

PSEUDODIFFERENTIAL OPERATORS ON MANIFOLDS WITH FIBRED CORNERS

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ABSTRACT. One way to geometrically encode the singularities of a stratified pseudomanifold is to endow its interior with an iterated fibred cusp metric. For such a metric, we develop and study a pseudodifferential calculus generalizing the Φ -calculus of Mazzeo and Melrose. Our starting point is the observation, going back to Melrose, that a stratified pseudomanifold can be ‘resolved’ into a manifold with fibred corners. This allows us to define pseudodifferential operators as conormal distributions on a suitably blown-up double space. Various symbol maps are introduced, leading to the notion of full ellipticity. This is used to construct refined parametrices and to provide criteria for the mapping properties of operators such as Fredholmness or compactness. We also introduce a semiclassical version of the calculus and use it to establish a Poincaré duality between the K -homology of the stratified pseudomanifold and the K -group of fully elliptic operators.

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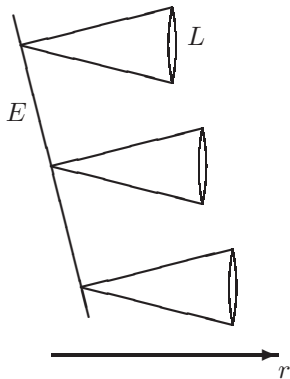
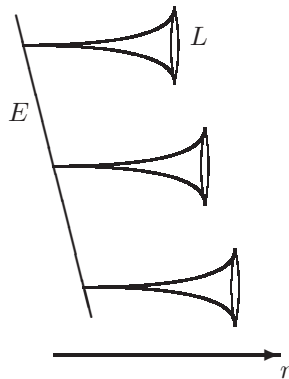
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INTRODUCTION

To study linear elliptic equations on singular spaces, it is very helpful to have a pseudodifferential calculus adapted to the geometry of the singularities. Indeed, such a tool allows one to construct refined parametrices to geometric operators like the Laplacian, leading to a precise description of the space of solutions and typically having important consequences and applications in spectral theory, scattering theory, index theory and regularity theory. This has also applications to study certain non-linear elliptic equations, see for instance [26],[15], [45] for recent works

in that direction. Over the years, various types of pseudodifferential calculi have been introduced on non-compact and singular spaces, see for instance [24], [8], [29], [48], [22], [23] [20], [18], [13] and [44]. Such a diversity of calculi comes from the fact that different types of singularities usually require quite different treatments.

Still, many of the examples above are concerned with a particular class of singular spaces: stratified pseudomanifolds. The notion of stratified pseudomanifold is relatively easy to describe and has the advantage of including many important examples of singular spaces, going from manifolds with corners to algebraic varieties. One could therefore hope for a relatively uniform treatment of pseudodifferential operators in this context. However, it is necessary to first choose a Riemannian metric geometrically encoding the singularities. There are two natural choices. To present these two choices, let us first consider a stratified pseudomanifold of depth one, that is, with only one singular stratum, see Figure 1 and Figure 2.

FIGURE 1. g_{ed} FIGURE 2. g_{fc}

The first choice, going back to Cheeger [7], is to consider an incomplete edge metric, a prototypical example being a metric which in a neighborhood of the singular stratum takes the form

$$(1) \quad g_{ed} = dr^2 + g_E + r^2 g_L,$$

where r is the distance to the singular stratum, g_E is a Riemannian metric on the singular stratum (the edge) and g_L is a choice of metric on the link. In this setting, a pseudodifferential calculus was developed independently by Mazzeo [22] and Schulze [48]. In [22], the metric which is really used as a starting point is in fact the conformally related metric

$$(2) \quad \tilde{g}_{ed} = \frac{g_{ed}}{r^2} = \frac{dr^2}{r^2} + \frac{g_E}{r^2} + g_L,$$

a complete edge metric, which has the virtue of defining a Lie algebra of vector fields ‘generating’ the pseudodifferential calculus.

Alternatively, one can consider a fibred cusp metric to encode the singularity, which is a certain type of complete Riemannian metric of finite volume on the

regular stratum. A prototypical example of such metric is one which near the singular stratum takes the form

$$(3) \quad g_{\text{fc}} = \frac{dr^2}{r^2} + g_E + r^2 g_L.$$

For such a metric, a pseudodifferential calculus was introduced by Mazzeo and Melrose [23] starting with a Lie algebra of smooth vector fields associated to the conformally related metric

$$(4) \quad g_{\text{fb}} = \frac{g_{\text{fc}}}{r^2} = \frac{dr^2}{r^4} + \frac{g_E}{r^2} + g_L.$$

Both (1) and (3) have analogs on general stratified pseudomanifolds by iterating the definition. The generalization of (1) is called an iterated edge metric [1]. An important source of examples of iterated edge metrics is given by certain constant curvature metrics [26] and by Kähler-Einstein metrics singular along a divisor [15]. On the other hand, we call the analog of (3) for a general stratified pseudomanifold an *iterated fibred cusp metric*, see Definition 2.3 below. For stratified pseudomanifolds of depth one, natural examples of such metrics are given by certain fibred cusp Kähler-Einstein metrics, see [45].

For iterated edge metrics, an associated pseudodifferential calculus has been introduced in [37] and [38] for operators of order zero and was used in [36]. There is also a recent survey [47] by Schulze giving a nice description of how his methods can be adapted to stratified pseudomanifolds of higher depth. Adopting a Lie groupoid point of view, one can obtain a pseudodifferential calculus by applying the general method of [39] and [2], which works for both iterated edge metrics and iterated fibred cusp metrics. This latter approach is suitable for certain applications in index theory, but the properness condition on the support of the operators makes it less appealing for the construction of refined parametrices. Still, in certain cases, this can be avoided by introducing a length function, see [43].

In this paper, we propose to systematically develop and study a calculus of pseudodifferential operators on stratified pseudomanifolds equipped with an iterated fibred cusp metric. We call it the \mathcal{S} -calculus. Our approach takes its inspiration from [23], which deals with the case of a stratified pseudomanifold of depth 1. In particular, we start with a Lie algebra of smooth vector fields associated to **iterated fibred corner metrics**, a type of metrics conformally related to iterated fibred cusp metrics. To be able to consider stratified pseudomanifolds of arbitrary depth, our starting point is the idea, going back to Melrose (see [1]), that a stratified pseudomanifold can be resolved by a manifold with fibred corners. This allows us to use blow-up techniques in a systematic way to construct the double space on which the Schwartz kernels of the operators can naturally be defined.

To prove that this pseudodifferential calculus is closed under composition, we diverge from [23] and follow an approach closer in spirit to [18]. Beside the ‘usual’ principal symbol, we introduce a ‘noncommutative’ symbol for each singular stratum of the stratified pseudomanifold by restricting on a corresponding front face in the double space. This lead to a simple Fredholm criterion for polyhomogeneous pseudodifferential operators: an operator is Fredholm when acting on suitable Sobolev spaces if and only if it is elliptic and its ‘noncommutative’ symbols are invertible for each stratum. We say such operators are fully elliptic. For fully elliptic operators, we are able to construct a refined parametrix giving rise to a

corresponding regularity result. This refined parametrix can also be used to show that our calculus is spectrally invariant, namely, that an invertible operator (when acting on suitable Sobolev spaces) has its inverse also contained in the calculus.

Along the way, we have a parallel discussion that keeps track of the underlying Lie groupoid and relates our approach with the one of [39] and [2]. This becomes particularly useful at the end of the paper, where we establish a Poincaré duality between the fully elliptic \mathbb{S} -operators and the K -homology of the stratified pseudomanifold. In [36], such a Poincaré duality was obtained using the pseudodifferential operators of [37] and [38]. Using instead groupoids, the first two authors introduced in [10] the noncommutative tangent space of a stratified pseudomanifold and showed its K -theory is Poincaré dual to the K -homology of the stratified pseudomanifold. In fact, they showed more generally that the C^* -algebra of the noncommutative tangent space is dual in the sense of KK -theory to the C^* -algebra of continuous functions of the underlying stratified pseudomanifold.

A key feature of our approach is the introduction of the semiclassical \mathbb{S} -double space and its associated semiclassical \mathbb{S} -calculus. This allows us to define a continuous family groupoid $T^{\text{FC}}X$ playing the role in our context of the noncommutative tangent space of [10]. By looking at the associated algebra of pseudodifferential operators, we can then provide a simple way of relating classes of fully elliptic \mathbb{S} -operators with elements of the K -theory of $T^{\text{FC}}X$ (see Theorem 10.6 below). This allows us to use a hybrid combination of the operator theoretic methods of [36] (see also [33]) and the groupoid approach of [10] to obtain our Poincaré duality result in KK -theory, Theorem 11.1. In Theorem 11.4, we also provide an interpretation of this result in terms of a quantization map for full symbols of fully elliptic \mathbb{S} -operators, relating in this way the points of view of [10] and [36].

The paper is organized as follows. In Section 1, we introduce the definition of manifolds with fibered corners and recall from [1] how they can be used to ‘resolve’ stratified pseudomanifolds. In Section 2, we introduce a natural class of vector fields defined on a manifold with fibered corners X . This leads to the notion of \mathbb{S} -pseudodifferential operators in Section 3. In Section 4, we review the definition of groupoid and explain its relevance to the present context. In Section 5, we describe how \mathbb{S} -pseudodifferential operators act on smooth functions. Section 6 is about suspended operators, which are used in Section 7 to introduce various symbol maps for \mathbb{S} -operators. In section 8, we prove that the composition of two \mathbb{S} -operators is again a \mathbb{S} -operator. In Section 9, we introduce natural Sobolev spaces on which \mathbb{S} -operators act and provide criteria to determine when a \mathbb{S} -operator is bounded, compact or Fredholm. In Section 10, we introduce the semiclassical \mathbb{S} -double space and the associated semiclassical \mathbb{S} -calculus, as well as the Lie groupoid $T^{\text{FC}}X$. This is used to obtain a relationship between classes of fully elliptic \mathbb{S} -operators and elements of the K -theory of $T^{\text{FC}}X$. Finally, in Section 11, we establish a Poincaré duality in KK -theory between $T^{\text{FC}}X$ and the stratified pseudomanifold ${}^{\mathbb{S}}X$ associated to X and interpret it in terms of a quantization map.

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1. MANIFOLDS WITH FIBERED CORNERS AND STRATIFIED PSEUDOMANIFOLDS

Let X be a manifold with corners as defined in [27]. In particular, we are assuming that each boundary hypersurface $H \subset X$ is embedded in X . This means

that there exists a boundary defining function $x_H \in \mathcal{C}^\infty(X)$ such that $x_H^{-1}(0) = H$, x_H is positive on $X \setminus H$ and the differential dx_H is nowhere zero on \bar{H} . In such a situation, one can choose a smooth retraction $r_H : \mathcal{N}_H \rightarrow H$, where \mathcal{N}_H is a (tubular) neighborhood of H in X such that $(r_H, x_H) : \mathcal{N}_H \rightarrow H \times [0, \infty)$ is a diffeomorphism on its image. We call $(\mathcal{N}_H, r_H, x_H)$ a **tube system** for H . A smooth map $\phi : X \rightarrow Y$ between manifolds with corners is said to be a **fibration**¹ if it is a locally trivial surjective submersion.

Definition 1.1. Let X be a compact manifold with corners and H_1, \dots, H_k an exhaustive list of its set of boundary hypersurfaces M_1X . Suppose that each boundary hypersurface H_i is the total space of a smooth fibration $\pi_i : H_i \rightarrow S_i$ where the base S_i is also a compact manifold with corners. The collection of fibrations $\pi = (\pi_1, \dots, \pi_k)$ is said to be an **iterated fibration structure** if there is a partial order on the set of hypersurfaces such that

- (i) for any subset $I \subset \{1, \dots, k\}$ with $\bigcap_{i \in I} H_i \neq \emptyset$, the set $\{H_i \mid i \in I\}$ is totally ordered.
- (ii) If $H_i < H_j$, then $H_i \cap H_j \neq \emptyset$, $\pi_i(H_i \cap H_j) = S_i$ with $\pi_i : H_i \cap H_j \rightarrow S_i$ a surjective submersion and $S_{ji} := \pi_j(H_i \cap H_j) \subset S_j$ is one of the boundary hypersurfaces of the manifold with corners S_j . Moreover, there is a surjective submersion $\pi_{ji} : S_{ji} \rightarrow S_i$ such that on $H_i \cap H_j$ we have $\pi_{ji} \circ \pi_j = \pi_i$.
- (iii) The boundary hypersurfaces of S_j are exactly the S_{ji} with $H_i < H_j$. In particular if H_i is minimal, then S_i is a closed manifold.

A **manifold with fibred corners** is a manifold with corners X together with an iterated fibration structure π . A smooth map $\psi : X \rightarrow X'$ between two manifolds with fibred corners (X, π) and (X', π') is said to be a **diffeomorphism of manifolds with fibred corners** if it is a diffeomorphism of manifolds with corners and if for each $H_i \in M_1X$, there is $H'_{\mu(i)} \in M_1X'$ and a diffeomorphism $\psi_i : S_i \rightarrow S'_{\mu(i)}$ inducing a commutative diagram

$$\begin{array}{ccc} H_i & \xrightarrow{\psi} & H'_{\mu(i)} \\ \downarrow \pi_i & & \downarrow \pi'_{\mu(i)} \\ S_i & \xrightarrow{\psi_i} & S'_{\mu(i)}. \end{array}$$

Remark 1.2. With the previous notation, for any j , S_j is naturally a manifold with fibred corners. The hypersurfaces are the S_{ji} with fibration $\pi_{ji} : S_{ji} \rightarrow S_i$ for any i such that $H_i < H_j$. The same goes for the fibres of π_i . Precisely, if x belongs to S_i let $L_i^x := \pi_i^{-1}(x)$. Then L_i^x is a manifold with fibred corners, where the boundary hypersurfaces are the $L_i^x \cap H_j$ with $H_i < H_j$ and the corresponding fibration comes from the restriction of π_j . Notice that in the special case where H_i is maximal, the fibre L_i^x is a closed manifold.

Definition 1.3. A family of tube system $(\mathcal{N}_i, r_i, x_i)$ for H_i , $i = 1, \dots, k$ is an **iterated fibred tube system** of the manifold with fibred corners X if the following condition holds for $H_i < H_j$,

$$r_j(\mathcal{N}_i \cap \mathcal{N}_j) \subset \mathcal{N}_i, \quad x_i \circ r_j = x_i, \quad \pi_i \circ r_i \circ r_j = \pi_i \circ r_i \text{ on } \mathcal{N}_i \cap \mathcal{N}_j,$$

¹A more standard terminology would be *smooth fibre bundle*.

and the restriction to H_j of the function x_i is constant on the fibres of π_j .

