REMARKS ON A COMPUTER PROGRAM WRITTEN BY KARL JOHANN SCHMIDT

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The aim of this review is to describe the mathematical background which might be helpful in order to appreciate the program of Karl Johann Schmidt. The program is part of Karl Johann Schmidt's diploma thesis (see [12]).

Let M be a 3-manifold obtained by Dehn surgery on a two bridge knot. The computer program calculates the representations $\rho : \pi_1(M) \to \mathrm{SU}(2)$ of the fundamental group in SU(2) and the Chern-Simons invariants $cs(\rho)$ for the corresponding flat SU(2)-connections. The calculation is based on the method developed by Paul Kirk and Eric Klassen (see [10, Section 5] for details). Of course this is not an exact calculation but the tolerance can be chosen (see [13]).

As a by-product one obtains the SO(3) and SU(2) representation curves for 2-bridge knot groups investigated by G. Burde and me (see [5] and [8]). Moreover, it is possible to get the *peripheral curve* of the knot, i.e. the image of the representation space on the *pillowcase*.

The program is written in C++ and was originally implemented on a SUN workstation. A Macintosh version realized by Alexander Pilz from Siegen is now available. The graphics are printed on the screen (X-windows) as well as in a Postscript EPS-file.

Each section of this review is dedicated to a feature of the program. Every section starts with an **outline** and an (hopefully instructive) **example**. Further information and references can be found in the **details**.

1. What is a 2–bridge knot?

Outline: To each rational number p/q, with $p > 0$, p odd and p, q coprime, there is associated the 2–bridge knot $\mathfrak{b}(p, q)$. In order to draw the knot $\mathfrak{b}(p, q)$ we choose a continued fraction

$$\frac{p}{q} = c_1 + \cfrac{1}{c_2 + \cdots + \cfrac{1}{c_k}}$$

1991 *Mathematics Subject Classification.* 57M25, 57M05, 57R57.

Key words and phrases. Chern-Simons invariants, gauge theory, knot groups, group representations, SU(2).

where $|c_i| \neq 0$. The knot $\mathfrak{b}(p,q)$ is then given by *Conway's normal form* (see figure 1) where c_i indicates $|c_i|$ crossing points with sign $\epsilon_i = c_i/|c_i|$. The knots $\mathfrak{b}(3,1)$ and $\mathfrak{b}(7,3)$ are shown in figure 2.

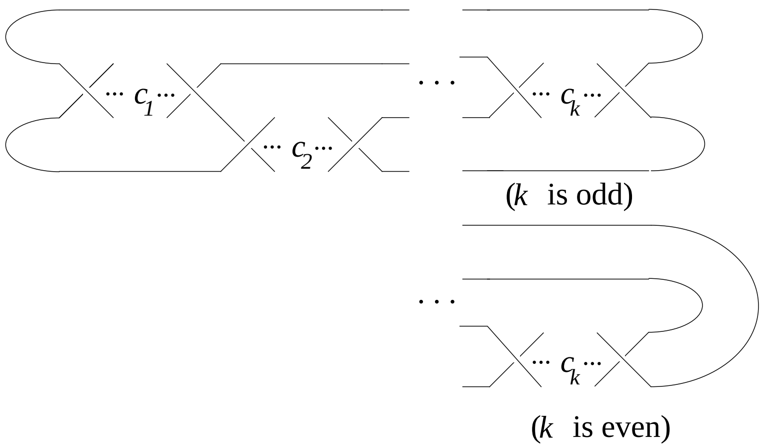


FIGURE 1. Conway's normal form of a 2–bridge knot.

Example:

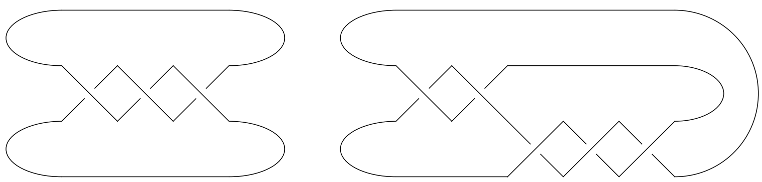


FIGURE 2. The 2–bridge knots $\mathfrak{b}(3,1)$ and $\mathfrak{b}(7,3)$.

Details: 2–bridge knots and links – sometimes called 4–plats (Viergeflechte) – were first investigated by C. Bankwitz and H. G. Schumann (see [2]) where they are shown to be alternating and invertible. The classification of 2–bridge knots and links is due to H. Schubert (see [14]). For general information about 2–bridge knots and links see [4], [9] and [15].

