BOUNDARY MULTIFRACTAL BEHAVIOUR FOR HARMONIC FUNCTIONS IN THE BALL

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Abstract. It is well known that if h is a nonnegative harmonic function in the ball of $\mathbb{R}^{d+1}$ or if h is harmonic in the ball with integrable boundary value, then the radial limit of h exists at almost every point of the boundary. In this paper, we are interested in the exceptional set of points of divergence and in the speed of divergence at these points. In particular, we prove that for generic harmonic functions and for any $\beta \in [0, d]$, the Hausdorff dimension of the set of points ξ on the sphere such that $h(r\xi)$ looks like $(1-r)^{-\beta}$ is equal to $d - \beta$.

1. Introduction

The story of this paper begins in 1906, when P. Fatou proved in [8] that bounded harmonic functions in the unit disk have nontangential limits almost everywhere on the circle. Later on, this result was improved by Hardy and Littlewood in dimension 2, and by Wiener, Bochner and many others in arbitrary dimension (a complete historical account can be found in [12]). Let us also mention R. Hunt and R. Wheeden who proved that a similar result holds for nonnegative harmonic functions in Lipschitz domains ([9, 10]). To state the result of the nontangential convergence of harmonic functions in the ball, we need to introduce some terminology.

Let $d \geq 1$ and let $\mathcal{S}_d$ (resp. B_{d+1}) be the (euclidean) unit sphere (resp. the unit ball) in $\mathbb{R}^{d+1}$. The euclidian norm in $\mathbb{R}^{d+1}$ will be denoted by $\|\cdot\|$. For $\mu \in \mathcal{M}(\mathcal{S}_d)$, the set of complex Borel measures on $\mathcal{S}_d$, the Poisson integral of μ, denoted by $P[\mu]$, is the function on B_{d+1} defined by

$$P[\mu](x) = \int_{\mathcal{S}_d} P(x,\xi)d\mu(\xi),$$

where $P(x,\xi)$ is the Poisson kernel,

$$P(x,\xi) = \frac{1-\|x\|^2}{\|x-\xi\|^{d+1}}.$$

When f is a function in $L^1(\mathcal{S}_d)$, we denote simply by $P[f]$ the function $P[fd\sigma]$. Here and elsewhere, $d\sigma$ denotes the normalized Lebesgue measure on $\mathcal{S}_d$. For any $\mu \in \mathcal{M}(\mathcal{S}_d)$, $P[\mu]$ is a harmonic function in B_{d+1} and it is well-known that, for instance, every bounded harmonic function in B_{d+1} is the Poisson integral $P[f]$ of a certain $f \in L^\infty(\mathcal{S}_d)$. It is also well known that every nonnegative harmonic function in B_{d+1} is the Poisson integral $P[\mu]$ of a positive finite measure $\mu \in \mathcal{M}(\mathcal{S}_d)$.

The Fatou theorem for Poisson integrals of L^1-functions says that, given a function $f \in L^1(\mathcal{S}_d)$, then $P[f](ry)$ tends to $f(y)$ for almost every $y \in \mathcal{S}_d$ when r increases to 1. More

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generally, if $\mu \in \mathcal{M}(\mathcal{S}_d)$, $P[\mu](ry)$ tends to $\frac{d\mu}{d\sigma}(y)$ almost everywhere and in fact, the limit exists for nontangential access.

In this paper, we are interested in the radial behaviour on exceptional sets, and especially in the following questions. How quick can grow $P[f](ry)$? For a prescribed growth $\tau(r)$, how big can be the sets of $y \in \mathcal{S}_d$ such that $\limsup_{r\to 1} |P[f](ry)|/\tau(r) = +\infty$? It is easy to see that the growth cannot be too fast. Indeed, the Poisson kernel satisfies, for any $y, \xi \in \mathcal{S}_d$,

$$P(ry,\xi) \leq \frac{2}{(1-r)^d},$$

so that for any $f \in L^1(\mathcal{S}_d)$, for any $y \in \mathcal{S}_d$ and any $r \in (0,1)$,

$$P[f](ry) \leq \frac{2\|f\|_1}{(1-r)^d}.$$

This motivates us to introduce, for a fixed $\beta \in (0,d)$ and any $f \in L^1(\mathcal{S}_d)$, the exceptional set

$$\mathcal{E}(\beta,f) = \left\{ y \in \mathcal{S}_d;\ \limsup_{r\to 1} \frac{|P[f](ry)|}{(1-r)^{-\beta}} = +\infty \right\},$$

and we ask for the size of $\mathcal{E}(\beta,f)$. To measure the size of subsets of $\mathcal{S}_d$, we shall use the notion of Hausdorff dimension (see Section 2 for precise definitions). Our first main result is the following.

Theorem 1.1. *Let $\beta \in [0,d]$ and let $f \in L^1(\mathcal{S}_d)$. Then $\dim_{\mathcal{H}}\big(\mathcal{E}(\beta,f)\big) \leq d-\beta$. Conversely, given a subset E of $\mathcal{S}_d$ such that $\dim_{\mathcal{H}}(E) < d-\beta$, there exists $f \in L^1(\mathcal{S}_d)$ such that $E \subset \mathcal{E}(\beta,f)$.*

Our second task is to perform a multifractal analysis of the radial behaviour of harmonic functions, as this is done in [3], [4] for the divergence of Fourier series. For a given function $f \in L^1(\mathcal{S}_d)$ and a given $y \in \mathcal{S}_d$, we define the real number $\beta(y)$ as the infimum of the real numbers β such that $|P[f](ry)| = O\big((1-r)^{-\beta}\big)$. The level sets of the function β are defined by

$$\begin{aligned} E(\beta,f) &= \{y \in \mathcal{S}_d;\ \beta(y) = \beta\} \\ &= \left\{ y \in \mathcal{S}_d;\ \limsup_{r\to 1} \frac{\log|P[f](ry)|}{-\log(1-r)} = \beta \right\}. \end{aligned}$$

We can ask for which values of β the sets $E(\beta,f)$ are non-empty. This set of values will be called the domain of definition of the spectrum of singularities of f. If β belongs to the domain of definition of the spectrum of singularities, it is also interesting to estimate the Hausdorff dimension of the sets $E(\beta,f)$. The function $\beta \mapsto \dim_{\mathcal{H}}(E(\beta,f))$ will be called the spectrum of singularities of the function f.