If X is equipped with an iterated fibred tube system, then for each $H_i \in M_1X$, we have an induced iterated fibred tube system on each fibre of $\pi_i : H_i \rightarrow S_i$. Similarly, there is an induced iterated fibred tube system on the base S_i .

To see that manifolds with fibred corners always admit iterated fibred tube systems, it is useful to describe tube systems in terms of vector fields. Given a tube system $(\mathcal{N}_H, r_H, x_H)$ for the boundary hypersurface H , one can naturally associate to it a vector field $\xi_H \in \mathcal{C}^\infty(\mathcal{N}_H; TX)$ such that

$$(r_H, x_H)_* \xi_H = \frac{\partial}{\partial x_H}.$$

Clearly, the tube system can be recovered from this vector field by considering its flow. More generally, if $\eta_H \in \mathcal{C}^\infty(X; TX)$ is a vector field which is inner pointing and nowhere vanishing on H , but tangent to all other boundary hypersurfaces, we can construct a tube system $(\mathcal{N}'_H, r'_H, x'_H)$ such that

$$(r'_H, x'_H)_* \eta_H = \frac{\partial}{\partial x'_H}$$

by considering the flow of η_H for some short period of time E_H . Thus, to obtain an iterated fibred tube system, it suffices to associate a vector field ξ_{H_i} to each boundary hypersurface H_i in such a way that,

- (i) The restriction $\xi_{H_i}|_{H_i}$ is inner pointing and nowhere vanishing on H_i ;
- (ii) If $H_i < H_j$, then ξ_{H_i} is tangent to H_j and there is a vector field $\xi_{j_i} \in \mathcal{C}^\infty(S_j, TS_j)$ such that $(\pi_j)_*(\xi_{H_i}|_{H_j}) = \xi_{j_i}$, while ξ_{H_j} is tangent to the fibres of the fibration $\pi_i : H_i \rightarrow S_i$ on H_i . Moreover, in a neighborhood of $H_i \cap H_j$, we have that $[\xi_{H_i}, \xi_{H_j}] = 0$.

Indeed, the flows of these vector fields generates tube system for each boundary hypersurface. The condition that $[\xi_{H_i}, \xi_{H_j}] = 0$ insures that the flows of ξ_{H_i} and ξ_{H_j} commute, so that in particular $x_i \circ r_j = x_i$ near $H_i \cap H_j$, while requiring ξ_{H_j} to be tangent to the fibres of $\pi_i : H_i \rightarrow S_i$ insures that $\pi_i \circ r_i \circ r_j = \pi_i \circ r_j$ near $H_i \cap H_j$. On the other hand, the condition $(\pi_j)_*(\xi_{H_i}|_{H_j}) = \xi_{j_i}$ insures that $x_i|_{H_j}$ is constant on the fibres of $\pi_j : H_j \rightarrow S_j$. Thus, by shrinking the tube systems if necessary, we can insure they form an iterated fibred tube system.

Lemma 1.4. *A manifold with fibred corners always admit an iterated fibred tube system.*

Proof. By the discussion above, it suffices to find vector fields $\xi_{H_i} \in \mathcal{C}^\infty(X; TX)$ for each boundary hypersurface $H_i \in M_1X$ in such a way that conditions (i) and (ii) above are satisfied. This requires to construct the vectors fields ξ_{ij} on S_j as well.

Recall that the depth of X is the highest codimension of a boundary face of X . If X has depth zero, that is, if X is a smooth manifold, the lemma is trivial. We can thus proceed by induction on the depth of X to prove the lemma. In particular, for each i , the base S_i of the fibration π_i is a manifold with fibred corners of depth smaller than the one of X and we can assume we have vector fields satisfying (i) and (ii) on S_i . We can denote the vector field associated to the boundary face S_{j_i} of S_j by ξ_{j_i} . Proceeding by recurrence on the partial order of the S_i to construct these vector fields, starting with S_i minimal, we can assume furthermore that for $H_i < H_j < H_k$, we have

(iii) $(\pi_{kj})_* \xi_{ki}|_{S_{kj}} = \xi_{ji}$.

To construct the vector fields ξ_{H_i} on X , we can proceed by recurrence on M_1X starting with maximal elements. For a maximal element H_i , we just choose $\xi_{H_i} \in \mathcal{C}^\infty(X; TX)$ such that $\xi_{H_i}|_{H_i}$ is inner pointing and nowhere vanishing on H_i and is tangent to the fibres of $\pi_j : H_j \rightarrow S_j$ for $H_j \neq H_i$.

Suppose now that H_i is a hypersurface such that for all $H_j \in M_1X$ with $H_j > H_i$, the corresponding vector field ξ_{H_j} has been constructed in such a way that conditions (i) and (ii) hold. To construct ξ_{H_i} , we first define it on the maximal elements in $\{H \in M_1X ; H > H_i\}$. Let H_j be such a maximal element. We choose $\xi_{H_i}|_{H_j} \in \mathcal{C}^\infty(H_j; TH_j)$ in such a way that on H_j , we have that $(\pi_j)_*(\xi_{H_i}|_{H_j}) = \xi_{ji}$. This can be done by also requiring at the same time that $[\xi_{H_i}|_{H_j}, \xi_{H_k}|_{H_j}] = 0$ near $H_i \cap H_j \cap H_k$ for H_k such that $H_j > H_k > H_i$. Indeed, we can do so by first constructing $\xi_{H_i}|_{H_j}$ recursively on the boundary faces of H_j , starting with the boundary faces of smallest dimension, and extending the definition at the next level using the flow of vector fields ξ_{H_k} for $H_j > H_k > H_i$ whenever possible. Thanks to the fact condition (ii) is satisfied by the vector fields ξ_{H_k} , this can be achieved consistently. In this process, we also require that $\xi_{H_i}|_{H_j}$ be tangent to the fibres of $\pi_l : H_l \rightarrow S_l$ for $H_l < H_i$. Using the flow of ξ_{H_j} , we can then extend the definition of ξ_{H_i} to a neighborhood of $H_i \cap H_j$.

Doing this for all maximal elements in $\{H \in M_1X ; H > H_i\}$, we then proceed recursively to extend the definition of ξ_{H_i} in a neighborhood of the other hypersurfaces of $\{H \in M_1X ; H > H_i\}$. Thus, let $H_k > H_i$ be given and suppose that ξ_{H_i} has already been defined in a neighborhood of H_j for $H_j > H_i$ such that $H_j > H_k$. Then $\xi_{H_i}|_{H_k}$ is already defined in a neighborhood of $H_j \cap H_k$. As before, proceeding by recurrence on the boundary faces of H_k , we can extend the definition of $\xi_{H_i}|_{H_k}$ to all of H_k in such a way that $(\pi_k)_*(\xi_{H_i}|_{H_k}) = \xi_{ki}$ and $[\xi_{H_i}|_{H_k}, \xi_{H_l}|_{H_k}] = 0$ near $H_k \cap H_l$ for $H_k > H_l > H_i$. We can also require $\xi_{H_i}|_{H_k}$ to be tangent to the fibres of $\pi_l : H_l \rightarrow S_l$ for $H_l < H_i$. Using the flow of ξ_{H_k} , we can then extend the definition of ξ_{H_i} to a neighborhood of $H_k \cap H_i$. Because the already defined vector fields satisfy condition (ii), this extension is consistent with the previous ones.

Thus, proceeding recursively, we can extend the definition of ξ_{H_i} to a neighborhood of

$$\bigcup_{\{H \in M_1X ; H > H_i\}} H_i \cup H.$$

We can then extend this definition further in such a way that ξ_{H_i} is tangent to the fibres of $\pi_k : H_k \rightarrow S_k$ for $H_k < H_i$. This give us a vector field ξ_{H_i} which together with the already existing vector fields satisfies conditions (i) and (ii), completing the inductive step and the proof. \square

As observed by Melrose and subsequently described in [1], there is a correspondence between manifolds with fibered corners and stratified pseudomanifolds. For the convenience of the reader and in order to set up the notation, we will review the main features of this correspondence and refer to [1] for further details.

Let us first recall what are pseudomanifolds. We will use the notations and equivalent descriptions given by A. Verona in [53] and used by the first two authors in [10].

Let ${}^S X$ be a locally compact separable metrizable space. Recall that a C^∞ -stratification of ${}^S X$ is a pair (S, N) such that,

- (1) $S = \{s_i\}$ is a locally finite partition of ${}^S X$ into locally closed subsets of ${}^S X$, called the strata, which are smooth manifolds such that

$$s_0 \cap \bar{s}_1 \neq \emptyset \text{ if and only if } s_0 \subset \bar{s}_1.$$

In that case we will write $s_0 \leq s_1$ and $s_0 < s_1$ if moreover $s_0 \neq s_1$.

- (2) $N = \{{}^S \mathcal{N}_s, \pi_s, \rho_s\}_{s \in S}$ is the set of control data, namely ${}^S \mathcal{N}_s$ is an open neighborhood of s in ${}^S X$, $\pi_s : {}^S \mathcal{N}_s \rightarrow s$ is a continuous retraction and $\rho_s : {}^S \mathcal{N}_s \rightarrow [0, +\infty[$ is a continuous map such that $s = \rho_s^{-1}(0)$. The map ρ_s is either surjective or the constant zero function.

Moreover if ${}^S \mathcal{N}_{s_0} \cap s_1 \neq \emptyset$, then the map

$$(\pi_{s_0}, \rho_{s_0}) : {}^S \mathcal{N}_{s_0} \cap s_1 \rightarrow s_0 \times]0, +\infty[$$

is a smooth proper submersion.

- (3) For any strata s, t such that $s < t$, the set $\pi_t({}^S \mathcal{N}_s \cap {}^S \mathcal{N}_t)$ is included in ${}^S \mathcal{N}_s$ and the equalities

$$\pi_s \circ \pi_t = \pi_s \text{ and } \rho_s \circ \pi_t = \rho_s$$

hold on ${}^S \mathcal{N}_s \cap {}^S \mathcal{N}_t$.

- (4) For any two strata s_0 and s_1 the following two statements hold,

$$s_0 \cap \bar{s}_1 \neq \emptyset \text{ if and only if } {}^S \mathcal{N}_{s_0} \cap s_1 \neq \emptyset;$$

$${}^S \mathcal{N}_{s_0} \cap {}^S \mathcal{N}_{s_1} \neq \emptyset \text{ if and only if } s_0 \subset \bar{s}_1, s_0 = s_1 \text{ or } s_1 \subset \bar{s}_0.$$

A stratification gives rise to a filtration. Indeed, if ${}^S X_j$ is the union of strata of dimension $\leq j$, then,

$$\emptyset \subset {}^S X_0 \subset \dots \subset {}^S X_n = {}^S X.$$

We call n the dimension of ${}^S X$ and $X^\bullet := {}^S X \setminus {}^S X_{n-1}$ the regular part of ${}^S X$. The strata included in X^\bullet are called *regular* while strata included in ${}^S X \setminus X^\bullet$ are called *singular*. The set of singular (resp. regular) strata is denoted S_{sing} (resp. S_{reg}).

Definition 1.5. A **stratified pseudomanifold** is a triple $({}^S X, S, N)$ where ${}^S X$ is a locally compact separable metrizable space, (S, N) is a C^∞ -stratification on ${}^S X$ and the regular part X^\bullet is a dense open subset of ${}^S X$.

Given a stratified pseudomanifold, notice that the closure of each of its strata is also naturally a stratified pseudomanifold. Given a manifold with fibred corners X , there is a simple way of extracting a stratified pseudomanifold. Let H_1, \dots, H_k an exhaustive list of its boundary hypersurfaces and let $\pi = (\pi_1, \dots, \pi_k)$ be the iterated fibration structure on X . On X , consider the equivalence relation

$$x \sim y \iff x = y \text{ or } x, y \in H_i \text{ with } \pi_i(x) = \pi_i(y) \text{ for some } H_i.$$

We denote by ${}^S X$ the quotient space of X by the previous equivalence relation and $q : X \rightarrow {}^S X$ the quotient map. By construction, the restriction of q to $X \setminus \partial X$ is a homeomorphism. We claim that ${}^S X$ is naturally a stratified pseudomanifold. Indeed, for any $i \in I$ let $\sigma_i := \pi_i(H_i \setminus \bigcup_{H_k < H_i} H_k)$, that is $\sigma_i = S_i \setminus \partial S_i$. Then σ_i

is a manifold and $S = \{X \setminus \partial X, \sigma_i\}$ is a locally finite partition of ${}^S X$ into strata. Choose an iterated fibred tube system $\{\mathcal{N}_i, r_i, x_i\}$ for the manifold with fibred corners X . One can easily check that the map $(\pi_i \circ r_i, x_i) : r_i^{-1}(H_i \setminus \bigcup_{H_k < H_i} H_k) \rightarrow$

$\sigma_i \times [0, \infty[$ factors through a map ${}^S\mathcal{N}_i \rightarrow \sigma_i \times [0, \infty[$ where ${}^S\mathcal{N}_i$ is the image of $r_i^{-1}(H_i \setminus \bigcup_{H_k < H_i} H_k) \subset \mathcal{N}_i$ in ${}^S X$. This enables us to define a set of control data on ${}^S X$ making it a stratified pseudomanifold. We call ${}^S X$ the stratified pseudomanifold **associated to** the manifold with fibered corners X .

Conversely, to any stratified pseudomanifold, one can associate a natural manifold with corners. To see this, we need to recall the notion of depth, which is a good measure of complexity of a stratified pseudomanifold.

Definition 1.6. Let $({}^S X, \mathcal{S}, N)$ be a stratified pseudomanifold. Then the depth $d(s)$ of a stratum s is the biggest k such that one can find k different strata s_0, \dots, s_{k-1} such that

$$s_0 < s_1 < \dots < s_{k-1} < s_k := s.$$

The depth of the stratification (\mathcal{S}, N) of X is

$$d(X) := \sup\{d(s), s \in \mathcal{S}\}.$$

A stratum whose depth is 0 will be called minimal.

Remark 1.7. This definition is consistent with the notion of depth for manifolds with corners, which constitute a particular type of stratified pseudomanifolds. Moreover, if X is a manifold with fibered corners of depth k , then its associated stratified pseudomanifold ${}^S X$ also has depth k . Notice that different conventions for the depth are also common, see for instance [1].