2–bridge knots are classified by their 2-fold branched covering – a method due to H. Seifert. It is easy to prove (see [4, Chapter 12]) that the 2-fold branched covering space of $\mathfrak{b}(p,q) \subset S^3$ is the lens spaces $L(p,q)$. Therefore, we obtain a classification of 2-bridge knots by the classification of lens spaces: $\mathfrak{b}(p,q) \cong \mathfrak{b}(p',q')$ if and only if $p' = p$ and $q' \equiv q^{\pm 1} \bmod p$.

2. SU(2)- and SO(3)-representations of 2–bridge knot groups

Outline: Equivalence classes of SU(2)- and SO(3)-representations of 2–bridge knot groups are determined by two real parameters. The space of equivalence classes of such representations is a semi-algebraic set in $\mathbb{R}^2$. For a given 2–bridge knot the program draws these semi-algebraic sets.
Example:

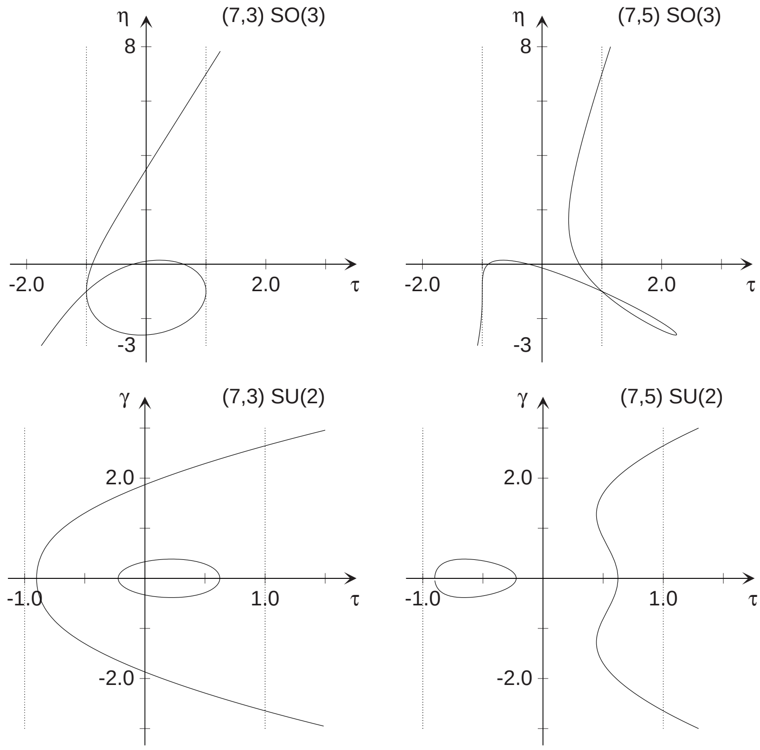


Figure 3. The SO(3)-representation curves $\widehat{\mathcal{C}}_{7,3}$, $\widehat{\mathcal{C}}_{7,5}$ and the SU(2)-representation curves $\mathcal{C}_{7,3}$ and $\mathcal{C}_{7,5}$.

Details: Given $\mathfrak{b}(p,q) \subset S^3$ we denote the fundamental group of its complement by $G(p,q)$. Using the normal form of the 2–bridge knots we can get a Wirtinger presentation:

$$G(p,q) = \langle S, T \mid L_S S = T L_S \rangle, \quad L_S = S^{\varepsilon_1} T^{\varepsilon_2} \dots S^{\varepsilon_{p-2}} T^{\varepsilon_{p-1}}$$

where $\varepsilon_i = (-1)^{[iq/p]}$ (for every real x let $[x]$ be the greatest integer n such that $n \leq x$).

Notice that $\mathrm{SU}(2)$ is the same as the unit quaternions $\mathrm{Sp}(1) \subset \mathbb{H}$; the identification is given by

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mapsto a + b\boldsymbol{j}.$$

The set of pure quaternions will be denoted by $\mathbb{E} \cong \mathbb{R}^3$. Every $q \in \mathrm{Sp}(1)$ can be written in *polar coordinates* $q = \cos\frac{\varphi}{2} + \sin\frac{\varphi}{2} Q$ where $0 \leq \varphi \leq 2\pi$ and $Q \in \mathrm{Sp}(1) \cap \mathbb{E}$ is a pure unit quaternion. Let (Q, φ) be short for $\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2} Q$.