Theorem 1.1 ensures that $\dim_{\mathcal{H}}(E(\beta,f)) \leq d-\beta$ and our second main result is that a *typical* function $f \in L^1(\mathcal{S}_d)$ satisfies $\dim_{\mathcal{H}}(E(\beta,f)) = d-\beta$ for *any* $\beta \in [0,d]$. In particular, such a function f has a multifractal behavior, in the sense that the domain of definition of its spectrum of singularities contains an interval with non-empty interior.

Theorem 1.2. *For quasi-all functions $f \in L^1(\mathcal{S}_d)$, for any $\beta \in [0,d]$,*

$$\dim_{\mathcal{H}}\big(E(\beta,f)\big) = d-\beta.$$

The terminology "quasi-all" used here is relative to the Baire category theorem. It means that this property is true for a residual set of functions in $L^1(\mathcal{S}_d)$.

NOTATIONS. Throughout the paper, $\mathbf{N} = (0, \dots, 0, 1)$ will denote the north pole of $\mathcal{S}_d$. The letter C will denote a positive constant whose value may change from line to line. This value may depend on the dimension d, but it will never depend on the other parameters which are involved.

2. PRELIMINARIES

In this section, we survey some results regarding Hausdorff measures. We refer to [7] and to [11] for more on this subject. Let (X, d) be a metric space such that, for every $\rho > 0$, the space X can be covered by a countable number of balls with diameter less than ρ. If $B = B(x, r)$ is a ball in X and $\lambda > 0$, $|B|$ denotes the diameter of B whereas λB denotes the ball B scaled by a factor λ, i.e. $\lambda B = B(x, \lambda r)$.
A dimension function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function satisfying $\phi(0) = 0$. Given $E \subset X$, the ϕ-Hausdorff outer measure of E is defined by

$$\mathcal{H}^\phi(E) = \lim_{\varepsilon \to 0} \inf_{r \in R_\varepsilon(E)} \sum_{B \in r} \phi(|B|),$$

where $R_\varepsilon(E)$ is the set of countable coverings of E with balls B with diameter $|B| \le \varepsilon$. When $\phi_s(x) = x^s$, we write for short $\mathcal{H}^s$ instead of $\mathcal{H}^{\phi_s}$. The Hausdorff dimension of a set E is

$$\dim_{\mathcal{H}}(E) := \sup\{s > 0;\ \mathcal{H}^s(E) > 0\} = \inf\{s > 0;\ \mathcal{H}^s(E) = 0\}.$$

We will need to construct on $\mathcal{S}_d$ a family of subsets with prescribed Hausdorff dimension. To this intention, we shall use results of [5]. Recall that a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is doubling provided there exists $\lambda > 1$ such that, for any $x > 0$, $\phi(2x) \le \lambda \phi(x)$. From now on, we suppose that the metric space (X, d) supports a doubling dimension function ϕ such that

$$\frac{1}{C}\phi(|B|) \le \mathcal{H}^\phi(B) \le C\phi(|B|)$$

where C is a positive constant independant of B.
The previous assumption is satisfied when $X = \mathcal{S}_d$, endowed with the distance inherited from $\mathbb{R}^{d+1}$, and $\phi(x) = x^d$.
Given a dimension function ψ and a ball $B = B(x, r)$, we denote by B^ψ the ball $B^\psi = B(x, \psi^{-1} \circ \phi(r))$. The following mass transference principle of [5] will be used.

Lemma 2.1 (The mass transference principle). *Let (B_i) be a sequence of balls in X whose radius goes to zero. Let ψ be a dimension function such that $\psi(x)/\phi(x)$ is monotonic and suppose that, for any ball B in X,*

$$\mathcal{H}^\phi\left(B \cap \limsup_{i \to +\infty} B_i\right) = \mathcal{H}^\phi(B).$$

Then, for any ball B in X,

$$\mathcal{H}^\psi\left(B \cap \limsup_{i \to +\infty} B_i^\psi\right) = \mathcal{H}^\psi(B).$$

Finally, the following basic covering lemma due to Vitali will be required (see [11]).

Lemma 2.2 (The $5r$-covering lemma)**.** *Every family $\mathcal{F}$ of balls with uniformly bounded diameter in a separable metric space (X,d) contains a disjoint subfamily $\mathcal{G}$ such that*

$$\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{B\in\mathcal{G}} 5B.$$

3. Majoration of the Hausdorff dimension

Let $f\in L^1(\mathcal{S}_d)$. One intend to show that $P[f](r\cdot)$ cannot grow too fast on sets with large Hausdorff dimension. More generally, we shall do this for $\mu\in\mathcal{M}(\mathcal{S}_d)$ and $P[\mu]$ instead of $P[f]$. If $y\in\mathcal{S}_d$ and $\delta>0$, we introduce

$$\kappa(y,\delta)=\big\{\xi\in\mathcal{S}_d;\ \|\xi-y\|<\delta\big\}$$

the open spherical cap on $\mathcal{S}_d$ with center y and radius $\delta>0$. The set $\kappa(y,\delta)$ is nothing else but the ball with center y and radius δ in the metric space $(\mathcal{S}_d,\|\cdot\|)$. Let us also define the slice

$$\mathcal{S}(y,\delta_1,\delta_2)=\big\{\xi\in\mathcal{S}_d;\ \delta_1\le\|\xi-y\|<\delta_2\big\}$$

where $0\le\delta_1<\delta_2$.

The starting point of our argument is a result linking the radial behaviour of $P[\mu]$ to the Hardy-Littlewood maximal function. More precisely, it is well known that if $y\in\mathcal{S}_d$, then

$$\sup_{r\in(0,1)}\big|P[\mu](ry)\big|\le\sup_{\delta>0}\frac{|\mu|(\kappa(y,\delta))}{\sigma(\kappa(y,\delta))}$$

(see for example [2]). Our aim is to control, for a fixed r close to 1, the minimal size of the caps which come into play on the right handside.