Let $({}^S X, \mathcal{S}, N)$ be a stratified pseudomanifold. For any singular stratum s , set $L_s := \rho_s^{-1}(1)$. From [53], we know there is an isomorphism between ${}^S\mathcal{N}_s$ and $L_s \times [0, +\infty[\sim_s$ where \sim_s is the equivalence relation induced by $(x, 0) \sim_s (y, 0)$ when $\pi_s(x) = \pi_s(y)$. This local triviality around strata enables to make the unfolding process of [6] (see also [10] for a complete description). If s is minimal, one can construct a pseudomanifold

$$({}^S X \setminus s) \cup L_s \times [-1, 1] \cup ({}^S X \setminus s)$$

using the gluing coming from the trivialization of the neighborhood ${}^S\mathcal{N}_s$ of s . If M is the set of minimal strata of ${}^S X$ and $m = \bigcup_{s \in M} s$ is the union of the minimal strata of ${}^S X$, then one can more generally construct the double stratified pseudomanifold

$$2X = ({}^S X \setminus m) \cup \left(\bigsqcup_{s \in M} L_s \times [-1, 1] \right) \cup ({}^S X \setminus m)$$

by gluing $L_s \times [-1, 1]$ for $s \in M$ via the trivialization of the neighborhood ${}^S\mathcal{N}_s$ of s . Since all the minimal strata of ${}^S X$ are involved, notice that the depth of $2X$ is one less than the one of ${}^S X$. The stratified pseudomanifold $2X$ also comes with an involution τ_1 interchanging the two copies of ${}^S X \setminus m$ with fixed point set identified with $L_m = \bigsqcup_{s \in M} L_s \times \{0\}$. This fixed point set is naturally a stratified pseudomanifold and come with a surjective map $L_m \rightarrow m$ induced by the retractions in each neighborhood ${}^S\mathcal{N}_s$ for $s \in M$.

If ${}^S X$ has depth k , we can repeat this process k times to obtain a stratified pseudomanifold $2^k X$ of depth 0, in other words, a smooth manifold. The manifold $2^k X$ comes with a continuous surjective map $p : 2^k X \rightarrow {}^S X$. At the j th step of this unfolding process, we get a stratified pseudomanifold $2^j X$ with an involution τ_j and a fixed point set equipped with a surjective map as before. This lift to $2^k X$ to

give k involutions τ_1, \dots, τ_k with k fixed point sets given by smooth hypersurfaces R_1, \dots, R_k equipped with smooth fibrations on each of their connected components. The various bases of these fibrations are simply the smooth manifolds corresponding to the unfolded strata of ${}^S X$. The complement $2^k X \setminus (\bigcup_{j=1}^k R_j)$ consists of 2^k copies of X^\bullet . The closure of any one of these copies is naturally a manifold with corners ${}^{\text{FC}}X$ with boundary hypersurfaces given by parts of the hypersurfaces R_1, \dots, R_k and have corresponding induced fibrations with bases given by manifolds with corners. These fibrations give ${}^{\text{FC}}X$ a structure of manifold with fibred corners. We call ${}^{\text{FC}}X$ the manifold with fibred corners **associated to** the pseudomanifold ${}^S X$.

Up to the identifications described below, the two previous operations are mutually inverse. Precisely, starting with a stratified pseudomanifold ${}^S X$ and letting ${}^{\text{FC}}X$ be the associated manifold with fibred corners, we have for any x, y in ${}^{\text{FC}}X$ that $x \sim y$ if and only if $p(x) = p(y)$. In other words, the map p factors through a homeomorphism ${}^{\text{FC}}X / \sim \xrightarrow{\sim} {}^S X$ which is a diffeomorphism in restriction to each strata and with respect to the control data. Conversely, starting with a manifold with fibred corners X and letting ${}^S X$ be its associated stratified pseudomanifold, it can be seen that ${}^{\text{FC}}X$ is isomorphic to the original manifold with fibred corners X by noticing that the unfolding process described above has an analog for manifolds with fibred corners obtained by gluing along boundary hypersurfaces and with the same resulting smooth manifold $2^k X$.

2. VECTOR FIELDS ON MANIFOLDS WITH FIBRED CORNERS

Let X be a manifold with corners with H_1, \dots, H_k an exhaustive list of the boundary hypersurfaces of X . For each $i \in \{1, \dots, k\}$, let $x_i \in \mathcal{C}^\infty(X)$ be a boundary defining function for H_i . Recall that

$$\mathcal{V}_b(X) := \{\xi \in \Gamma(TX) ; \xi x_i \in x_i \mathcal{C}^\infty(X) \forall i\}$$

is the Lie algebra of b -vector fields. Notice in particular that a b -vector field $\xi \in \mathcal{V}_b(X)$ is necessarily tangent to all the boundary hypersurfaces of X . Suppose that $\pi = (\pi_1, \dots, \pi_k)$ is an iterated fibration structure on X .

Definition 2.1. The space $\mathcal{V}_S(X)$ of **S-vector fields** on the manifold with fibred corners (X, π) is

$$(2.1) \quad \mathcal{V}_S(X) := \{\xi \in \mathcal{V}_b(X); \xi|_{H_i} \text{ is tangent to the fibres of } \pi_i : H_i \rightarrow S_i \\ \text{and } \xi x_i \in x_i^2 \mathcal{C}^\infty(X) \forall i\}.$$

Remark 2.2. This definition depends on the choice of boundary defining functions. To lighten the discussion, we might sometime not mention explicitly the choice of boundary defining functions, but the use of **S-vector fields** and related concepts always presuppose such a choice has been made to start with. Moreover, we will usually assume the boundary defining functions are induced by an iterated fibred tube system.

As can be seen directly from the definition, $\mathcal{V}_S(X)$ is a $\mathcal{C}^\infty(X)$ module and is closed under the Lie bracket. It is therefore a Lie subalgebra of $\Gamma(TX)$. More generally, the space $\text{Diff}_S^k(X)$ of **S-differential operators of order k** is the space of operators on $\mathcal{C}^\infty(X)$ generated by $\mathcal{C}^\infty(X)$ and product of up to m elements of $\mathcal{V}_S(X)$.

Away from the boundary, a \mathcal{S} -vector field is just like a usual vector field in $\mathcal{C}^\infty(X; TX)$. On the other hand, near a point $p \in \partial X$, it is useful to introduce a system of coordinates in which \mathcal{S} -vector fields admits a simple description. To this end, let $H_{i_1}, \dots, H_{i_\ell}$ be the boundary hypersurfaces of X containing the point $p \in \partial X$. After relabelling if necessary, we can assume H_1, \dots, H_ℓ are the boundary hypersurfaces containing p and that

$$(2.2) \quad H_1 < H_2 < \dots < H_\ell.$$

Let $x_i \in \mathcal{C}^\infty(X)$ be the chosen boundary defining function for H_i . Consider a small neighborhood of p where for each $i \in \{1, \dots, \ell\}$, the fibration $\pi_i : H_i \rightarrow S_i$ restricts to be trivial. Consider then tuples of functions $y_i = (y_i^1, \dots, y_i^{k_i})$ for $1 \leq i \leq \ell$ and $z = (z^1, \dots, z^q)$ such that

$$(2.3) \quad (x_1, y_1, \dots, x_\ell, y_\ell, z)$$

form coordinates near p with the property that on H_i , $(x_1, \dots, x_{i-1}, y_1, \dots, y_i)$ induce coordinates on the base S_i of the fibration $\pi_i : H_i \rightarrow S_i$ such that π_i corresponds to the map

$$(2.4) \quad (x_1, \dots, \widehat{x}_i, \dots, x_\ell, y, z) \rightarrow (x_1, \dots, x_{i-1}, y_1, \dots, y_i),$$

where the $\widehat{}$ notation above the variable x_i means it is omitted. Thus, the coordinates $(x_{i+1}, \dots, x_\ell, y_{i+1}, \dots, y_\ell, z)$ restrict to give coordinates on the fibres of the fibration π_i . With such coordinates, the Lie algebra $\mathcal{V}_{\mathcal{S}}(X)$ is locally spanned over $\mathcal{C}^\infty(X)$ by the vector fields

$$(2.5) \quad \frac{\partial}{\partial z^j}, w_i x_i \frac{\partial}{\partial x_i}, w_i \frac{\partial}{\partial y_i^{n_i}},$$

for $j \in \{1, \dots, q\}$, $i \in \{1, \dots, \ell\}$, $n_i \in \{1, \dots, k_i\}$, where $w_i = \prod_{m=i}^\ell x_m$. Thus, in these coordinates, a \mathcal{S} -vector field $\xi \in \mathcal{V}_{\mathcal{S}}(X)$ is of the form

$$(2.6) \quad \xi = \sum_{i=1}^\ell a_i x_i w_i \frac{\partial}{\partial x_i} + \sum_{i=1}^\ell \sum_{j=1}^{k_i} b_{ij} w_i \frac{\partial}{\partial y_i^j} + \sum_{m=1}^q c_m \frac{\partial}{\partial z^m},$$

with $a_i, b_{ij}, c_m \in \mathcal{C}^\infty(X)$.

Since $\mathcal{V}_{\mathcal{S}}(X)$ is a $\mathcal{C}^\infty(M)$ -module, there exists a smooth vector bundle ${}^\pi TX \rightarrow X$ and a natural map $\iota_\pi : {}^\pi TX \rightarrow TX$ which restricts to an isomorphism on $X \setminus \partial X$ such that

$$(2.7) \quad \mathcal{V}_{\mathcal{S}}(X) = \iota_\pi \mathcal{C}^\infty(X; {}^\pi TX).$$

At a point $p \in X$, the fibre of ${}^\pi TX$ above p can be defined by

$$(2.8) \quad {}^\pi T_p X = \mathcal{V}_{\mathcal{S}} / \mathcal{I}_p \cdot \mathcal{V}_{\mathcal{S}},$$

where $\mathcal{I}_p \subset \mathcal{C}^\infty(X)$ is the ideal of smooth functions vanishing at p . Although the map $\iota_\pi : {}^\pi TX \rightarrow TX$ fails to be an isomorphism of vector bundles, notice that ${}^\pi TX$ is nevertheless isomorphic to TX , but not in a natural way.

Unless the hypersurface H_i has no boundary, notice that the kernel of the natural map ${}^\pi TX|_{H_i} \rightarrow TX|_{H_i}$ does not form a vector bundle over H_i . To obtain a vector bundle on H_i , we need to introduce an intermediate vector bundle in between ${}^\pi TX$ and TX . Let

$$X_i = X \cup_{H_i} X$$

be the manifold with corners obtained by gluing two copies of X along the boundary hypersurface H_i . The manifold X_i naturally has an iterated fibration structure induced from the one of X . Hoping this will lead to no confusion, we will also denote this iterated fibration by π . We then have a corresponding Lie algebra $\mathcal{V}_{\mathbb{S}}(X_i)$ of \mathbb{S} -vector fields as well as an associated \mathbb{S} -tangent vector bundle ${}^{\pi}TX_i$. Consider then the restriction of this vector bundle to one of the two copies of X inside X_i ,

$$\pi \setminus \pi_i TX = {}^{\pi}TX_i|_X.$$

Away from H_i , the vector bundle ${}^{\pi \setminus \pi_i}TX$ is canonically isomorphic to ${}^{\pi}TX$. However, seen as a subspace $\mathcal{C}^{\infty}(X; TX)$, the space of sections $\mathcal{C}^{\infty}(X; {}^{\pi \setminus \pi_i}TX)$ is slightly bigger than $\mathcal{V}_{\mathbb{S}}(X)$. We have in fact the following natural sequence of maps

$${}^{\pi}TX \longrightarrow {}^{\pi \setminus \pi_i}TX \xrightarrow{a_{\pi \setminus \pi_i}} TX.$$

When restricted to the boundary hypersurface H_i , the first map ${}^{\pi}TX \rightarrow {}^{\pi \setminus \pi_i}TX$ is such that its kernel ${}^{\pi}NH_i$ is naturally a vector bundle over H_i . This vector bundle is the pullback of a vector bundle on S_i . To see this, set

$${}^{\pi}TH_i := \{\xi \in {}^{\pi \setminus \pi_i}TX \Big|_{H_i} ; a_{\pi \setminus \pi_i}(\xi) \in TH_i\}.$$

The fibration $\pi_i : H_i \rightarrow S_i$ induces the short exact sequence

$$0 \longrightarrow {}^{\pi}T(H_i \setminus S_i) \longrightarrow {}^{\pi}TH_i \xrightarrow{(\pi_i)_*} \pi_i^* {}^{\pi}TS_i \longrightarrow 0,$$

where ${}^{\pi}TS_i$ is the \mathbb{S} -tangent bundle of S_i and ${}^{\pi}T(H_i \setminus S_i)$ is such that its restriction to each fibre F_i of the fibration π_i is the \mathbb{S} -tangent bundle ${}^{\pi}TF_i$ of that fibre. In particular, this induces a canonical identification $\pi_i^* {}^{\pi}TS_i = {}^{\pi}TH_i / {}^{\pi}T(H_i/S_i)$. Now, using the vector field $x_i^2 \frac{\partial}{\partial x_i}$ induced by a tube system for H_i , we have a natural decomposition

$$(2.9) \quad {}^{\pi}NH_i \cong ({}^{\pi}TH_i / {}^{\pi}T(H_i/S_i)) \times \mathbb{R}.$$

This means a tube system for H_i induces an isomorphism of vector bundles

$$(2.10) \quad {}^{\pi}NH_i \cong \pi_i^* {}^{\pi}NS_i$$

where ${}^{\pi}NS_i = {}^{\pi}TS_i \times \mathbb{R}$, and thus, a corresponding fibration

$$(2.11) \quad \begin{array}{ccc} F_i & \longrightarrow & {}^{\pi}NH_i \\ & & \downarrow (\pi_i)_* \\ & & {}^{\pi}NS_i. \end{array}$$

An **iterated fibred corner metric** (or **S-metric**) is a choice of metric g_{π} for the vector bundle ${}^{\pi}TX$. Via the map $\iota_{\pi} : {}^{\pi}TX \rightarrow TX$, it restricts to give a complete Riemannian metric on $X \setminus \partial X$. In the local coordinates (2.3), a special example of such a metric would be

$$(2.12) \quad g_{\pi} = \sum_{i=1}^{\ell} \frac{dx_i^2}{(x_i w_i)^2} + \sum_{i=1}^{\ell} \sum_{j=1}^{k_i} \frac{(dy_i^j)^2}{w_i^2} + \sum_{m=1}^q dz_m^2.$$

The Laplacian associated to a \mathbb{S} -metric is an example of \mathbb{S} -differential operator of order 2. Similarly, if ${}^{\pi}TX \rightarrow X$ has a spin structure, then the corresponding Dirac operator associated to a \mathbb{S} -metric is a \mathbb{S} -differential operator of order 1.