There is a twofold covering $\delta : \mathrm{Sp}(1) \to \mathrm{SO}(3)$ given by $(P, \varphi) \mapsto \delta(P, \varphi)$. The element $\delta(P, \varphi) \in \mathrm{SO}(3)$ is a rotation of angle φ with axis P.

Let $G = G(p, q)$ be a given 2–bridge knot group. We now consider non-abelian representations $\varrho : G \to \mathrm{SU}(2)$ resp. $\widehat{\varrho} : G \to \mathrm{SO}(3)$.

$$\varrho : \begin{cases} S \mapsto (P, \varphi) \\ T \mapsto (Q, \varphi) \end{cases} \quad \text{resp.} \quad \widehat{\varrho} : \begin{cases} S \mapsto \delta(P, \varphi) \\ T \mapsto \delta(Q, \varphi) \end{cases} \qquad (P \neq Q) \tag{1}$$

which assign the same angle φ to S and T. Every representation $\widehat{\varrho} : G \to \mathrm{SO}(3)$ factors through $\mathrm{SU}(2)$ (see [5]). Moreover, S and T are conjugate in $G(p, q)$ and so $S \mapsto (P, \varphi)$ and $T \mapsto (Q, \varphi)$ holds for every representation $\varrho : G \to \mathrm{SU}(2)$.

The equivalence class of ϱ (resp. $\widehat{\varrho}$), given by (1), is determined by the parameters $\tau := \langle P, Q \rangle = \cos\psi$, $\psi = \sphericalangle(P, Q)$ and $\gamma = \cot\frac{\varphi}{2}$ (resp. $\eta := \gamma^2$). Here $\langle P, Q \rangle$ denotes the scalar product in $\mathbb{E}$ (for details see [5]).

There is a restriction of the parameters and we denote by $D \subseteq \mathbb{R}^2$ and $\widehat{D} \subseteq \mathbb{R}^2$ the following subsets of $\mathbb{R}^2$

$$D := \{(\tau, \gamma) \in \mathbb{R}^2 \mid \ -1 < \tau < 1\}, \quad \widehat{D} := \{(\tau, \eta) \in \mathbb{R}^2 \mid \eta \geq 0,\ -1 < \tau < 1\}.$$

Theorem 1. (G. Burde) *Given $G(p, q)$, there exists a polynomial $z_{p,q}(\tau, \eta) \in \mathbb{Z}[\tau, \eta]$, $\deg z_{p,q} = (p-1)/2$, such that a pair $(\tau_0, \gamma_0) \in D$ (resp. $(\tau_0, \eta_0) \in \widehat{D}$) determines an equivalence class of* $\mathrm{SU}(2)$*- (resp.* $\mathrm{SO}(3)$*-) representations, given by (1), if and only if $z_{p,q}(\tau_0, \gamma_0^2) = 0$ (resp. $z_{p,q}(\tau_0, \eta_0) = 0$).*

Proof. An algorithm for calculating $z_{p,q}(\tau, \eta)$ is given by G. Burde in [5]. $\square$

Definition 1. We call the affine real algebraic sets

$$\mathcal{C}_{p,q} = \{(\tau, \gamma) \in \mathbb{R}^2 \mid z_{p,q}(\tau, \gamma^2) = 0\} \text{ resp. } \widehat{\mathcal{C}}_{p,q} = \{(\tau, \eta) \in \mathbb{R}^2 \mid z_{p,q}(\tau, \eta) = 0\}$$

the $\mathrm{SU}(2)$- (resp. $\mathrm{SO}(3)$-) representation curve of $\mathfrak{b}(p, q)$.

For more informations about the representation curves of 2–bridge knots see [5] and [8].

3. The peripheral curve

Outline: We obtain the *peripheral curve* of a knot by restricting the SU(2)-representations of the knot group on the peripheral subgroup. The peripheral curve is contained in the *pillowcase*. For a given 2–bridge knot the program draws its peripheral curve.