Lemma 3.1. *Let $\mu\in\mathcal{M}(\mathcal{S}_d)$, $r\in(0,1)$ and $y\in\mathcal{S}_d$. There exists $\delta\ge1-r$ such that*

$$\big|P[\mu](ry)\big|\le C\frac{|\mu|(\kappa(y,\delta))}{\sigma(\kappa(y,\delta))},$$

where C is a constant independent of μ, r and y.

Proof. Replacing μ by $|\mu|$, we may assume that μ is positive. Moreover, without loss of generality, we may assume that $y=\mathbf{N}$ is the north pole. Observe that

$$P[\mu](r\mathbf{N})=\int_{\mathcal{S}_d}P(r\mathbf{N},\xi)d\mu(\xi),$$

with

$$\begin{aligned}P(r\mathbf{N},\xi)&=\frac{1-r^2}{\|r\mathbf{N}-\xi\|^{d+1}}\\&=\frac{1-r^2}{(1-2r\xi_{d+1}+r^2)^{(d+1)/2}}.\end{aligned}$$

Observe also that $\|\xi-\mathbf{N}\|^2=2(1-\xi_{d+1})$ if $\xi\in\mathcal{S}_d$. In particular, $P(r\mathbf{N},\xi)$ just depends on $\|\xi-\mathbf{N}\|$ and r. Moreover, $P(r\mathbf{N},\xi)$ decreases when $\|\xi-\mathbf{N}\|$ increases, ξ keeping on $\mathcal{S}_d$.

We shall approach $\xi\mapsto P(r\mathbf{N},\xi)$ by functions which are constant on slices. The function $\xi\mapsto P(r\mathbf{N},\xi)$ is harmonic and non negative in the ball $\{\xi\in\mathbb{R}^{d+1};\ \|\xi-\mathbf{N}\|<1-r\}$. By the Harnack inequality, there exists $C_0>0$ (which does not depend on r) such that, for any $\xi\in\mathbb{R}^{d+1}$ with $\|\xi-\mathbf{N}\|\leq(1-r)/2$,

$$P(r\mathbf{N},\xi)\geq C_0P(r\mathbf{N},\mathbf{N}).$$

Necessarily, C_0 belongs to $(0,1)$. We then define an integer $k>0$ and a finite sequence $\delta_0,\dots,\delta_k$ by

- $\delta_0=0$;
- $\delta_1=(1-r)/2$;
- δ_{j+1} (if it exists) is the real number in $[\delta_j,2]$ such that $P(r\mathbf{N},\xi^{j+1})=C_0P(r\mathbf{N},\xi^j)$ where ξ^j (resp. ξ^{j+1}) is an arbitrary point of $\mathcal{S}_d$ such that $\|\xi^j-\mathbf{N}\|=\delta_j$ (resp. $\|\xi^{j+1}-\mathbf{N}\|=\delta_{j+1}$) (remember that $P(r\mathbf{N},\xi)$ only depends on $\|\xi-\mathbf{N}\|$);
- $\delta_{j+1}=2$ and $k=j+1$ otherwise.

Observe that the sequence is well-defined and that, by compactness, the process ends up after a finite number of steps. We set $c_j=P(r\mathbf{N},\xi^j)$, $0\leq j\leq k-1$ where ξ^j is an arbitrary point in $\mathcal{S}_d$ such that $\|\mathbf{N}-\xi^j\|=\delta_j$. Let us also remark that, if $\xi\in\mathcal{S}_d$, $\xi\neq-\mathbf{N}$,

$$C_0\sum_{j=0}^{k-1}c_j\mathbf{1}_{\mathcal{S}(\mathbf{N},\delta_j,\delta_{j+1})}(\xi)\leq P(r\mathbf{N},\xi)\leq\sum_{j=0}^{k-1}c_j\mathbf{1}_{\mathcal{S}(\mathbf{N},\delta_j,\delta_{j+1})}(\xi).$$

The sequence $(c_j)_{j\geq0}$ is decreasing. Thus, we can rewrite the step function using only caps as

$$\sum_{j=0}^{k-1}c_j\mathbf{1}_{\mathcal{S}(\mathbf{N},\delta_j,\delta_{j+1})}=\sum_{j=1}^{k}d_j\mathbf{1}_{\kappa(\mathbf{N},\delta_j)}$$

where the real numbers d_j are *positive*. In fact, $d_1=c_0$ and $d_j=c_{j-1}-c_j$ if $j\geq2$. Then we get

$$C_0\sum_{j=1}^{k}d_j\mathbf{1}_{\kappa(\mathbf{N},\delta_j)}\leq P(r\mathbf{N},\xi)\leq\sum_{j=1}^{k}d_j\mathbf{1}_{\kappa(\mathbf{N},\delta_j)}. \tag{1}$$

We integrate the right inequality with respect to μ to obtain

$$\begin{aligned}P[\mu](r\mathbf{N})&\leq\sum_{j=1}^{k}d_j\mu(\kappa(\mathbf{N},\delta_j))\\&\leq\sup_{j=1,\dots,k}\frac{\mu(\kappa(\mathbf{N},\delta_j))}{\sigma(\kappa(\mathbf{N},\delta_j))}\sum_{j=1}^{k}d_j\sigma(\kappa(\mathbf{N},\delta_j))\\&\leq C_0^{-1}\sup_{j=1,\dots,k}\frac{\mu(\kappa(\mathbf{N},\delta_j))}{\sigma(\kappa(\mathbf{N},\delta_j))}\int_{\mathcal{S}_d}P(r\mathbf{N},\xi)d\sigma(\xi)\end{aligned}$$

where the last inequality is obtained by integrating the left part of (1) over $\mathcal{S}_d$ with respect to the surface measure σ. This yields the lemma, since $\int_{\mathcal{S}_d}P(r\mathbf{N},\xi)d\sigma(\xi)=1$, except that we have found a cap with radius greater than $(1-r)/2$ instead of $1-r$. Fortunately, it is easy to dispense with the factor $1/2$. Indeed,

$$\frac{\mu(\kappa(\mathbf{N},\delta))}{\sigma(\kappa(\mathbf{N},\delta))}\leq C\frac{\mu(\kappa(\mathbf{N},\delta))}{\sigma(\kappa(\mathbf{N},2\delta))}\leq C\frac{\mu(\kappa(\mathbf{N},2\delta))}{\sigma(\kappa(\mathbf{N},2\delta))}.$$

□

The previous lemma is the main step to obtain an upper bound of the Hausdorff dimension of the sets where $P[\mu](r\cdot)$ behave badly.