The **S-density bundle** associated to a manifold with fibred corners is the smooth real line bundle $\pi\Omega$ with fibre above $p \in X$ given by

$$(2.13) \quad \pi\Omega_p = \{u : \Lambda^{\dim X}(\pi T_p X) \rightarrow \mathbb{R} ; u(t\omega) = |t|u(\omega), \quad \forall \omega \in \Lambda^{\dim X}(\pi T_p X), t \neq 0\}.$$

A **S-density** is an element of $\mathcal{C}^\infty(X; \pi\Omega)$. In particular, the volume form of a **S-metric** is naturally a **S-density**. Via the map $\iota_\pi : \pi TX \rightarrow TX$, a **S-density** restricts to give a density on the interior of X . Let $\nu \in \mathcal{C}^\infty(X; \Omega)$ be a non-vanishing density on X , where $\Omega = \Omega(TX)$ is the density bundle associated to TX . On $X \setminus \partial X$, a **S-density** ν_π can be written in terms of ν as

$$(2.14) \quad \nu_\pi = \left(\prod_{i=1}^k x_i^{2+\dim S_i} \right)^{-1} h\nu, \quad \text{for some } h \in \mathcal{C}^\infty(X).$$

As indicated in the introduction, **S-metrics** are conformally related to another type of metrics geometrically encoding the singularities of the stratified pseudo-manifold.

Definition 2.3. On a manifold with fibred corners (X, π) with a boundary defining function x_H specified for each boundary hypersurface $H \in M_1 X$, an **iterated fibred cusp metric** g_{ifc} is a metric of the form

$$g_{\text{ifc}} = x^2 g_\pi, \quad x = \prod_{H \in M_1 X} x_H,$$

where g_π is a **S-metric**.

3. THE DEFINITION OF S-PSEUDODIFFERENTIAL OPERATORS

Let X be a manifold with fibred corners. Let H_1, \dots, H_k be an exhaustive list of its boundary hypersurfaces with $x_1, \dots, x_k \in \mathcal{C}^\infty(X)$ a choice of corresponding boundary defining functions and $\pi_i : H_i \rightarrow S_i$ the corresponding fibrations. Consider the Cartesian product $X^2 = X \times X$ with the projections $\text{pr}_R : X \times X \rightarrow X$ and $\text{pr}_L : X \times X \rightarrow X$ on the right and left factors respectively. Then $x'_i := \text{pr}_R^* x_i$ and $x_i := \text{pr}_L^* x_i$ are boundary defining functions for $X \times H_i$ and $H_i \times X$ respectively. The b -double space X_b^2 is the space obtained from X^2 by blowing up the p -submanifolds $H_1 \times H_1, \dots, H_k \times H_k$,

$$(3.1) \quad X_b^2 := [X^2; H_1 \times H_1; \dots; H_k \times H_k],$$

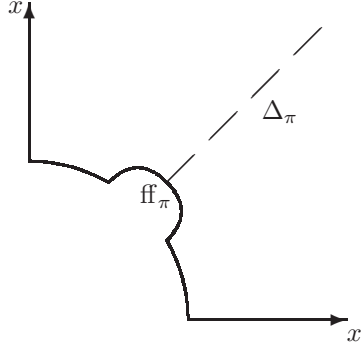
with blow-down map $\beta_b : X_b^2 \rightarrow X^2$. Near $H_i \times H_i$, this amounts to the introduction of polar coordinates

$$r_i := \sqrt{x_i^2 + (x'_i)^2}, \quad \omega_i = \frac{x_i}{r_i}, \quad \omega'_i = \frac{x'_i}{r_i},$$

where r_i is a boundary defining function for the ‘new’ hypersurface

$$B_i := \beta_b^{-1}(H_i \times H_i) \subset X_b^2$$

introduced by the blow-up, while near B_i , the functions ω_i and ω'_i are boundary defining functions for the lifts of the ‘old’ boundary hypersurfaces. Notice that since $H_i \times H_i$ and $H_j \times H_j$ are transversal as p -submanifolds for $i \neq j$, the diffeomorphism class of X_b^2 stays the same if we change the order in which we blow up (cf. Proposition 5.8.2 in [27] or [25, p.21]).

FIGURE 3. The π -double space

To define the π -double space, consider the fibre diagonal on the p -submanifold $H_i \times H_i$,

$$(3.2) \quad D_{\pi_i} = \{(h, h') \in H_i \times H_i ; \pi_i(h) = \pi_i(h')\}.$$

To lift this p -submanifold to the front face B_i , notice that

$$B_i = SN^+(H_i \times H_i)$$

is by definition a quarter of circle bundle on $H_i \times H_i$ giving a canonical decomposition

$$(3.3) \quad B_i = (H_i \times H_i) \times [-1, 1]_{s_i}, \text{ with } s_i := \omega_i - \omega'_i.$$

Thus, we can define a lift of D_{π_i} to B_i by

$$(3.4) \quad \Delta_i := \{(h, h', 0) \in H_i \times H_i \times [-1, 1]_{s_i} ; \pi_i(h) = \pi_i(h')\}.$$

The space Δ_i is a p -submanifold of B_i and X_b^2 . To obtain the π -double space, it suffices to blow up Δ_i in X_b^2 for $i \in \{1, \dots, k\}$. As opposed to the definition of X_b^2 , the order in which the blow-ups are performed is important, different orders leading to different diffeomorphism classes of manifolds with corners. Fortunately, our assumptions on the partial order of hypersurfaces of X give us a systematic way to proceed.

More precisely, assume that the hypersurfaces of X are labeled in such a way that

$$(3.5) \quad i < j \implies H_i < H_j \text{ or } H_i \cap H_j = \emptyset.$$

With this convention, we define the π -double space by

$$(3.6) \quad X_\pi^2 := [X_b^2; \Delta_1; \dots; \Delta_k].$$

See Figure 3 for a picture of X_π^2 when X is a manifold with boundary. Notice that the order in which we blow up is not completely determined by (3.5), but a different choice of ordering satisfying (3.5) would amount in commuting the blow-ups of p -submanifolds which do not intersect, an operation which does not affect the diffeomorphism class of X_π^2 .

We have corresponding blow-down maps

$$(3.7) \quad \beta_{\pi-b} : X_\pi^2 \rightarrow X_b^2, \quad \beta_\pi := \beta_b \circ \beta_{\pi-b} : X_\pi^2 \rightarrow X^2.$$

We denote the ‘new’ hypersurface introduced by blowing up Δ_i by

$$(3.8) \quad \text{ff}_{\pi_i} := (\beta_{\pi-b})^{-1}(\Delta_i) \subset X_\pi^2.$$

The p -submanifold ff_{π_i} is called the **front face** associated to the boundary hypersurface H_i . Let also

$$\Delta_\pi := \overline{\beta_\pi^{-1}(\overset{\circ}{\Delta}_X)}$$

denote the lift of the diagonal $\Delta_X \subset X \times X$ to X_π^2 , where $\overset{\circ}{\Delta}_X$ is the interior of Δ_X . On X_π^2 , we are particularly interested in the lift of the Lie algebra $\mathcal{V}_S(X)$ with respect to the natural maps

$$(3.9) \quad \pi_L = \text{pr}_L \circ \beta_\pi : X_\pi^2 \rightarrow X, \quad \pi_R = \text{pr}_R \circ \beta_\pi : X_\pi^2 \rightarrow X.$$

Lemma 3.1. *The lifted diagonal Δ_π is a p -submanifold of X_π^2 . Furthermore, the lifts to X_π^2 of the Lie algebra $\mathcal{V}_S(X)$ via the maps π_L and π_R are transversal to the lifted diagonal Δ_π .*

Proof. The result is trivial in the interior of X_π^2 . Thus, let $p \in \Delta_\pi \cap \partial X_\pi^2$ be given. We need to show that the lemma holds in a neighborhood of p in X_π^2 . Moreover, by symmetry, we only have to prove the result for the lift of $\mathcal{V}_S(X)$ with respect to the map π_L . Let $H_{i_1}, \dots, H_{i_\ell}$ be the boundary hypersurfaces of X containing the point $\pi_L(p) \in \partial X$. After relabelling if necessary, we can assume H_1, \dots, H_ℓ are the hypersurfaces containing $\pi_L(p)$ and that

$$(3.10) \quad H_1 < H_2 < \dots < H_\ell.$$

Near $\pi_L(p) \in \partial X$, let (x_i, y_i, z) be coordinates as in (2.3). Recall that in such coordinates, the Lie algebra $\mathcal{V}_S(X)$ is locally spanned over $\mathcal{C}^\infty(X)$ by the vector fields

$$(3.11) \quad \frac{\partial}{\partial z^j}, w_i x_i \frac{\partial}{\partial x_i}, w_i \frac{\partial}{\partial y_i^{n_i}},$$

for $j \in \{1, \dots, q\}$, $i \in \{1, \dots, \ell\}$, $n_i \in \{1, \dots, k_i\}$, where $w_i = \prod_{m=i}^\ell x_m$.

On X^2 , we can then consider the coordinates

$$(3.12) \quad x_i, y_i, z, x'_i, y'_i, z',$$

where (x_i, y_i, z) is seen as the pullback of our coordinates from the left factor of X^2 , while (x'_i, y'_i, z') is the pullback of our coordinates from the right factor of X^2 . On the b -double space X_b^2 , we can then consider the local coordinates

$$(3.13) \quad r_i = x'_i, s_i = \frac{x_i - x'_i}{x'_i}, y_i, y'_i, z, z',$$

In these coordinates, we have that

$$x_i = r_i(s_i + 1), \quad w_i = w'_i \sigma_i, \quad \text{where } w'_i = \prod_{j=i}^\ell r_j, \quad \sigma_i = \prod_{j=i}^\ell (s_j + 1).$$

Thus, under the map $\text{pr}_L \circ \beta_b$, the vector fields of (3.11) lift to

$$(3.14) \quad \frac{\partial}{\partial z^j}, w'_i \sigma_i (s_i + 1) \frac{\partial}{\partial s_i}, w'_i \sigma_i \frac{\partial}{\partial y_i^{n_i}}.$$

Finally, on X_π^2 , we can consider the local coordinates near p given by

$$(3.15) \quad r_i = x'_i, S_i = \frac{s_i}{w'_i}, Y_i = \frac{y_i - y'_i}{w'_i}, y'_i, z, z'.$$

In these coordinates, the lifted diagonal Δ_π is given by the subset $\{S_i = 0, Y_i = 0, z = z'\}$. In particular, this shows it is a p -submanifold. From (3.14), we also see that under the map π_L , the vector fields of (3.11) lift to

$$(3.16) \quad \frac{\partial}{\partial z^j}, \sigma_i(S_i w'_i + 1) \frac{\partial}{\partial S_i}, \sigma_i \frac{\partial}{\partial Y_i^{n_i}}, \quad \text{where } \sigma_i = \prod_{j=i}^{\ell} (S_j w'_j + 1).$$

These vector fields are clearly transversal to the lifted diagonal $\Delta_\pi = \{S_i = 0, Y_i = 0, z = z'\}$, which completes the proof. \square

Corollary 3.2. *The natural diffeomorphism $\Delta_\pi \cong X$ induced by the map π_L (or alternatively by the map π_R) is covered by natural identifications*

$$N\Delta_\pi \cong {}^\pi T X, \quad N^* \Delta_\pi \cong {}^\pi T^* X,$$

where $N\Delta_\pi$ is the normal bundle of Δ_π in X_π^2 .

Remark 3.3. It can also be proved that the Lie algebra $\mathcal{V}_S(X)$ lifts via π_L or π_R to give a Lie subalgebra of $\mathcal{V}_b(X_\pi^2)$. Near Δ_π , this follows from the local description (3.16). Since we do not need this fact elsewhere on X_π^2 , we omit the proof.

On the π -double space, the Schwartz kernels of S -differential operators admit a simple description. Let us first describe the Schwartz kernel of the identity operator. In the local coordinates (3.12) near $\Delta_X \cap (\partial X \times \partial X)$, the Schwartz kernel of the identity operator can be written as

$$(3.17) \quad \begin{aligned} K_{\text{Id}} &= \left(\prod_{i=1}^{\ell} \delta(x_i - x'_i) \delta(y_i - y'_i) dx_i dy'_i \right) \delta(z - z') dz' \\ &= \delta(x - x') \delta(y - y') \delta(z - z') dx' dy' dz', \end{aligned}$$

where in the second line we suppressed the subscripts to lighten the notation. Using the coordinates (3.13) on X_b^2 , this Schwartz kernel becomes

$$(3.18) \quad K_{\text{Id}} = \delta(s) \delta(y - y') \delta(z - z') \frac{dx' dy' dz'}{w'_1}, \quad w'_1 = \prod_{j=1}^{\ell} x'_j.$$

Finally, using the local coordinates (3.15) on the π -double space X_π^2 , this becomes

$$(3.19) \quad \begin{aligned} K_{\text{Id}} &= \delta(S) \delta(Y) \delta(z - z') \frac{dx' dy' dz'}{\prod_{i=1}^{\ell} (x'_i) (w'_i)^{k_i+1}}, \quad w'_i = \prod_{j=i}^{\ell} x'_j, \\ &= \delta(S) \delta(Y) \delta(z - z') \frac{dx' dy' dz'}{\prod_{i=1}^{\ell} (x'_i)^{2+\dim S_i}}, \\ &= \delta(S) \delta(Y) \delta(z - z') \pi_R^*(\nu_\pi), \end{aligned}$$

where $\nu_\pi \in \mathcal{C}^\infty(X; {}^\pi \Omega)$ is a non-vanishing S -density on X . Let

$$\mathcal{D}^0(\Delta_\pi) = \mathcal{C}^\infty(X_\pi^2) \cdot \mu$$

be the space of smooth δ -functions on the p -submanifold $\Delta_\pi \subset X_\pi^2$, where μ is a non-vanishing delta function with smooth coefficient as in (3.19). From the local computation (3.19), we see that

$$(3.20) \quad K_{\text{Id}} \in \mathcal{D}^0(\Delta_\pi) \cdot \pi_R^*(\nu_\pi).$$

Thus, if $P \in \text{Diff}_S^k(X)$ is a S -differential operator of order k , we see from Lemma 3.1 that

$$(3.21) \quad K_P = \pi_L^* P \cdot K_{\text{Id}} \in \mathcal{D}^k(\Delta_\pi) \cdot \pi_R^*(\nu_\pi),$$

where $\mathcal{D}^k(\Delta_\pi)$ is the space of delta functions of order at most k , namely

$$(3.22) \quad \mathcal{D}^k(\Delta_\pi) = \text{Diff}^k(X_S^2) \cdot \mathcal{D}^0(\Delta_\pi).$$

In fact, since $\pi_L^*(\mathcal{V}_S(X))$ is transversal to Δ_π by Lemma 3.1, the space of Schwartz kernels of S -differential operators of order k is precisely given by

$$(3.23) \quad \mathcal{D}^k(\Delta_\pi) \cdot \pi_R^*(\nu_\pi).$$

If E and F are smooth complex vector bundles on X and $\text{Diff}_S^k(X; E, F)$ is the space of S -differential operators of order k acting from $\mathcal{C}^\infty(X; E)$ to $\mathcal{C}^\infty(X; F)$, then working in local trivializations, we can in a similar way identify the corresponding space of Schwartz kernels with

$$(3.24) \quad \mathcal{D}^k(\Delta_\pi) \cdot \mathcal{C}^\infty(X_\pi^2; \beta_\pi^* \text{Hom}(E, F) \otimes \pi_R^*(\pi\Omega)),$$

where $\text{Hom}(E, F) = \text{pr}_L^*(F) \otimes \text{pr}_R^*(E^*)$.