Example:

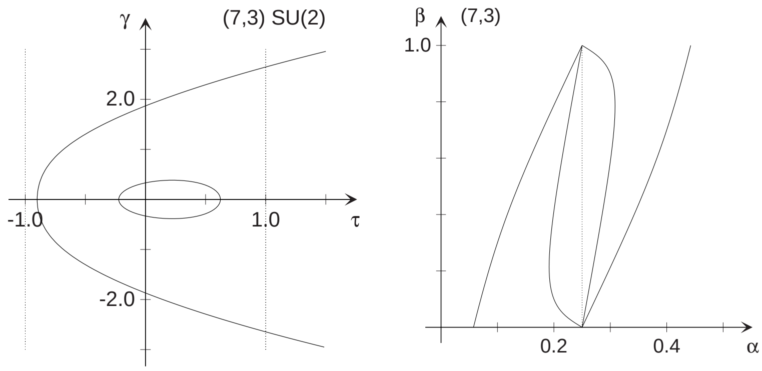


FIGURE 4. The curve $\mathcal{C}_{7,3}$ and the *peripheral curve* of the knot $\mathfrak{b}(7,3)$ on the pillowcase.

Details: While the representation curves are not invariants for the knot, $\mathcal{C}_{7,3} \neq \mathcal{C}_{7,5}$ but $3 \cdot 5 \equiv 1 \bmod 7$, there is the *peripheral curve* which is determined by the knot type only.

Let in general $\mathfrak{k} \subset S^3$ be a non trivial knot and denote by X the complement of a regular neighborhood of $\mathfrak{k}$ in S^3. The boundary ∂X is a torus and since $\mathfrak{k}$ is non trivial the inclusion induces an injection $\pi_1(\partial X) \to \pi_1(X)$ (the peripheral subgroup $\pi_1(\partial X) \subset \pi_1(X)$ is unique up to conjugation).

A given representation $\rho\colon\ \pi_1(X) \to \mathrm{SU}(2)$ can be restricted to the peripheral subgroup $res(\rho) := \rho\,|_{\pi_1(\partial X)}\colon\ \pi_1(\partial X) \to \mathrm{SU}(2)$. Let μ be the *meridian* and λ the *longitude* of the knot $\mathfrak{k}$. The pair (μ, λ) is unique up to common conjugation and gives us a distinguished pair of generators for $\pi_1(\partial X)$. The group $\pi_1(\partial X) \cong \mathbb{Z}\mu \oplus \mathbb{Z}\lambda$ is abelian and the image of each homomorphism $\varrho\colon\ \pi_1(\partial X) \to \mathrm{SU}(2)$ is contained (after conjugation) in the maximal torus $S^1 \subset \mathrm{SU}(2)$. Therefore, ϱ is determined by two real numbers α, β, $0 \le \alpha, \beta \le 1$,

$$\varrho\colon\ \mu \mapsto e^{2\boldsymbol{i}\alpha\pi} \text{ and } \varrho\colon\ \lambda \mapsto e^{2\boldsymbol{i}\beta\pi}.$$

It is clear that the two representations determined by (α, β) and $(1-\alpha, 1-\beta)$ are equivalent (conjugation by $\boldsymbol{j}$). The space of conjugacy classes of representations of $\pi_1(\partial X)$ into SU(2) is hence parameterized by the *pillowcase* P (see figure 5).

$$P := (S^1 \times S^1)/\sim$$

where $(e^{2i\alpha\pi}, e^{2i\beta\pi}) \sim (e^{-2i\alpha\pi}, e^{-2i\beta\pi})$.

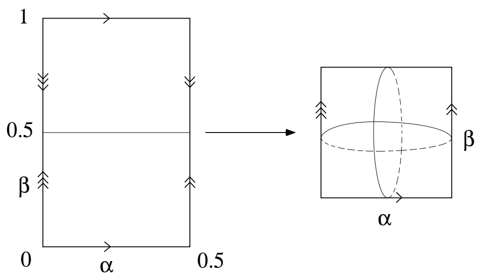


FIGURE 5. The *pillowcase* and its parameterization.

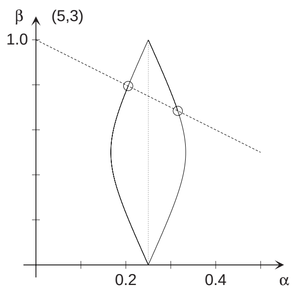
The restriction defines a map $res : \mathcal{R} \to P$ where $\mathcal{R}$ is the space of conjugacy classes of representations of $\pi_1(X)$ to SU(2). The image $res(\mathcal{R}) \subset P$ is called the *peripheral curve* of the knot $\mathfrak{k}$. For more information and related topics see [6] and [7].