Theorem 3.2. *Let $\mu \in \mathcal{M}(\mathcal{S}_d)$ and let $\tau : (0,1) \to (0,+\infty)$ be nonincreasing, with $\lim_{x\to 0^+} \tau(x) = +\infty$. Let us define*

$$\mathcal{E}(\tau,\mu) = \left\{ y \in \mathcal{S}_d;\ \limsup_{r\to 1} \frac{|P[\mu](ry)|}{\tau(1-r)} = +\infty \right\}.$$

Let $\phi : (0,+\infty) \to (0,+\infty)$ be a dimension function satisfying $\phi(s) = O(\tau(s)s^d)$. Then

$$\mathcal{H}^\phi\big(\mathcal{E}(\tau,\mu)\big) = 0.$$

Proof. For any $M > 1$, we introduce

$$\mathcal{E}_M = \left\{ y \in \mathcal{S}_d;\ \limsup_{r\to 1} \frac{|P[\mu](ry)|}{\tau(1-r)} > M \right\}.$$

Let $\varepsilon > 0$ and $y \in \mathcal{E}_M$. The definition of $\mathcal{E}_M$ and Lemma 3.1 ensure that we can find $r_y \in (0,1)$, as close to 1 as we want, and a cap $\kappa_y = \kappa(y,\delta_y)$ such that $\delta_y \geq 1 - r_y$ satisfying

$$M\tau(1-r_y) \leq |P[\mu](r_y y)| \leq C\frac{|\mu|(\kappa_y)}{\sigma(\kappa_y)}. \tag{2}$$

Observe that

$$\sigma(\kappa_y) \leq \frac{C|\mu|(\mathcal{S}_d)}{M\tau(1-r_y)}.$$

It follows that $\delta_y \to 0$ when $r_y \to 1$. We can then always ensure that $|\kappa_y| \leq \varepsilon$. The family $(\kappa_y)_{y\in\mathcal{E}_M}$ is an ε-covering of $\mathcal{E}_M$. By the 5r-covering lemma, one can extract from it a countable family of disjoint caps $(\kappa_{y_i})_{i\in\mathbb{N}}$ such that $\mathcal{E}_M \subset \bigcup_i 5\kappa_{y_i}$. Inequality (2) implies that

$$M\sum_i \tau(1-r_{y_i})\sigma(\kappa_{y_i}) \leq C\|\mu\|.$$

If we remark that $|5\kappa_{y_i}| \geq \delta_{y_i} \geq 1 - r_{y_i}$, we can conclude that

$$\sum_i \tau(|5\kappa_i|)|5\kappa_i|^d \leq \frac{C}{M}\|\mu\|.$$

Our assumption on ϕ ensures that $\mathcal{H}^\phi(\mathcal{E}_M) \leq C(\phi,\mu)/M$. The result follows from the equality $\mathcal{E}(\tau,\mu) = \bigcap_{M>1} \mathcal{E}_M$. □

Applying this to the function $\tau(s) = s^{-\beta}$, we get the first half of Theorem 1.1.

Corollary 3.3. *For any $\beta \in [0,d]$, for any $\mu \in \mathcal{M}(\mathcal{S}_d)$, $\dim_{\mathcal{H}}\big(\mathcal{E}(\beta,\mu)\big) \leq d - \beta$.*

Remark 3.4. The corresponding result for the divergence of Fourier series was obtained in [1] using the Carleson-Hunt theorem (see also [4] for the L^1-case). Our proof in this context is much more elementary, since we do not need the maximal inequality for the Hardy-Littlewood maximal function.

4. Minoration of the Hausdorff dimension

In this section, we prove the converse part of Theorem 1.1. We first need a technical lemma on the Poisson kernel. It is a quantitative way to express that it is an approximate identity.

Lemma 4.1. *There exists a constant $C > 0$ such that, for any $r \in (1/2, 1)$ and any $y \in \mathcal{S}_d$,*

$$\int_{\kappa(y,1-r)} P(ry, \xi) d\sigma(\xi) \geq C.$$

Proof. We may assume $y = \mathbf{N}$. Let $\rho = 1 - r$. A generic point $x = (x_1, \cdots, x_{d+1}) \in \mathbb{R}^{d+1}$ will be noted $x = (x', x_{d+1})$ with $x' \in \mathbb{R}^d$. In particular, $x \in \kappa(\mathbf{N}, \rho)$ if and only if $\|x'\|^2 + x_{d+1}^2 = 1$ and $\|x'\|^2 + (1 - x_{d+1})^2 < \rho^2$. Let $\mathcal{C}$ be the cylinder

$$\mathcal{C} = \left\{ x \in \mathbb{R}^{d+1} \ ; \ \|x'\|^2 < \rho^2/2 \text{ and } 1 - 2\rho < x_{d+1} < 1 \right\}.$$

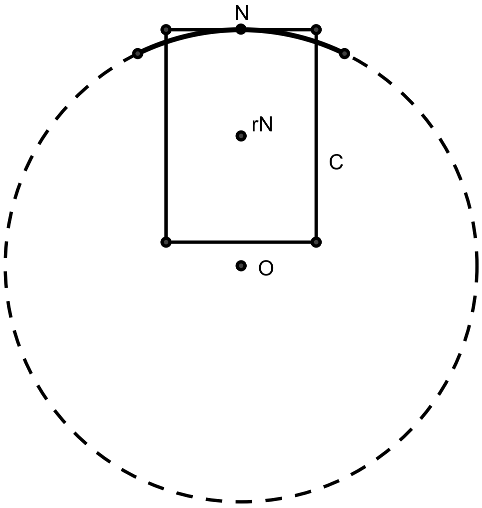
It is not hard to show that $\mathcal{S}_d \cap \overline{\mathcal{C}} \subset \kappa(\mathbf{N}, \rho)$ when $1/2 < r < 1$. We now define two harmonic functions. h is the harmonic function in $\mathcal{C}$ such that $h(x) = 1$ if $x \in \partial C \cap \{x_{d+1} = 1\}$ and $h(x) = 0$ if $x \in \partial C \cap \{x_{d+1} < 1\}$; u is the harmonic function in B_{d+1} such that $u = 1$ on $\kappa(\mathbf{N}, \rho)$ and $u = 0$ elsewhere on $\mathcal{S}_d$. We claim that $h \leq u$ on $\partial(\mathcal{C} \cap B_{d+1})$. Indeed, we can decompose $\partial(\mathcal{C} \cap B_{d+1})$ into $E \cup F$, with $E \subset S_d \cap \overline{\mathcal{C}}$ and $F \subset \partial C \cap \{x_{d+1} < 1\}$. Now, $u = 1 \geq h$ on E and $u \geq 0 = h$ on F. By the maximum principle in $\mathcal{C} \cap B_{d+1}$, we deduce that $u(x) \geq h(x)$ for any $x \in \mathcal{C} \cap B_{d+1}$. This in particular the case if $x = (1 - \rho)\mathbf{N} = r\mathbf{N}$, so that