Since delta functions are a special type of conormal distributions, this suggests to define S -pseudodifferential operators of order k acting from sections of E to sections of F by

$$(3.25) \quad \Psi_S^m(X; E, F) := \{K \in I^m(X_\pi^2; \Delta_\pi; \beta_\pi^*(\text{Hom}(E; F)) \otimes \pi_R^*(\pi\Omega)); \\ K \equiv 0 \text{ at } \partial X_\pi^2 \setminus \text{ff}_\pi\}.$$

Here, $\text{ff}_\pi := \cup_{i=1}^k \text{ff}_{\pi_i}$ and $I^m(X_\pi^2; \Delta_\pi; \beta_\pi^*(\text{Hom}(E, F)) \otimes \pi_R^*(\pi\Omega))$ is the space of conormal distributions of order m at Δ_π . The notation $K \equiv 0$ at $\partial X_\pi^2 \setminus \text{ff}(X_\pi^2)$ means that its Taylor series identically vanishes at $\partial X_\pi^2 \setminus \text{ff}(X_\pi^2)$. We can similarly define the space of polyhomogeneous (or classical) S -pseudodifferential operators of order m by

$$(3.26) \quad \Psi_{S\text{-ph}}^m(X; E, F) := \{K \in I_{\text{ph}}^m(X_\pi^2; \Delta_\pi; \beta_\pi^*(\text{Hom}(E; F)) \otimes \pi_R^*(\pi\Omega)); \\ K \equiv 0 \text{ at } \partial X_\pi^2 \setminus \text{ff}_\pi\},$$

where $I_{\text{ph}}^m(X_\pi^2; \Delta_\pi; \beta_\pi^*(\text{Hom}(E, F)) \otimes \pi_R^*(\pi\Omega))$ is the space of polyhomogeneous conormal distributions of order m at Δ_π .

With these definitions, notice that there are natural inclusions

$$(3.27) \quad \text{Diff}_S^k(X; E, F) \subset \Psi_{S\text{-ph}}^k(X; E, F) \subset \Psi_S^k(X; E, F).$$

4. GROUPOIDS

We refer to [42, 21] for the classical definitions and constructions related to groupoids and their Lie algebroids. We recall here the basic definitions needed for this paper.

A **groupoid** is a small category in which every morphism is an isomorphism. Let us make this notion more explicit. A groupoid \mathcal{G} is a pair $(\mathcal{G}^{(0)}, \mathcal{G}^{(1)})$ of sets together with structural morphisms: the unit $u : \mathcal{G}^{(0)} \rightarrow \mathcal{G}^{(1)}$, the source and range $s, r : \mathcal{G}^{(1)} \rightarrow \mathcal{G}^{(0)}$, the inverse $\iota : \mathcal{G}^{(1)} \rightarrow \mathcal{G}^{(1)}$, and the multiplication μ which is defined on the set $\mathcal{G}^{(2)}$ of pairs $(\alpha, \beta) \in \mathcal{G}^{(1)} \times \mathcal{G}^{(1)}$ such that $s(\alpha) = r(\beta)$. Here, the set $\mathcal{G}^{(0)}$ denotes the set of objects (or units) of the groupoid, whereas the set $\mathcal{G}^{(1)}$ denotes the set of morphisms of \mathcal{G} . The identity morphism of any object of

\mathcal{G} enables one to identify that object with a morphism of \mathcal{G} . This leads to the injective map $u : \mathcal{G}^{(0)} \rightarrow \mathcal{G}$. Each morphism $g \in \mathcal{G}$ has a “source” and a “range.” The inverse of a morphism α is denoted by $\alpha^{-1} = \iota(\alpha)$. The structural maps satisfy the following properties,

- (i) $r(\alpha\beta) = r(\alpha)$ and $s(\alpha\beta) = s(\beta)$, for any pair (α, β) in $\mathcal{G}^{(2)}$,
- (ii) $s(u(x)) = r(u(x)) = x$, $u(r(\alpha))\alpha = \alpha$, $\alpha u(s(\alpha)) = \alpha$,
- (iii) $r(\alpha^{-1}) = s(\alpha)$, $s(\alpha^{-1}) = r(\alpha)$, $\alpha\alpha^{-1} = u(r(\alpha))$, and $\alpha^{-1}\alpha = u(s(\alpha))$,
- (iv) the partially defined multiplication μ is associative.

We shall need groupoids with additional structures.

Definition 4.1. A **Lie groupoid** (resp. **locally compact groupoid**) is a groupoid

$$\mathcal{G} = (\mathcal{G}^{(0)}, \mathcal{G}^{(1)}, s, r, \mu, u, \iota)$$

such that $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$ are manifolds with corners (resp. locally compact spaces), the structural maps s, r, μ, u , and ι are differentiable (resp. continuous), the source map s is a submersion (resp. surjective and open) and $\mathcal{G}_x := s^{-1}(x)$, $x \in M$, are all Hausdorff manifolds without corners (resp. locally compact Hausdorff spaces).

We will also encounter the notion of continuous family groupoid ([41]).

Definition 4.2. A locally compact groupoid \mathcal{G} is a **continuous family groupoid** when it is covered by open sets U with homeomorphisms $\Phi_f = (f, \phi_f) : U \rightarrow f(U) \times U_f$ where $f \in \{r, s\}$ and $U_f \subset \mathbb{R}^n$ such that the following holds:

- (1) for all (U, Φ_f) and (V, Ψ_f) as above such that $W = U \cap V \neq \emptyset$, the map $\Psi_f \circ \Phi_f^{-1} : \Phi_f(W) \rightarrow \Psi_f(W)$ is of class $C^{0,\infty}$, which means that $x \mapsto \psi_f \circ \phi_f^{-1}(x, \cdot)$ is continuous from $f(W)$ to $C^\infty(\phi_f(W), \psi_f(W))$ (which has the topology of uniform convergence on compacta of all derivatives);
- (2) The inversion and product maps are locally $C^{0,\infty}$ in the above sense.

We say that (U, Φ_f) is a $C^{0,\infty}$ local chart for (\mathcal{G}, f) .

A simple example of Lie groupoid is the **pair groupoid** associated to a smooth manifold M . It is obtained by taking $\mathcal{G}^{(0)} = M$, $\mathcal{G}^{(1)} = M \times M$, $s(x, y) = y$, $r(x, y) = x$, $(x, y)(y, z) = (x, z)$, $u(x) = (x, x)$ and with inverse $\iota(x, y) = (y, x)$.

Like vector bundles, groupoids can be pulled back. More precisely, let $G \rightrightarrows M$ be a locally compact Hausdorff groupoid with source s and range r . If $f : N \rightarrow M$ is a surjective map, the **pullback** groupoid $*f^*(G) \rightrightarrows N$ of G by f is by definition the set

$$(4.1) \quad *f^*(G) := \{(x, \gamma, y) \in N \times G \times N \mid r(\gamma) = f(x), s(\gamma) = f(y)\}$$

with the structural morphisms given by

- (1) the unit map $x \mapsto (x, f(x), x)$,
- (2) the source map $(x, \gamma, y) \mapsto y$ and range map $(x, \gamma, y) \mapsto x$,
- (3) the product $(x, \gamma, y)(y, \eta, z) = (x, \gamma\eta, z)$ and inverse $(x, \gamma, y)^{-1} = (y, \gamma^{-1}, x)$.

The results of [35] apply to show that the groupoids G and $*f^*(G)$ are Morita equivalent when f is surjective and open.

The infinitesimal object associated to a Lie groupoid is its Lie algebroid, which we define next.

Definition 4.3. A **Lie algebroid** \mathcal{A} over a manifold M is a vector bundle $\mathcal{A} \rightarrow M$, together with a Lie algebra structure on the space $\Gamma(\mathcal{A})$ of smooth sections of \mathcal{A} and a bundle map $\varrho : \mathcal{A} \rightarrow TM$ whose extension to sections of these bundles satisfies

- (i) $\varrho([X, Y]) = [\varrho(X), \varrho(Y)]$, and
- (ii) $[X, fY] = f[X, Y] + (\varrho(X)f)Y$,

for any smooth sections X and Y of \mathcal{A} and any smooth function f on M .

The map ϱ is called the **anchor map** of \mathcal{A} . Note that we allow the base M in the definition above to be a manifold with corners.

Now, let $\mathcal{G} = \mathcal{G}^{(1)} \xrightarrow[r]{s} \mathcal{G}^{(0)}$ be a Lie groupoid. We denote by $T^s\mathcal{G}$ the subbundle of $T\mathcal{G}^{(1)}$ of s -vertical tangent vectors. In other words, $T^s\mathcal{G}$ is the kernel of the differential Ts of s .

For any α in $\mathcal{G}^{(1)}$, let $R_\alpha : \mathcal{G}_{r(\alpha)} \rightarrow \mathcal{G}_{s(\alpha)}$ be the right multiplication by α . A tangent vector field Z on $\mathcal{G}^{(1)}$ is **right invariant** if it satisfies,

- Z is s -vertical, namely $Ts(Z) = 0$.
- For all (α, β) in $\mathcal{G}^{(2)}$, $Z(\alpha\beta) = TR_\beta(Z(\alpha))$.

The Lie algebroid \mathcal{AG} of a Lie groupoid \mathcal{G} is defined as follows [21],

- The fibre bundle $\mathcal{AG} \rightarrow \mathcal{G}^{(0)}$ is the restriction of $T^s\mathcal{G}$ to $\mathcal{G}^{(0)}$. In other words, $\mathcal{AG} = \cup_{x \in \mathcal{G}^{(0)}} T_x\mathcal{G}_x$ is the union of the tangent spaces to the s -fibre at the corresponding unit.
- The anchor $\rho : \mathcal{AG} \rightarrow T\mathcal{G}^{(0)}$ is the restriction of the differential Tr of r to \mathcal{AG} .
- If $Y : U \rightarrow \mathcal{AG}$ is a local section of \mathcal{AG} , where U is an open subset of $\mathcal{G}^{(0)}$, we define the local **right invariant vector field** Z_Y **associated** with Y by

$$Z_Y(\alpha) = TR_\alpha(Y(r(\alpha))) \quad \text{for all } \alpha \in \mathcal{G}^U := r^{-1}(U).$$

The Lie bracket is then defined by

$$\begin{aligned} [,] : \Gamma(\mathcal{AG}) \times \Gamma(\mathcal{AG}) &\longrightarrow \Gamma(\mathcal{AG}) \\ (Y_1, Y_2) &\longmapsto [Z_{Y_1}, Z_{Y_2}]_{\mathcal{G}^{(0)}} \end{aligned}$$

where $[Z_{Y_1}, Z_{Y_2}]$ denotes the s -vertical vector field obtained with the usual bracket and $[Z_{Y_1}, Z_{Y_2}]_{\mathcal{G}^{(0)}}$ is the restriction of $[Z_{Y_1}, Z_{Y_2}]$ to $\mathcal{G}^{(0)}$.

Remark 4.4. When \mathcal{G} is a continuous family groupoid, the vector bundle $\mathcal{AG} \rightarrow \mathcal{G}^{(0)}$ as defined above still exists. Indeed, the fibres \mathcal{G}_x , $x \in \mathcal{G}^{(0)}$ are smooth manifolds and we still can set

$$(4.2) \quad \mathcal{AG} = \bigsqcup_{x \in \mathcal{G}^{(0)}} T_x\mathcal{G}_x.$$

This vector bundle is smooth in the sense of [41] and it is called the Lie algebroid of \mathcal{G} again.