For a given knot $\mathfrak{b}(p,q)$ we obtain a map $res\colon \mathcal{C}_{p,q} \cap D \to P$. The program draws the peripheral curve $res(\mathcal{C}_{p,q} \cap D) \subset P$ (see figure 4).

4. Chern-Simons invariants

Outline: Let M be a three manifold obtained by Dehn surgery on a 2–bridge knot. The computer program calculates the representations $\rho : \pi_1(M) \to \mathrm{SU}(2)$ of the fundamental group in SU(2) and the Chern-Simons invariants $cs(\rho)$ for the corresponding flat SU(2)-connections.

Example:



p,q	r/s	c	τ	γ	α	β	$\int \beta\alpha'$	cs
$5,3$	1	1	-0.75892	0.28882	0.20525	0.79475	0.10978	0.14881=25/168
$5,3$	1	1	0.073574	-0.43770	0.31566	-0.31566	0.09006	0.72024=121/168

FIGURE 6. The representations of the manifold obtained by 1-surgery on the knot $\mathfrak{b}(5,3)$ and the data produced by the program.

Details: Let $\mathfrak{b}(p,q)$ be a given 2–bridge knot and let r/s be a rational number (r,s coprime). We are interested in the closed 3-dimensional manifold M which is obtained by r/s-surgery on $\mathfrak{b}(p,q)$. We denote the 2-dimensional disk in $\mathbb{R}^2$ by D^2.

$$M := X(p,q) \cup_h (S^1 \times D^2)$$

here $X(p,q)$ denotes the complement of a regular neighborhood of $\mathfrak{b}(p,q) \subset S^3$ and $h\colon \partial X(p,q) \to \partial(S^1 \times D^2)$ is a homeomorphism such that $h(\mu^r\lambda^s) = \{*\} \times \partial D^2$ (note that $\mu^r\lambda^s$ is a simple closed curve in $\partial X(p,q)$).

We have a surjection $\pi_1(X(p,q)) \to \pi_1(M)$ by van Kampen's theorem. Hence the $\mathrm{SU}(2)$-representations of $\pi_1(M)$ are exactly the representations $\rho\colon \pi_1(X(p,q)) \to \mathrm{SU}(2)$ which satisfies $\rho(\mu^r\lambda^s) = \mathbf{1}$. Given a representation $\rho_0\colon \pi_1(M) \to \mathrm{SU}(2)$ we denote by $cs(\rho_0) \in \mathbb{Q}/\mathbb{Z}$ the Chern–Simons invariant of the corresponding flat connection on the trivial bundle $M \times \mathrm{SU}(2)$. For more information about gauge theory see [1], [3] and [10].

The program calculates the Chern–Simons invariant $cs(\rho_0)$ by using the method developed by Paul Kirk and Eric Klassen:

1. The program determines the coordinates (τ,γ) of those points of $\mathcal{C}_{p,q}$ which correspond to the representations of $\pi_1(M)$ (see figure 6).
2. Given a representation $\rho_1\colon \pi_1(M) \to \mathrm{SU}(2)$ the program constructs a piecewise smooth path $\rho_t\colon \pi_1(X(p,q)) \to \mathrm{SL}_2(\mathbb{C})$, $0 \le t \le 1$, connecting the trivial representation ρ_0 with ρ_1. Here the $\mathrm{SL}_2(\mathbb{C})$-representation polynomial of R. Riley is used (see [11]). This step is more complicated than it seems to be (see [12] and [13] for more information).

   Following Kirk and Klassen the program calculates $\alpha(t)$ and $\beta(t)$ such that

   $$\rho_t\colon\ \mu \mapsto e^{2i\alpha(t)\pi} \text{ and } \rho_t\colon\ \lambda \mapsto e^{2i\beta(t)\pi}.$$

3. In the last step the integral $\int_0^1 \beta(t)\alpha'(t)\,dt$ is calculated by a numerical integration. According to Theorem 4.2 of [10] the Chern–Simons invariant is given by

   $$cs(\rho_1) = -2\int_0^1 \beta(t)\alpha'(t)\,dt - ur\,\alpha^2(1) - vs\,\beta^2(1) - 2us\,\alpha(1)\beta(1)$$

   where $u,v \in \mathbb{Z}$ are integers such that $rv - su = 1$. Details can be found in [10, Theorem 4.2 and Section 5, application to surgery on the figure 8 knot] and [12].

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