$$\int_{\kappa(\mathbf{N},\rho)} P(r\mathbf{N}, \xi) d\sigma(\xi) \geq h(r\mathbf{N}).$$

On the other hand, $\mathcal{C}$ is just the translation and dilation of a fixed domain : $\mathcal{C} = \mathbf{N} + \rho\mathcal{U}$, where

$$\mathcal{U} = \left\{ x \in \mathbb{R}^{d+1} \ ; \ \|x'\|^2 < 1/2 \text{ and } -2 < x_{d+1} < 0 \right\}.$$

Thus the quantity $h(r\mathbf{N})$ is strictly positive and independent of r. We can then take $C = h(r\mathbf{N})$.



□

Here is the converse part of Theorem 1.1.

Theorem 4.2. *Let $E\subset\mathcal{S}_d$, let ϕ be a dimension function and let $\tau:(0,1)\to(0,+\infty)$ be nonincreasing with $\lim_{x\to 0^+}\tau(x)=+\infty$. Suppose that $\mathcal{H}^\phi(E)=0$ and that $\tau(s)=O\big(s^{-d}\phi(s)\big)$. Then there exists $f\in L^1(\mathcal{S}_d)$ such that, for any $y\in E$,*

$$\limsup_{r\to 1}\frac{P[f](ry)}{\tau(1-r)}=+\infty.$$

A remarkable feature of Theorem 3.2 and Theorem 4.2 is that they are sharp: if $\phi(s)=\tau(s)s^d$ is a dimension function and

$$\mathcal{E}(\tau,f)=\left\{y\in\mathcal{S}_d;\ \limsup_{r\to 1}\frac{|P[f](ry)|}{\tau(1-r)}=+\infty\right\},$$

then

(1) for any $f\in L^1(\mathcal{S}_d)$, $\mathcal{H}^\phi\big(\mathcal{E}(\tau,f)\big)=0$;
(2) if E is a set satisfying $\mathcal{H}^\phi(E)=0$, we can find $f\in L^1(\mathcal{S}_d)$ such that $\mathcal{E}(\tau,f)\supset E$.

Proof of Theorem 4.2. Let $j\geq 1$. Since $\mathcal{H}^\phi(E)=0$, we can find a covering $\mathcal{R}_j$ of E by caps with diameter less than 2^{-j} and such that $\sum_{\kappa\in\mathcal{R}_j}\phi(|\kappa|)\leq 2^{-j}$. We collect together the caps with approximatively the same size. Precisely, if $n\geq 1$, let

$$\mathcal{C}_n=\left\{\kappa\in\bigcup_j\mathcal{R}_j;\ 2^{-(n+1)}<|\kappa|\leq 2^{-n}\right\}.$$

Let also $E_n=\bigcup_{\kappa\in\mathcal{C}_n}\kappa$ so that $E\subset\limsup_n E_n$ and

$$\sum_{n\geq 1}\sum_{\kappa\in\mathcal{C}_n}\phi(|\kappa|)\leq\sum_{j\geq 1}\sum_{\kappa\in\mathcal{R}_j}\phi(|\kappa|)\leq 1.$$

In particular, there exists a sequence $(\omega_n)_{n\geq 1}$ tending to infinity such that

$$\sum_{n\geq 1}\sum_{\kappa\in\mathcal{C}_n}\omega_n\phi(|\kappa|)<+\infty.$$

For any $n\geq 1$, let $x_{n,1},\dots,x_{n,m_n}$ be the centers of the caps appearing in $\mathcal{C}_n$ and let $\kappa_{n,i}=\kappa(x_{n,i},2\cdot 2^{-n})$. We define

$$f=\sum_{n\geq 1}\sum_{i=1}^{m_n}\omega_n\tau(2^{-n})\mathbf{1}_{\kappa_{n,i}}.$$

f belongs to $L^1(\mathcal{S}_d)$. Indeed,

$$\begin{aligned}\|f\|_1 &\leq C\sum_{n\geq 1}\sum_{i=1}^{m_n}\omega_n\tau(2^{-n})(2^{-n})^d\\ &\leq C\sum_{n\geq 1}\sum_{i=1}^{m_n}\omega_n\phi(2^{-n})\\ &\leq C\sum_{n\geq 1}\sum_{\kappa\in\mathcal{C}_n}\omega_n\phi(|\kappa|)<+\infty.\end{aligned}$$

Moreover, let $y\in E_n$ and let $r=1-2^{-n}$. Let also $\kappa_y=\kappa(x_{n,i},\delta_{n,i})\in\mathcal{C}_n$ such that y belongs to κ_y. It is clear that $\|y-x_{n,i}\|\leq\delta_{n,i}\leq 2^{-n}$ so that $\kappa(y,2^{-n})\subset\kappa_{n,i}$. By the positivity of f and of the Poisson kernel,

$$\begin{aligned}P[f](ry) &\geq \int_{\kappa(y,2^{-n})}\omega_n\tau(2^{-n})P(ry,\xi)d\sigma(\xi)\\ &\geq C\omega_n\tau(1-r)\end{aligned}$$

where C is the constant that appears in Lemma 4.1. Thus, provided y belongs to $\limsup_n E_n$, we get

$$\limsup_{r\to 1}\frac{P[f](ry)}{\tau(1-r)}=+\infty,$$

what is exactly what we need. □

5. Construction of saturating functions

In this section, we turn to the construction of functions in $L^1(\mathcal{S}_d)$ having multifractal behaviour. Our first step is a construction of a sequence of nets in $\mathcal{S}_d$ which play the same role as dyadic numbers in the interval.