In this paper, a central example of Lie algebroid is given by ${}^\pi TX$ with anchor map given by the natural map $\iota_\pi : {}^\pi TX \rightarrow TX$. In fact, the space $X_\pi^2 \cup \overset{\circ}{\text{ff}}_\pi$ has a natural structure of Lie groupoid with Lie algebroid naturally identified with ${}^\pi TX$ under the identification $X \cong \Delta_\pi$. More precisely, we set

$$(4.3) \quad \mathcal{G}_\pi^{(0)} = \Delta_\pi, \quad \mathcal{G}_\pi^{(1)} = X_\pi^2 \cup \overset{\circ}{\text{ff}}_\pi.$$

For $\alpha \in \mathcal{G}_\pi^{(1)}$ with $\beta_\pi(\alpha) = (x_1, x_2)$, we define the source and range of α by

$$(4.4) \quad s(\alpha) = x_2, \quad r(\alpha) = x_1.$$

The map

$$(4.5) \quad \begin{aligned} \iota : \overset{\circ}{X} \times \overset{\circ}{X} &\rightarrow \overset{\circ}{X} \times \overset{\circ}{X} \\ (x, x') &\mapsto (x', x) \end{aligned}$$

extends in a unique way to a smooth map $\iota : \mathcal{G}_\pi^{(1)} \rightarrow \mathcal{G}_\pi^{(1)}$ defining on $\mathcal{G}_\pi^{(1)}$ the inverse map. Similarly, the natural multiplication map on the groupoid $\overset{\circ}{X} \times \overset{\circ}{X}$ extends to give a composition map

$$(4.6) \quad \mu : \mathcal{G}_\pi^{(2)} \rightarrow \mathcal{G}_\pi^{(1)}$$

where

$$(4.7) \quad \mathcal{G}_\pi^{(2)} = \{(\alpha, \beta) \in \mathcal{G}_\pi^{(1)} \times \mathcal{G}_\pi^{(1)}; r(\beta) = s(\alpha)\}.$$

To see that the Lie algebroid of \mathcal{G}_π is precisely ${}^\pi T X$, it suffices to use Corollary 3.2 and to notice that $\mathcal{A}\mathcal{G}_\pi$ is isomorphic to $N\Delta_\pi$, a fact that follows from the observation that the source map of \mathcal{G}_π is a surjective submersion equal to the identity map when restricted to units. The Lie groupoid \mathcal{G}_π admits a decomposition into simpler groupoids. Indeed, for each boundary hypersurface H_i of X , notice that the subgroupoid

$$(\text{ff}_{\pi_i} \cap \overset{\circ}{\text{ff}}_\pi) \setminus \left(\bigcup_{H_i < H_j} (\text{ff}_{\pi_j} \cap \text{ff}_{\pi_i} \cap \overset{\circ}{\text{ff}}_\pi) \right)$$

is naturally isomorphic to the pull-back groupoid ${}^*\pi_i^*({}^\pi NS_i)$. Since these subgroupoids give a partition of $\overset{\circ}{\text{ff}}_\pi$, this means that, forgetting about the Lie structure, the groupoid \mathcal{G}_π can be written as a disjoint union of groupoids,

$$\mathcal{G}_\pi \cong (\overset{\circ}{X} \times \overset{\circ}{X}) \bigsqcup \left(\bigsqcup_{i=1}^k {}^*\pi_i^*({}^\pi NS_i) \right).$$

Since \mathcal{G}_π is a smooth groupoid, one can consider the space $\Psi(\mathcal{G}_\pi)$ of pseudodifferential operators on \mathcal{G}_π [34, 39, 2]. Conditions on the supports of these operators are necessary to perform their products. They corresponds to those pseudodifferential operators of (3.25) whose Schwartz kernels are compactly supported in $\mathcal{G}_\pi^{(1)}$. Thus, the space of operators in (3.25) is not much larger. It replaces the condition of compact support in $\mathcal{G}_\pi^{(1)}$ by a condition of Schwartz decay at ‘infinity’ with the infinity and the decay condition specified by the natural compactification X_π^2 of $\mathcal{G}_\pi^{(1)}$. Still, it has an important consequence. In the larger space (3.25), the inverse of an invertible pseudodifferential is automatically contained in the same space (with minus the order of the original operator), a fact which is not true for the smaller space $\Psi(\mathcal{G}_\pi)$. This is intimately related with the well-known fact that the Fourier transform is an automorphism of the space of Schwartz functions on \mathbb{R}^n , but does not preserve the space smooth functions with compact support.

For \mathcal{G} a locally compact groupoid, we need a Haar system to define an associated C^* -algebra. Recall first that a (right continuous) Haar system [42] on \mathcal{G} is a family $(\lambda_x)_{x \in \mathcal{G}^{(0)}}$ of (positive, regular, Borel) measures λ_x on \mathcal{G} such that

- (1) the support of λ_x is equal to (the whole of) \mathcal{G}_x ,

- (2) for all $f \in \mathcal{C}_c(\mathcal{G})$, the map $x \mapsto \int_{\mathcal{G}_x} f(\gamma) d\lambda_x$ is continuous,
 (3) for all $\gamma \in \mathcal{G}$ and $f \in \mathcal{C}_c(\mathcal{G})$, one has $\int_{\mathcal{G}_{r(\gamma)}} f(\gamma' \gamma) d\lambda_{r(\gamma)} = \int_{\mathcal{G}_{s(\gamma)}} f(\gamma') d\lambda_{s(\gamma)}$.

Haar systems always exist on continuous family groupoids. Given a continuous Haar system, the space $\mathcal{C}_c(\mathcal{G})$ is endowed with natural product and involution, and this involutive algebra is then automatically represented in an appropriate Hilbert space as well. This leads to the notions of reduced C^* -algebra of \mathcal{G} , usually denoted by $\mathcal{C}_r^*(\mathcal{G})$, and of universal (maximal) C^* -algebra of \mathcal{G} , usually denoted by $\mathcal{C}^*(\mathcal{G})$. Different Haar systems provide up to isomorphism the same reduced and universal C^* -algebras[42, 41]. Moreover, if E is a Hermitian vector bundle on $\mathcal{G}^{(0)}$, then the space of sections $\mathcal{C}_c(\mathcal{G}, r^*E)$ has also natural Hilbert $\mathcal{C}_r^*(\mathcal{G})$ and $\mathcal{C}^*(\mathcal{G})$ -modules completions denoted by $\mathcal{C}_r^*(\mathcal{G}, E)$ and $\mathcal{C}^*(\mathcal{G}, E)$. Often, the reduced and universal completions coincide, in which case we omit the subscript r in \mathcal{C}_r^* . For the groupoids considered in this paper, this can be seen using the following general criterion.

Lemma 4.5. *If a measured groupoid \mathcal{G} is the finite disjoint union of measurewise amenable (see [3, Definition 3.3.1]) groupoids \mathcal{G}_i , that is, $\mathcal{G} = \sqcup_{i \in I} \mathcal{G}_i$ and $\mathcal{G}^{(0)} = \sqcup_{i \in I} \mathcal{G}_i^{(0)}$, where everything is assumed to be borelian, then \mathcal{G} is measurewise amenable. In particular, $\mathcal{C}^*(\mathcal{G})$ is nuclear and equal to $\mathcal{C}_r^*(\mathcal{G})$.*

Proof. The fact \mathcal{G} is measurewise amenable follows from [3, Proposition 5.3.4] applied to the Borel map $q : \mathcal{G}^{(0)} \rightarrow I$ defined by $q(x) = i$ if $x \in \mathcal{G}_i^{(0)}$. By [3, 6.2.14], we then have that $\mathcal{C}^*(\mathcal{G})$ is nuclear and equal to $\mathcal{C}_r^*(\mathcal{G})$. \square

For instance, this criterion can be applied to the groupoid \mathcal{G}_π .

Lemma 4.6. *The groupoid \mathcal{G}_π is measurewise amenable. In particular, $\mathcal{C}^*(\mathcal{G}_\pi)$ is nuclear and equal to $\mathcal{C}_r^*(\mathcal{G}_\pi)$.*

Proof. By Lemma 4.5, it suffices to observe that \mathcal{G}_π can be written as a disjoint union of topologically amenable (and thus measurewise amenable, by [3]) groupoids,

$$(4.8) \quad \mathcal{G}_\pi = (\mathring{X} \times \mathring{X}) \bigsqcup \left(\bigsqcup_{i=1}^k (H_i \times_{\pi_i} {}^\pi T S_i \times_{\pi_i} H_i) |_{G_i} \times \mathbb{R} \right),$$

where $G_i = H_i \setminus (\cup_{j>i} H_j)$. The topological amenability of the various subgroupoids on the right-hand side can be justified as follows,

- (i) A vector bundle is topologically amenable as a bundle of abelian groups;
- (ii) Topological amenability is preserved under equivalence of groupoids ([3]).
 For instance, given a vector bundle $E \rightarrow S$ and a locally trivial fibre bundle $p : H \rightarrow S$, the groupoid $(H \times_p E \times_p H) \rightrightarrows H$ is equivalent as a groupoid to the vector bundle E , and thus is topologically amenable;
- (iii) The cartesian product of amenable groupoids is amenable.

\square

5. ACTION OF S-PSEUDODIFFERENTIAL OPERATORS

Let us first consider the space $\Psi_S^m(X; E, F)$ in (3.25) in the simpler situation where $E = F = \underline{\mathbb{C}}$. Notice that (3.25) can alternatively be rewritten as

$$(5.1) \quad \Psi_S^m(X) = I^m(X_\pi^2; \Delta_\pi) \cdot \mathcal{C}_{\text{ff}_\pi}^\infty(X_\pi^2; \pi_R^*(\pi\Omega)),$$

where $\mathcal{C}_{\text{ff}_\pi}^\infty(X_\pi^2; \pi_R^*(\pi\Omega))$ is the space of smooth sections vanishing with all their derivatives at all boundary faces except those contained in ff_π . To describe the action of \mathcal{S} -operators on functions, we will need the following result about the pushforward of conormal distributions.

Lemma 5.1. *The map $\pi_L = \text{pr}_L \circ \beta_\pi : X_\pi^2 \rightarrow X$ induces a continuous linear map*

$$(\pi_L)_* : I^m(X_\pi^2; \Delta_\pi) \cdot \mathcal{C}_{\text{ff}_\pi}^\infty(X_\pi^2; \Omega) \rightarrow \mathcal{C}^\infty(X; \Omega).$$

Proof. If $K \in I^m(X_\pi^2; \Delta_\pi) \cdot \mathcal{C}_{\text{ff}_\pi}^\infty(X_\pi^2; \Omega)$ is supported near the lifted diagonal, then the result follows from general properties of conormal distributions together with the fact the map π_L is transversal to Δ_π . Thus, using a cut-off function, we can assume $K \in \mathcal{C}_{\text{ff}_\pi}^\infty(X_\pi^2; \Omega)$. To proceed further, notice that π_L is a b -fibration (we refer to [27] for a definition). Indeed, as a blow-down map, β_π is a surjective b -submersion. Since the projection $\text{pr}_L : X^2 \rightarrow X$ is also clearly a surjective b -submersion, so is the composite $\pi_L = \text{pr}_L \circ \beta_\pi$. Thus, according to Proposition 2.4.2 in [27], π_L is a b -fibration provided no boundary hypersurface of X_π^2 is mapped to a boundary face of X of codimension greater than one. This is clear for the ‘old’ hypersurfaces in X_π^2 , while the ‘new’ hypersurfaces are mapped under β_π to boundary faces of X^2 of codimension 2 which are then mapped under pr_L to boundary faces of codimension 1 under the projection pr_L .

The lemma can then be seen as a special case of the Push-forward Theorem of [28] for b -fibrations. Precisely, the lemma is a consequence of this theorem combined with the fact

$$\pi_L^{-1}(H_i) \cap \text{ff}_\pi = \text{ff}_{\pi_i}$$

for all boundary hypersurfaces $H_i \subset X$. □

Since the previous lemma is dealing with smooth densities, it cannot be applied directly to the space of conormal distributions $\Psi_S^m(X)$.

Lemma 5.2. *The tensor product identification $\text{pr}_L^* \Omega \otimes \text{pr}_R^* \pi\Omega \equiv \Omega$ on the interior of X^2 extends to give an isomorphism of spaces of sections*

$$\mathcal{C}_{\text{ff}_\pi}^\infty(X_\pi^2; \beta_\pi^*(\text{pr}_L^* \Omega \otimes \text{pr}_R^* \pi\Omega)) = \mathcal{C}_{\text{ff}_\pi}^\infty(X_\pi^2; \Omega)$$

Proof. It suffices to notice that the singular factors of sections of $\beta_\pi^*(\text{pr}_L^* \Omega \otimes \text{pr}_R^* \pi\Omega)$ all arise at faces not contained in ff_π , and so are absorbed by the infinite order vanishing at these faces. This can be seen using the local coordinates. Indeed, in the coordinates (3.12), an element of $\mathcal{C}^\infty(X_\pi^2; \text{pr}_L^* \Omega \otimes \text{pr}_R^* \pi\Omega)$ is of the form

$$\frac{h dx dy dz dx' dy' dz'}{\prod_{i=1}^\ell (x'_i)^{2+\dim S_i}} = \frac{h dx dy dz dx' dy' dz'}{\prod_{i=1}^\ell x'_i (w'_i)^{k_i+1}}, \quad \text{for some } h \in \mathcal{C}^\infty(X_\pi^2).$$

Thus, in the coordinates of (3.15), it takes the form

$$\tilde{h} dS dY dz dx' dy' dz' \quad \text{for some } \tilde{h} \in \mathcal{C}^\infty(X_\pi^2),$$

and the only possible singular terms occur when $S_i \rightarrow \infty$ or $Y_i \rightarrow \infty$, that is, at faces not contained in ff_π . □

We can then define a push-forward map

$$(5.2) \quad (\pi_L)_* : I^m(X_\pi^2; \Delta_\pi) \cdot \mathcal{C}_{\text{ff}\pi}^\infty(X_\pi^2; \pi_R^* \pi \Omega) \rightarrow \mathcal{C}^\infty(X)$$

by requiring that for $K \in I^m(X_\pi^2; \Delta_\pi) \cdot \mathcal{C}_{\text{ff}\pi}^\infty(X_\pi^2; \pi_R^* \pi \Omega)$ and any non-vanishing section $v \in \mathcal{C}^\infty(X; \Omega)$,

$$(5.3) \quad v \cdot (\pi_L)_* K = (\pi_L)_* (\pi_L^* v \cdot K),$$

where the right hand side of (5.3) is in $\mathcal{C}^\infty(X; \Omega)$ thanks to Lemma 5.1 and Lemma 5.2. This push-forward map provides a way to make \mathbb{S} -pseudodifferential operators act on functions. To state the main result of this section, we still need to introduce some notation. If $M_1 X$ is the set of boundary hypersurfaces and $A \subset M_1 X$ is a subset, then set

$$x_A = \prod_{H \in A} x_H$$

where $x_H \in \mathcal{C}^\infty(X)$ is a choice of boundary defining function for H . For any $A \subset M_1 X$, consider the space

$$\dot{\mathcal{C}}_A^\infty(X; E) = \bigcap_{k \in \mathbb{N}} x_A^k \mathcal{C}^\infty(X; E)$$

of smooth sections on X vanishing with all their derivatives on each boundary hypersurface $H \in A$. When $A = M_1 X$, this gives the space

$$\dot{\mathcal{C}}^\infty(X; E) = \dot{\mathcal{C}}_{M_1 X}^\infty(X; E)$$

of smooth sections vanishing with all their derivatives on ∂X . It is also useful to use the notation $\mathcal{C}_A^\infty(X; E) = \dot{\mathcal{C}}_{M_1 X \setminus A}^\infty(X; E)$. Thus, for $A = M_1 X$, we have $\mathcal{C}_{M_1 X}^\infty(X; E) = \mathcal{C}^\infty(X; E)$.

Each space $\dot{\mathcal{C}}_A^\infty(X; E)$ comes with a natural structure of Fréchet space induced from the one of $\mathcal{C}^\infty(X; E)$. The corresponding space of distributions $\dot{\mathcal{C}}_A^{-\infty}(X; E)$ is defined to be the dual of $\mathcal{C}_A^\infty(X; E^* \otimes \Omega)$. Similarly, we use the notation $\mathcal{C}_A^{-\infty}(X; E)$ to denote the dual of $\dot{\mathcal{C}}_A^\infty(X; E^* \otimes \Omega)$.

