null in $\mathcal{C}(\mathbb{T})$. □

Remark: We only proved that a proper subspace in a complete metric vector space is Haar-null. This property is probably well-known.

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