Lemma 5.1. *There exists a sequence $(\mathcal{R}_n)_{n\geq 1}$ of finite subsets of $\mathcal{S}^d$ satisfying*

- $\mathcal{R}_n\subset\mathcal{R}_{n+1}$;
- $\bigcup_{x\in\mathcal{R}_n}\kappa(x,2^{-n})=\mathcal{S}_d$;
- $\mathrm{card}(\mathcal{R}_n)\leq C2^{nd}$;
- *For any x,y in $\mathcal{R}_n$, $x\neq y$, then $|x-y|\geq 2^{-n}$.*

Proof. Let $\mathcal{R}_0=\varnothing$ and let us explain how to construct $\mathcal{R}_{n+1}$ from $\mathcal{R}_n$. $\mathcal{R}_{n+1}$ is a maximal subset of $\mathcal{S}_d$ containing $\mathcal{R}_n$ and such that any distinct points in $\mathcal{R}_{n+1}$ have their distance greater than or equal to $2^{-(n+1)}$. Then $\bigcup_{x\in\mathcal{R}_{n+1}}\kappa\left(x,2^{-(n+1)}\right)=\mathcal{S}_d$ by maximality of

$\mathcal{R}_{n+1}$. Then, taking the surface and using that the caps $\kappa\left(x, 2^{-(n+2)}\right)$, $x \in \mathcal{R}_{n+1}$, are pairwise disjoint, we get

$$\mathrm{card}(\mathcal{R}_{n+1}) \times C 2^{-(n+2)d} \leq 1.$$

□

From now on, we fix a sequence $(\mathcal{R}_n)_{n\geq 0}$ as in the previous lemma. Our sets with big Hausdorff dimension will be based on open caps centered at points of $\mathcal{R}_n$. Precisely, let $\alpha > 1$ and let $N_{n,\alpha} = [n/\alpha] + 1$ where $[n/\alpha]$ denotes the integer part of n/α. We introduce

$$D_{n,\alpha} = \bigcup_{x \in \mathcal{R}_{N_{n,\alpha}}} \kappa\left(x, 2^{-n}\right).$$

Lemma 5.2. *Let $\alpha > 1$ and let $(n_k)_{k\geq 0}$ be a sequence of integers growing to infinity. Then*

$$\mathcal{H}^{d/\alpha}\left(\limsup_{k\to+\infty} D_{n_k,\alpha}\right) = +\infty.$$

Proof. This follows from an application of the mass transference principle (Lemma 2.1), applied with the function $\psi(x) = x^{d/\alpha}$ and $\phi(x) = x^d$. The key points are that

$$\bigcup_{x \in \mathcal{R}_{N_{n,\alpha}}} \kappa\left(x, 2^{-N_{n,\alpha}}\right) = \mathcal{S}_d$$

and that $\kappa\left(x, 2^{-n}\right) \supset \kappa\left(x, \psi^{-1} \circ \phi(2^{-N_{n,\alpha}})\right)$ since $\alpha N_{n,\alpha} \geq n$. □

We now construct saturating functions step by step.

Lemma 5.3. *Let $n \geq 1$. There exists a nonnegative fonction $f_n \in L^1(\mathcal{S}_d)$, satisfying $\|f_n\|_1 = 1$, such that, for any $\alpha > 1$, for any $y \in D_{n,\alpha}$,*

$$P[f_n](r_n y) \geq \frac{C}{n} 2^{(n-N_{n,\alpha})d},$$

where $1 - r_n = 2^{-n}$, $N_{n,\alpha} = [n/\alpha] + 1$ and C is independant of n and α.

Proof. We define $\tilde{f}_n$ by

$$\tilde{f}_n := \frac{1}{n+1} \sum_{N=1}^{n+1} \sum_{x \in \mathcal{R}_N} 2^{(n-N)d} \mathbf{1}_{\kappa(x, 2\cdot 2^{-n})}.$$

The triangle inequality ensures that

$$\begin{aligned} \|\tilde{f}_n\|_1 &\leq \frac{C}{n+1} \sum_{N=1}^{n+1} \mathrm{card}(\mathcal{R}_N) 2^{(n-N)d} 2^{-nd} \\ &\leq C. \end{aligned}$$

Let $y \in D_{n,\alpha}$ and let $x \in \mathcal{R}_{N_{n,\alpha}}$ such that $y \in \kappa\left(x, 2^{-n}\right)$. Observe that $\kappa\left(y, 2^{-n}\right) \subset \kappa\left(x, 2.2^{-n}\right)$. Moreover, $1 \leq N_{n,\alpha} \leq n+1$. Using the positivity of the Poisson kernel, we get

$$P[\tilde{f}_n](ry) \geq \int_{\kappa(y, 2^{-n})} \frac{2^{(n-N_{n,\alpha})d}}{n+1} P(ry, \xi) d\sigma(\xi).$$

Lemma 4.1 ensures that
$$P[\tilde{f}_n](r_n y)\geq \frac{C}{n+1}2^{(n-N_{n,\alpha})d}$$
and it suffices to take $f_n=\frac{\tilde{f}_n}{\|\tilde{f}_n\|_1}$. □

We are now ready for the proof of our second main theorem.