Proposition 5.3. *Via the push-forward map (5.2), an element $P \in \Psi_{\mathbb{S}}^m(X; E, F)$ defines a continuous linear map*

$$P : \mathcal{C}^\infty(X; E) \rightarrow \mathcal{C}^\infty(X; F).$$

For each subset $A \subset M_1 X$, this map restricts to give a continuous linear map

$$P : \dot{\mathcal{C}}_A^\infty(X; E) \rightarrow \dot{\mathcal{C}}_A^\infty(X; F).$$

These maps extend by continuity in the distributional topology to linear maps

$$P : \dot{\mathcal{C}}_A^{-\infty}(X; E) \rightarrow \dot{\mathcal{C}}_A^{-\infty}(X; F)$$

for all subsets $A \subset M_1 X$.

Proof. The first assertion is a consequence of Lemma 5.1 and Lemma 5.2. Using a partition of unity subordinate to a covering by open sets over which E and F restrict to be trivial, we can reduce to the case $E = F = \underline{\mathbb{C}}$ to prove the second assertion.

Let $A \subset M_1 X$ be given. Since the function $(\frac{x_A}{x'_A}) \in \mathcal{C}^\infty(X^2 \setminus \partial X^2)$ pulls back to X_π^2 to give a function which is smooth on ff_π and has only finite order singularities at hypersurfaces not in ff_π , we see that

$$P \in \Psi_S^m(X; E, F) \implies \tilde{P}_k = x_A^k \circ P \circ x_A^{-k} \in \Psi_S^m(X; E, F)$$

for all $k \in \mathbb{N}$. On the other hand, given $u \in \dot{\mathcal{C}}_A^\infty(X; E)$, we can write it as $u = x_A^k \tilde{u}_k$ for some $\tilde{u}_k \in \mathcal{C}^\infty(X; E)$, so that

$$x_A^{-k} P u = \tilde{P}_k \tilde{u}_k \in \mathcal{C}^\infty(X; F) \implies P u \in x_A^k \mathcal{C}^\infty(X; F).$$

Since $k \in \mathbb{N}$ is arbitrary, this means $P u \in \dot{\mathcal{C}}_A^\infty(X; F)$.

For the proof of the last assertion, choose a non-vanishing density in $\mathcal{C}^\infty(X; \pi\Omega)$ as well as Hermitian metrics for E and F . These then define a L^2 -inner product for sections of E and F . To see the action of $P \in \Psi^m(X; E, F)$ extends to distributions, it suffices to notice that from (3.25), the formal adjoint of $P \in \Psi^m(X; E, F)$ with respect to this L^2 -inner product is an element of $\Psi_S^m(X; F, E)$, so that the action of P on distributions can be defined by duality. \square

The following proposition can be interpreted as a dual statement to the Schwartz kernel theorem.

Proposition 5.4. *A continuous linear operator $A : \dot{\mathcal{C}}^\infty(X) \rightarrow \mathcal{C}^{-\infty}(X)$ induces a continuous linear map $A : \mathcal{C}^{-\infty}(X) \rightarrow \dot{\mathcal{C}}^\infty(X)$ if and only if its Schwartz kernel K_A is an element of $\dot{\mathcal{C}}^\infty(X \times X; \text{pr}_R^* \Omega X)$ where ΩX is the density bundle on X and $\text{pr}_R : X \times X \rightarrow X$ is the projection on the right factor.*

Proof. One proceeds as in the proof of Propositione 1.2 in [40]. Namely, it suffices to notice that if $\mathcal{L}(\mathcal{C}^{-\infty}(X), \dot{\mathcal{C}}^\infty(X))$ denotes the space of continuous linear maps (with $\mathcal{C}^{-\infty}(X)$ equipped with the strong dual topology), then (see [52])

$$\begin{aligned} \mathcal{L}(\mathcal{C}^{-\infty}(X), \dot{\mathcal{C}}^\infty(X)) &\cong \dot{\mathcal{C}}^\infty(X; \Omega X) \hat{\otimes} \dot{\mathcal{C}}^\infty(X) \\ &\cong \dot{\mathcal{C}}^\infty(X \times X; \text{pr}_R^* \Omega X). \end{aligned}$$

\square

Let us denote by $\dot{\Psi}_S^{-\infty}(X)$ the space of operators with Schwartz kernel in $\dot{\mathcal{C}}^\infty(X \times X; \text{pr}_R^* \Omega X)$. From the definition of \mathcal{S} -operators, it is clear that we have the identification $x^\infty \Psi_S^{-\infty}(X) = \dot{\Psi}_S^{-\infty}(X)$ where $x = \prod_{i=1}^k x_i$. From Proposition 5.4, we immediately obtain the following.

Corollary 5.5. *For $A \in \dot{\Psi}_S^{-\infty}(X)$ and $B \in \Psi_S^m(X)$, we have*

$$AB \in \dot{\Psi}_S^{-\infty}(X), \quad BA \in \dot{\Psi}_S^{-\infty}(X).$$

6. SUSPENDED \mathcal{S} -OPERATORS

Before describing the symbol maps associated to \mathcal{S} -operators, we first need to discuss how to suspend them in the sense of [30]. To this end, let (X, π) be a manifold with fibred corners and let H_1, \dots, H_k be its boundary hypersurfaces with corresponding boundary defining functions x_1, \dots, x_k . Let V be a Euclidean

vector space, that is, a finite dimensional real vector space with inner product $\langle \cdot, \cdot \rangle_V$. Consider on V the function

$$\rho_V(v) = (1 + \langle v, v \rangle_V)^{-\frac{1}{2}}, \quad v \in V.$$

Let \overline{V} be the radial compactification of V as defined in [31], so that ρ_V extends to be a boundary defining function for $\partial\overline{V} \subset \overline{V}$. We can regard \overline{V} as a manifold with fibred corners, the fibration on the boundary being given by the identity map $\text{Id} : \partial\overline{V} \rightarrow \partial\overline{V}$. We can get a new manifold with fibred corners $(\overline{V} \times X, \varpi)$ by taking the Cartesian product of \overline{V} and X . The iterated fibration structure ϖ of $\overline{V} \times X$ is naturally induced from those of \overline{V} and X as follows. The fibration ϖ_0 on the boundary hypersurface $Z_0 = \partial\overline{V} \times X$ is given by the projection on $\partial\overline{V}$, while the fibration of the boundary hypersurface $Z_i = \overline{V} \times H_i$ is given by $\varpi_i = \text{Id} \times \pi_i$. The partial order on the boundary hypersurfaces of $\overline{V} \times X$ is specified by requiring that for all $i, j \in \{1, \dots, k\}$,

$$(6.1) \quad Z_0 < Z_i, \quad Z_i < Z_j \iff H_i < H_j.$$

Finally, the boundary defining function of $Z_i = \overline{V} \times H_i$ is taken to be the pullback of x_i to $\overline{V} \times X$, while we choose the boundary defining function x_0 of Z_0 to be the pullback of ρ_V to $\overline{V} \times X$.

Let E and F be smooth complex vector bundles on $\overline{V} \times X$ obtained by pulling back complex vector bundles on X to $\overline{V} \times X$. Consider then the space $\Psi_{\mathbb{S}}^m(\overline{V} \times X; E, F)$ of \mathbb{S} -operators of order m acting from sections of E to sections of F . From the previous section, we know that an operator $P \in \Psi_{\mathbb{S}}^m(\overline{V} \times X; E, F)$ induces a continuous linear map

$$(6.2) \quad P : \mathcal{S}(V \times X; E) \rightarrow \mathcal{S}(V \times X; F)$$

where $\mathcal{S}(V \times X; E) = \dot{C}_{Z_0}^\infty(\overline{V} \times X; E)$ is the space of smooth sections of E vanishing with all their derivatives at the boundary hypersurface $Z_0 = \partial\overline{V} \times X$, and similarly $\mathcal{S}(V \times X; F) = \dot{C}_{Z_0}^\infty(\overline{V} \times X; F)$. Given $v \in V$, consider the diffeomorphism

$$(6.3) \quad \begin{aligned} T_v : V \times X &\rightarrow V \times X \\ (w, p) &\mapsto (w + v, p) \end{aligned}$$

obtained by translating by v . Since E is the pullback of a vector bundle defined on X , we have a corresponding action

$$(6.4) \quad \begin{aligned} T_v^* : \mathcal{S}(V \times X; E) &\rightarrow \mathcal{S}(V \times X; E) \\ \psi &\mapsto \psi \circ T_v \end{aligned}$$

For the same reason, we have an action $T_v^* : \mathcal{S}(V \times X; F) \rightarrow \mathcal{S}(V \times X; F)$.

Definition 6.1. The space $\Psi_{\mathbb{S}\text{-sus}(V)}^m(X; E, F)$ of V -suspended \mathbb{S} -operators of order m on X acting from sections of E to sections of F is the subspace of operators P in $\Psi_{\mathbb{S}}^m(\overline{V} \times X; E, F)$ such that for all $v \in V$,

$$T_{-v}^* \circ P \circ T_v^* = P.$$

When $V = \mathbb{R}^p$, we use the notation $\Psi_{\mathbb{S}\text{-sus}(p)}^m(X; E, F) = \Psi_{\mathbb{S}\text{-sus}(\mathbb{R}^p)}^m(X; E, F)$ and say the corresponding operators are p -suspended.

In terms of the Schwartz kernel K_P seen as a distribution on $V^2 \times X^2$, the translation invariance in this definition means that for all $v \in V$,

$$(6.5) \quad T_{(v,v)}^* K_P = K_P$$

where $T_{(v,v)}$ is the diffeomorphism

$$(6.6) \quad \begin{aligned} T_v : \quad V^2 \times X^2 &\rightarrow V^2 \times X^2 \\ (w, w', p, p') &\mapsto (w + v, w' + v, p, p'). \end{aligned}$$

If

$$(6.7) \quad \begin{aligned} a : \quad V^2 &\rightarrow V \\ (v, v') &\mapsto (v - v') \end{aligned}$$

denotes the projection onto the anti-diagonal of V^2 , this means that K_P is the pullback via the map $a \times \text{Id} : V^2 \times X^2 \rightarrow V \times X^2$ of a distribution on $V \times X^2$.

To accurately describe this distribution, notice first that parallel transport with respect to the Euclidean metric on V gives a canonical identification of vector bundles $TV = V \times T_0V = V \times V$ extending naturally to a trivialization

$$(6.8) \quad \text{Id}T\bar{V} \cong \bar{V} \times V.$$

Using this identification and Corollary 3.2, one can see that the linear isomorphism

$$(6.9) \quad \begin{aligned} L : \quad V \times V &\rightarrow V \times V \\ (v', w) &\mapsto (v' + w, v') \end{aligned}$$

naturally extends to give an identification $\text{Id}T\bar{V} \cong \mathcal{G}_{\text{Id}}^{(1)}(\bar{V})$ of non-compact manifolds with boundary, where

$$\mathcal{G}_{\text{Id}}^{(1)}(\bar{V}) = (\bar{V}_{\text{Id}}^2 \setminus \partial\bar{V}_{\text{Id}}^2) \cup (\text{ff}_{\text{Id}} \setminus \partial\text{ff}_{\text{Id}})$$

is the Lie groupoid associated to \bar{V}_{Id}^2 . Since $a \circ L(v', w) = w$, this means the map a can be extended to a map

$$(6.10) \quad \bar{a} : \mathcal{G}_{\text{Id}}^{(1)}(\bar{V}) \rightarrow V$$

by composing the identification $\mathcal{G}_{\text{Id}}^{(1)}(\bar{V}) \cong \text{Id}T\bar{V} \cong \bar{V} \times V$ with the projection $\text{pr}_2 : \bar{V} \times V \rightarrow V$ on the second factor.

On the other hand, the ϖ -double space is naturally given by

$$(6.11) \quad (\bar{V} \times X)_{\varpi}^2 = \bar{V}_{\text{Id}}^2 \times X_{\pi}^2,$$

where \bar{V}_{Id}^2 is the Id-double space of the manifold with fibred boundary \bar{V} . Consider then the map

$$(6.12) \quad \alpha = \bar{a} \times \text{Id} : \mathcal{G}_{\text{Id}}^{(1)}(\bar{V}) \times X_{\pi}^2 \rightarrow V \times X_{\pi}^2.$$

In terms of this map, the translation invariance condition in Definition 6.1 means that as a distribution on $\mathcal{G}_{\text{Id}}^{(1)}(\bar{V}) \times X_{\pi}^2$, the Schwartz kernel of a V -suspended S -operator is the pullback of a distribution on $V \times X_{\pi}^2$. More precisely, we have obtained the following.

Lemma 6.2. *The space of Schwartz kernels of V -suspended S -operators of order m acting from sections of E to sections of F is given by*

$$\begin{aligned} \Psi_{S\text{-sus}(V)}^m(X; E, F) &= \{\alpha^* K ; K \in I^m(\bar{V} \times X_{\pi}^2; \{0\} \times \Delta_{\pi}; \mathcal{V}), \\ &\quad K \equiv 0 \text{ at } (\bar{V} \times \partial X_{\pi}^2 \setminus \text{ff}_{\pi}) \cup (\partial\bar{V} \times X_{\pi}^2)\}, \end{aligned}$$

where $\mathcal{V} = \text{pr}_2^*(\beta_\pi^* \text{Hom}(E, F) \otimes \pi_R^* \Omega) \otimes \text{pr}_1^* \text{Id} \Omega$ with $\text{pr}_1 : \overline{V} \times X_\pi^2 \rightarrow \overline{V}$ and $\text{pr}_2 : \overline{V} \times X_\pi^2 \rightarrow X_\pi^2$ the natural projections. From that perspective, the action of an operator $P \in \Psi_{\mathbb{S}\text{-sus}(V)}^m(X; E, F)$ on a section $u \in \mathcal{S}(V \times X; E)$ is given by

$$Pu = (\varpi_L)_*(\alpha^* K_P \cdot \varpi_R^* u),$$

where ϖ_L and ϖ_R are the analog of the maps (3.9) for the manifold with fibred corners $\overline{V} \times X$.