Proof of Theorem 1.2. Let $(g_n)_{n\geq 1}$ be a dense sequence of $L^1(\mathcal{S}_d)$ such that each g_n is continuous and $\|g_n\|_\infty\leq n$. The maximum principle ensures that $|P[g_n](r\xi)|\leq n$ for any $r\in(0,1)$ and for any $\xi\in\mathcal{S}_d$. Let (f_n) be the sequence given by Lemma 5.3 and let us set
$$h_n=g_n+\frac{1}{n}f_n.$$
$(h_n)_{n\geq 1}$ remains dense in $L^1(\mathcal{S}_d)$. Moreover, if $r_n=1-2^{-n}$, $\alpha>1$ and $y\in D_{n,\alpha}$,
$$\begin{aligned}P[h_n](r_n y)&\geq C\frac{2^{(n-N_{n,\alpha})d}}{n^2}-n\\&\geq C\frac{2^{(n-N_{n,\alpha})d}}{2n^2}\end{aligned}$$
provided n is sufficiently large. Let us finally consider $\delta_n>0$ sufficiently small such that
$$\|P[f](r_n\cdot)\|_\infty\leq 1\quad\text{if}\quad \|f\|_1\leq\delta_n.$$
The residual set we will consider is the dense G_δ-set
$$A=\bigcap_{l\geq 1}\bigcup_{n\geq l}B_{L^1}(h_n,\delta_n).$$
Pick any $f\in A$. One can find an increasing sequence of integers (n_k) such that $f\in B_{L^1}(h_{n_k},\delta_{n_k})$ for any k. Let $\alpha>1$ and let $y\in\limsup_k D_{n_k,\alpha}=:D_\alpha(f)$. Then we can find integers n, picked in the sequence $(n_k)_{k\geq 1}$, as large as we want such that
$$P[f](r_n y)\geq P[h_n](r_n y)-1\geq C\frac{2^{(n-N_{n,\alpha})d}}{2n^2}-1.$$
Observe that for such values of n,
$$\frac{\log|P[f](r_n y)|}{-\log(1-r_n)}\geq\frac{(n-N_{n,\alpha})\,d}{n}+o(1).$$
Hence,
$$\limsup_{r\to 1}\frac{\log|P[f](ry)|}{-\log(1-r)}\geq\lim_{n\to+\infty}\left(1-\frac{N_{n,\alpha}}{n}\right)d=\left(1-\frac{1}{\alpha}\right)d.$$
Furthermore, Lemma 5.2 tells us that $\mathcal{H}^{d/\alpha}(D_\alpha(f))=+\infty$. We divide $D_\alpha(f)$ into two parts:
$$\begin{aligned}D_\alpha^{(1)}(f)&=\left\{y\in D_\alpha(f);\ \limsup_{r\to 1}\frac{\log|P[f](ry)|}{-\log(1-r)}=\left(1-\frac{1}{\alpha}\right)d\right\}\\D_\alpha^{(2)}(f)&=\left\{y\in D_\alpha(f);\ \limsup_{r\to 1}\frac{\log|P[f](ry)|}{-\log(1-r)}>\left(1-\frac{1}{\alpha}\right)d\right\}.\end{aligned}$$
Let $(\beta_n)_{n\geq 0}$ be a sequence of real numbers such that
$$\beta_n>\left(1-\frac{1}{\alpha}\right)d\quad\text{and}\quad\lim_{n\to+\infty}\beta_n=\left(1-\frac{1}{\alpha}\right)d.$$

Then

$$D_\alpha^{(2)}(f) \subset \bigcup_{n\geq 0} \mathcal{E}(\beta_n, f).$$

Observe that $\frac{d}{\alpha} > d-\beta_n$. Then, by Corollary 3.3, $\mathcal{H}^{d/\alpha}(\mathcal{E}(\beta_n, f)) = 0$. We get $\mathcal{H}^{d/\alpha}(D_\alpha^{(2)}(f)) = 0$ and $\mathcal{H}^{d/\alpha}(D_\alpha^{(1)}(f)) = +\infty$. Finally,

$$E\left(\left(1-\frac{1}{\alpha}\right)d, f\right) \supset D_\alpha^{(1)}(f)$$

and

$$\dim_{\mathcal{H}}\left(E\left(\left(1-\frac{1}{\alpha}\right)d, f\right)\right) \geq \frac{d}{\alpha}.$$

By Corollary 3.3 again, this inequality is necessarily an equality, and we conclude that f satisfies the conclusion of Theorem 1.2 by setting

$$\left(1-\frac{1}{\alpha}\right)d = \beta \iff \frac{d}{\alpha} = d-\beta.$$

□

One can also ask whether the Poisson integral of a typical Borel measure on $\mathcal{S}_d$ has a multifractal behaviour. Here, we have to take care of the topology on $\mathcal{M}(\mathcal{S}_d)$. We endow it with the weak-star topology, which turns the unit ball $B_{\mathcal{M}(\mathcal{S}_d)}$ of the dual space $\mathcal{M}(\mathcal{S}_d)$ into a compact space. We need the following folklore lemma:

Lemma 5.4. *The set of measures $fd\sigma$, with $f \in \mathcal{C}(\mathcal{S}_d)$, is weak-star dense in $\mathcal{M}(\mathcal{S}_d)$.*

Proof. The set of measures with finite support is weak-star dense in $\mathcal{M}(\mathcal{S}_d)$ (see for instance [6]). Thus, let $\xi \in \mathcal{S}_d$, let $\varepsilon > 0$ and let $g_1, \dots, g_n \in \mathcal{C}(\mathcal{S}_d)$. It suffices to prove that one can find $f \in \mathcal{C}(\mathcal{S}_d)$ such that, for any $\varepsilon > 0$, for any $i \in \{1, \dots, n\}$,

$$\left| g_i(\xi) - \int_{\mathcal{S}_d} g_i(y)f(y)d\sigma(y)\right| < \varepsilon.$$

Since each g_i is continuous at ξ, one can find $\delta > 0$ such that $|\xi - y| < \delta$ implies $|g_i(\xi) - g_i(y)| < \varepsilon$. Let f be a continuous and nonnegative function on $\mathcal{S}_d$ with support in $\kappa(\xi, \delta)$ and whose integral is equal to 1. Then

$$\begin{aligned}\left| g_i(\xi) - \int_{\mathcal{S}_d} g_i(y)f(y)d\sigma(y)\right| &\leq \int_{\kappa(\xi,\delta)} |g_i(\xi) - g_i(y)|f(y)d\sigma(y) \\ &\leq \varepsilon.\end{aligned}$$

□

Mimicking the proof of Theorem 1.2, we can prove the following result.

Theorem 5.5. *For quasi-all measures $\mu \in B_{\mathcal{M}(\mathcal{S}_d)}$, for any $\beta \in [0, d]$,*

$$\dim_{\mathcal{H}}\big(E(\beta, \mu)\big) = d - \beta.$$

Proof. Let $(g_n)_{n\geq 1}$ be a dense sequence of the unit ball of $\mathcal{C}(\mathcal{S}_d)$ such that $\|g_n\|_\infty \leq 1-\frac{1}{n}$. The sequence $(g_n d\sigma)_{n\geq 1}$ is weak-star dense in $B_{\mathcal{M}(\mathcal{S}_d)}$. Let $(f_n)_{n\geq 1}$ be the sequence given by Lemma 5.3 and let us set

$$h_n = g_n + \frac{1}{n} f_n$$

so that $(h_n d\sigma)_{n\geq 1}$ lives in the unit ball $B_{\mathcal{M}(\mathcal{S}_d)}$ and is always a weak-star dense sequence in $B_{\mathcal{M}(\mathcal{S}_d)}$. For any $\alpha > 1$ and any $y \in D_{n,\alpha}$,

$$P[h_n](r_n y) \quad \geq \quad C\frac{2^{(n-N_{n,\alpha})d}}{n^2} - 1$$

with $r_n = 1-2^{-n}$. The function $(y,\xi)\mapsto P(r_n y,\xi)$ is uniformly continuous on $\mathcal{S}_d\times\mathcal{S}_d$. In particular, using the compactness of $\mathcal{S}_d$, one may find $y_1,\dots,y_s\in\mathcal{S}_d$ such that, for any $y\in\mathcal{S}_d$, there exists $j\in\{1,\dots,s\}$ satisfying

$$\forall \xi\in\mathcal{S}_d,\quad \big|P(r_n y,\xi) - P(r_n y_j,\xi)\big| \leq 1.$$

Let $\mathcal{U}_n$ be the following weak-star open neighbourhood of $h_n d\sigma$ in $B_{\mathcal{M}(\mathcal{S}_d)}$:

$$\begin{aligned}\mathcal{U}_n \quad = \quad & \Bigg\{\mu\in B_{\mathcal{M}(\mathcal{S}_d)};\ \text{for all } j\in\{1,\dots,s\},\\ & \left|\int_{\mathcal{S}_d} P(r_n y_j,\xi)d\mu - \int_{\mathcal{S}_d} P(r_n y_j,\xi)h_n(\xi)d\sigma\right| < 1\Bigg\}.\end{aligned}$$

By the triangle inequality, for any $y\in\mathcal{S}_d$ and any $\mu\in\mathcal{U}_n$,

$$|P[\mu - h_n d\sigma](r_n y)| \leq 3.$$

We now define $A = \bigcap_{l\geq 1}\bigcup_{n\geq l}\mathcal{U}_n$ which is a dense G_δ-subset of $B_{\mathcal{M}(\mathcal{S}_d)}$, and we conclude like in the proof of Theorem 1.2. □

If we remember that $\mu\mapsto P[\mu]$ is a bijection between the set of nonnegative finite measures on the sphere $\mathcal{S}_d$ and the set of nonnegative harmonic functions in the ball B_{d+1} we can also obtain the following result.

Theorem 5.6. *For quasi-all nonnegative harmonic functions h in the unit ball B_{d+1}, for any $\beta\in[0,d]$,*

$$\dim_{\mathcal{H}}\big(E(\beta,h)\big) = d-\beta$$

where $E(\beta,h)$ is defined here by $E(\beta,h) = \left\{y\in\mathcal{S}_d\ ;\ \limsup_{r\to 1}\dfrac{\log h(ry)}{-\log(1-r)} = \beta\right\}$.

The set $\mathcal{H}^+(B_{d+1})$ of nonnegative harmonic functions in the unit ball B_{d+1} is endowed with the topology of the locally uniform convergence. It is a closed cone in the complete vector space of all continous functions in the ball. So it satisfies the Baire's property.

Proof of Theorem 5.6. We begin with the following lemma.

Lemma 5.7. *The set of nonnegative functions which are continuous in the closed unit ball $\overline{B_{d+1}}$ and harmonic in the open ball B_{d+1} is dense in $\mathcal{H}^+$.*

Proof. Let $h \in \mathcal{H}^+$ and $\rho_n < 1$ be a sequence of real number that increases to 1. Set $f_n(\xi) = h(\rho_n \xi)$ if $\xi \in \mathcal{S}_d$ and $h_n(x) = h(\rho_n x) = P[f_n](x)$ if $x \in B_{d+1}$. The functions h_n are nonnegative, harmonic and continuous on the closed ball $\overline{B_{d+1}}$. Moreover, let $\rho < 1$. The uniform continuity of h in the closed ball $\bar{B}(0,\rho) = \{x\ ;\ \|x\| \le \rho\}$ ensures that h_n converges uniformly to h in the compact set $\bar{B}(0,\rho)$. □

We can now prove Theorem 5.6, using the same way as in Theorem 1.2. Let $(g_n)_{n\ge1}$ be a dense sequence in the set of nonnegative continuous functions in $\mathcal{S}_d$. Lemma 5.7 ensures that the sequence $(P[g_n])_{n\ge1}$ is dense in $\mathcal{H}^+$. Moreover, we can suppose that $\|g_n\|_\infty \le n$ so that by the maximum principle, $0 \le P[g_n](x) \le n$ for any $x \in B_{d+1}$. Let $(f_n)_{n\ge1}$ be the sequence given by Lemma 5.3 and observe that if $\|x\| \le \rho$,

$$\left|\frac{1}{n}P[f_n](x)\right| \le \frac{2}{n(1-\rho)^d}\|f_n\|_1 = \frac{2}{n(1-\rho)^d}.$$

It follows that $\frac{1}{n}P[f_n]$ goes to 0 in $\mathcal{H}^+$. Define

$$h_n = P[g_n] + \frac{1}{n}P[f_n]$$

so that $(h_n)_{n\ge1}$ is always dense in $\mathcal{H}^+$. Let $\alpha > 1$, $y \in D_{n,\alpha}$ and $r_n = 1 - 2^{-n}$. Lemma 5.3 ensures that

$$h_n(r_n y) \quad \ge \quad C\frac{2^{(n-N_{n,\alpha})d}}{n^2} - n.$$

We can define

$$A = \bigcap_{l\ge1}\bigcup_{n\ge l}\left\{h \in \mathcal{H}^+\ ;\ \sup_{\|x\|\le r_n}|h(x) - h_n(x)| < 1\right\}$$

which is a dense G_δ-set in $\mathcal{H}^+$ and we can conclude like in the proof of Theorem 1.2. □

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