Seen as a distribution on $V \times X_\pi^2$, it is possible to take the Fourier transform in the V -factor of the Schwartz kernel K_P of a V -suspended \mathbb{S} -operator P ,

$$(6.13) \quad K_{\widehat{P}}(\Upsilon) = \int_V e^{-i\Upsilon \cdot v} K_P(v), \quad \Upsilon \in V^*.$$

We will call $\Upsilon \in V^*$ the **suspension parameter**. This gives for each $\Upsilon \in V^*$ the Schwartz kernel $K_{\widehat{P}}(\Upsilon)$ of a \mathbb{S} -operator $\widehat{P}(\Upsilon) \in \Psi_{\mathbb{S}}^m(X; E, F)$. Similarly, if ν denotes the translation invariant density on V associated to our choice of inner product $\langle \cdot, \cdot \rangle_V$, then we can define the Fourier transform

$$(6.14) \quad \mathcal{F}_E : \mathcal{S}(V \times X; E) \rightarrow \mathcal{S}(V^* \times X; E)$$

by

$$(6.15) \quad \widehat{u}(\Upsilon) = \mathcal{F}_E(u)(\Upsilon) = \int_V e^{-i\Upsilon \cdot v} u(v) \nu, \quad \Upsilon \in V^*,$$

with inverse Fourier transform given by

$$(6.16) \quad u(v) = \mathcal{F}_E^{-1}(\widehat{u})(v) = \frac{1}{(2\pi)^{\dim V}} \int_{V^*} e^{i\Upsilon \cdot v} \widehat{u}(\Upsilon) \nu^*,$$

where ν^* is the density on V^* dual to ν . With these definitions, we have as expected that the action of P on $\mathcal{S}(V \times X; E)$ can be described by

$$(6.17) \quad \widehat{P}u(\Upsilon) = \widehat{P}(\Upsilon)\widehat{u}(\Upsilon), \quad \forall \Upsilon \in V^*.$$

In other words, the Fourier transform of P is given by

$$(6.18) \quad \widehat{P} = \mathcal{F}_F \circ P \circ \mathcal{F}_E^{-1}.$$

If $Q \in \Psi_{\mathbb{S}\text{-sus}(V)}^m(X; G, E)$ is another V -suspended operator, where G is a complex vector bundle on $\overline{V} \times X$ given by the pullback of a complex vector bundle on X , then we have in particular that

$$(6.19) \quad \widehat{P \circ Q}(\Upsilon) = \widehat{P}(\Upsilon) \circ \widehat{Q}(\Upsilon).$$

That is, under the Fourier transform, the convolution product in the V -factor becomes pointwise composition. Since an operator P can be recovered from \widehat{P} by taking the inverse Fourier transform, we see that \widehat{P} completely describes the operator P . It is important however to notice that the Fourier transform of an operator $P \in \Psi_{\mathbb{S}\text{-sus}(V)}^m(X; E, F)$ is **not** an arbitrary smooth family of \mathbb{S} -operators. For instance, as can be readily seen by taking the Fourier transform of K_P in directions conormal to $\Delta_\pi \subset X_\pi^2$, we must have that

$$(6.20) \quad (D_\Upsilon^\alpha \widehat{P})(\Upsilon) \in \Psi_{\mathbb{S}}^{m-|\alpha|}(X; E, F), \quad \forall \alpha \in \mathbb{N}_0^{\dim V}, \quad \forall \Upsilon \in V^*.$$

For operators of order $-\infty$, we can completely characterize the image of the Fourier transform. It is given by smooth families of \mathbb{S} -operators

$$V^* \ni \Upsilon \mapsto \widehat{P}(\Upsilon) \in \Psi_{\mathbb{S}}^{-\infty}(X; E, F)$$

such that for any Fréchet semi-norm $\|\cdot\|$ of the space $\Psi_S^{-\infty}(X; E, F)$, we have

$$(6.21) \quad \sup_{\Upsilon} \|\Upsilon^\alpha D_\Upsilon^\beta \widehat{P}\| < \infty, \quad \forall \alpha, \beta \in \mathbb{N}_0^{\dim V}.$$

For operators of order $m \in \mathbb{R}$, one has more generally that for any Fréchet semi-norm $\|\cdot\|$ of $\Psi_S^m(X; E, F)$, the Fourier transform \widehat{P} of a suspended operator $P \in \Psi_{S-\text{sus}(V)}^m(X; E, F)$ must satisfy

$$(6.22) \quad \sup_{\Upsilon} \|(1 + |\Upsilon|^2)^{\frac{|\alpha|-m}{2}} D_\Upsilon^\alpha \widehat{P}\| < \infty \quad \forall \alpha \in \mathbb{N}_0^{\dim V}.$$

In this latter case however, these conditions are not sufficient to fully characterize the image of the Fourier transform.

The discussion above has a straightforward generalizations to families. Namely, consider a fibration

$$(6.23) \quad \begin{array}{ccc} F & \longrightarrow & H \\ & & \downarrow \phi \\ & & S \end{array}$$

where S is a manifold with corners and where the fibres are manifolds with fibred corners. We suppose that the fibration is locally trivial in the sense that for each $s \in S$, there is a neighborhood \mathcal{U} of s , a manifold with fibred corners F and a diffeomorphism $\psi : \phi^{-1}(\mathcal{U}) \rightarrow \mathcal{U} \times F$ inducing a commutative diagram

$$(6.24) \quad \begin{array}{ccc} \phi^{-1}(\mathcal{U}) & \xrightarrow{\psi} & \mathcal{U} \times F \\ & \searrow \phi & \swarrow \text{pr}_{\mathcal{U}} \\ & \mathcal{U} & \end{array}$$

such that for all $u \in \mathcal{U}$, the restriction

$$\psi : \phi^{-1}(u) \rightarrow \{u\} \times F$$

is a diffeomorphism of manifold with fibred corners. For such a fibration, we can consider the space of fibrewise S -operators of order m $\Psi_S^m(H/S; E, F)$ where E and F are smooth complex vector bundles on H . If moreover $V \rightarrow S$ is a smooth Euclidean vector bundle, that is, a smooth real vector bundle equipped with a fibrewise inner product, we can then consider the space of fibrewise V -suspended S -operators $\Psi_{S-\text{sus}(V)}^m(H/S; E, F)$. Thus, an operator $P \in \Psi_{S-\text{sus}(V)}^m(H/S; E, F)$ is a smooth family

$$S \ni s \mapsto P_s \in \Psi_{S-\text{sus}(V_s)}^m(\phi^{-1}(s); E, F)$$

of fibrewise V -suspended S -operators, where V_s is the fibre of V above $s \in S$.

7. SYMBOL MAPS

As for other calculi of pseudodifferential operators on singular spaces, various symbol maps can be defined. The ordinary symbol map can be defined in terms of the principal symbol map for conormal distributions introduced by Hörmander (see Theorem 18.2.11 in [14]),

$$(7.1) \quad I^m(Y, Z; \Omega_Y^{\frac{1}{2}}) \xrightarrow{\sigma_m} S^{[M]}(N^*Z; \Omega^{\frac{1}{2}}(N^*Z))$$

with $M = m - \frac{1}{4} \dim Y + \frac{1}{2} \dim Z$ where $\phi : N^*Z \rightarrow Z$ is the natural projection and

$$(7.2) \quad S^{[M]}(N^*Z) = S^M(N^*Z)/S^{M-1}(N^*Z),$$

where $S^M(N^*Z)$ is the usual space of functions $\psi \in C^\infty(N^*Z)$ such that in a local trivialization $N^*Z|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}_\xi^n$ with local variable u in \mathcal{U} ,

$$(7.3) \quad \sup_{u, \xi} \frac{|D_u^\alpha D_\xi^\beta \psi|}{(1 + |\xi|^2)^{\frac{M-|\beta|}{2}}} < \infty \quad \forall \alpha, \beta \in \mathbb{N}_0^n.$$

In our case, $Y = X_\pi^2$ and $Z = \Delta_\pi$. By Corollary 3.2, $N^*\Delta_\pi \cong \pi T^*X$. Since $\pi\Omega_R$ is naturally isomorphic to $\pi\Omega_L^{\frac{1}{2}} \otimes \pi\Omega_R^{\frac{1}{2}}$ when restricted to the diagonal and since the singular symplectic form of πT^*X provides a natural trivialization of $\Omega(\pi T^*X)$, we get a map

$$(7.4) \quad \Psi_S^m(X; E, F) \xrightarrow{\sigma_m} S^{[m]}(\pi T^*X; \phi^* \text{Hom}(E, F))$$

inducing a short exact sequence

$$(7.5) \quad 0 \longrightarrow \Psi_S^{m-1}(X; E, F) \longrightarrow \Psi_S^m(X; E, F) \xrightarrow{\sigma_m} S^{[m]}(\pi T^*X; \phi^* \text{Hom}(E, F)) \longrightarrow 0.$$

Here, $\phi : \pi T^*X \rightarrow X$ is the bundle projection. When we consider instead polyhomogeneous pseudodifferential operators of degree m , the principal symbol is a homogeneous section of degree m on $\pi T^*X \setminus \{0\}$, so it defines a map

$$(7.6) \quad \Psi_{S\text{-ph}}^m(X; E, F) \xrightarrow{\sigma_m} C^\infty(\pi S^*X; \Lambda^m \otimes \phi^* \text{Hom}(E, F))$$

where Λ is the dual of the tautological real line bundle of πS^*X .

Definition 7.1. An operator $P \in \Psi_S^m(X; E, F)$ is **elliptic** if its principal symbol $\sigma_m(P)$ is invertible.

To study the asymptotic behavior of S-operators at each boundary hypersurface, it is also useful to introduce other symbols, that is, normal operators in the terminology of [23]. Those additional symbol maps are defined by restricting the Schwartz kernel of the operator to the various front faces,

$$(7.7) \quad \sigma_{\partial_i} : \Psi_S^m(X; E, F) \rightarrow \Psi_{\text{ff}_{\pi_i}}^m(H_i; E, F)$$

with

$$(7.8) \quad \Psi_{\text{ff}_{\pi_i}}^m(H_i; E, F) = \left\{ K \in I^m(\text{ff}_{\pi_i}, \Delta_{\text{ff}_{\pi_i}}; \beta_\pi^*(\text{Hom}(E, F)) \otimes \pi_R^* \pi \Omega|_{\text{ff}_{\pi_i}}); \right. \\ \left. K \equiv 0 \text{ at } (\partial \text{ff}_{\pi_i} \cap \partial \text{ff}_\pi) \right\},$$

where $\Delta_{\text{ff}_{\pi_i}} = \text{ff}_{\pi_i} \cap \Delta_\pi$. The symbol map σ_{∂_i} clearly induces a short exact sequence

$$(7.9) \quad 0 \longrightarrow x_i \Psi_S^m(X; E, F) \longrightarrow \Psi_S^m(X; E, F) \xrightarrow{\sigma_{\partial_i}} \Psi_{\text{ff}_{\pi_i}}^m(H_i; E, F) \longrightarrow 0,$$

where x_i is the boundary defining function of H_i .

Remark 7.2. Since $\beta_\pi^*\left(\frac{x_i}{x_i}\right)$ is equal to 1 on ff_{π_i} , notice that for $z \in \mathbb{C}$,

$$P \in \Psi_S^m(X; E, F) \implies P_{i,z} := x_i^z P x_i^{-z} \in \Psi_S^m(X; E, F) \quad \text{with} \quad \sigma_{\partial_i}(P_{i,z}) = \sigma_{\partial_i}(P).$$

Clearly, the space $\mathcal{G}_{\text{ff}\pi_i}^{(1)} = \text{ff}\pi_i \setminus (\partial \text{ff}\pi_i \cap \partial \text{ff}\pi)$ has a natural Lie groupoid structure induced from the one of $\mathcal{G}_\pi^{(1)}$ with units given by $\mathcal{G}_{\text{ff}\pi_i}^{(0)} = \Delta_{\text{ff}\pi_i}$. The conormal distributions in $\Psi_{\text{ff}\pi}^m(X)$ which have compact support on $\mathcal{G}_{\text{ff}\pi_i}^{(1)}$ can be understood as elements of the algebra $\Psi^*(\mathcal{G}_{\text{ff}\pi_i}^{(1)})$ of pseudodifferential operators associated to the Lie groupoid $\mathcal{G}_{\text{ff}\pi_i}^{(1)}$.

The space $\Psi_{\text{ff}\pi_i}^m(H_i; E, F)$ can also be interpreted as a space of suspended S-operators. To see this, notice that since the fibres of the fibration $\pi_i : H_i \rightarrow S_i$ are naturally manifolds with fibre corners with typical fibre F_i having iterated fibration structure π_{F_i} , we can form the fibrewise π_{F_i} -double space

$$(7.10) \quad \begin{array}{ccc} (F_i)_{\pi_{F_i}}^2 & \xrightarrow{\quad} & (H_i \times_{\pi_i} H_i)_{\pi_{F_i}} \\ & & \downarrow \\ & & S_i. \end{array}$$

If $\overline{\pi NS_i}$ denotes the radial compactification of the vector bundle $\pi NS_i \rightarrow S_i$ defined in (2.10), then notice that the front face $\text{ff}\pi_i$ is naturally identified with the total space of the fibration obtained from the fibration (7.10) by pulling it back to $\overline{\pi NS_i}$. This means we have a natural fibration

$$(7.11) \quad \begin{array}{ccc} (F_i)_{\pi_{F_i}}^2 & \xrightarrow{\quad} & \text{ff}\pi_i \\ & & \downarrow \\ & & \overline{\pi NS_i}. \end{array}$$

With this identification, the Schwartz kernels in (7.8) corresponds to the Schwartz kernels of $\overline{\pi NS_i}$ -suspended S-operators associated to the fibration $\pi_i : H_i \rightarrow S_i$, that is,

$$(7.12) \quad \Psi_{\text{ff}\pi_i}^m(H_i; E, F) = \Psi_{\text{S-sus}(\overline{\pi NS_i})}^m(H_i/S_i; E, F).$$

Recalling the identification (2.10), we see that, as a suspended operator, the symbol $\sigma_{\partial_i}(P)$ has a natural action on Schwartz sections,

$$(7.13) \quad \sigma_{\partial_i}(P) : \mathcal{S}(\pi NH_i; E) \rightarrow \mathcal{S}(\pi NH_i; F).$$

Definition 7.3. An operator $P \in \Psi_{\text{S}}^m(X; E, F)$ is said to be **fully elliptic** if it is elliptic and if for all $i \in \{1, \dots, k\}$, $\sigma_{\partial_i}(P)$ is invertible as a map

$$\sigma_{\partial_i}(P) : \mathcal{S}(\pi NH_i; E) \rightarrow \mathcal{S}(\pi NH_i; F).$$

A V -suspended S-operator $P \in \Psi_{\text{S-sus}(V)}^m(X; E, F)$ is said to be **fully elliptic** if, as a S-operator in $\Psi_{\text{S}}^m(\overline{V} \times X; E, F)$, it is elliptic and if for all boundary hypersurfaces of the form $Z_i = \overline{V} \times H_i$, the corresponding symbol $\sigma_{\partial_i}(P)$ is invertible as a map

$$\sigma_{\partial_i}(P) : \mathcal{S}(\varpi NF_i; E) \rightarrow \mathcal{S}(\varpi NF_i; F).$$

If H_i and H_j are two hypersurfaces such that $H_i < H_j$, then the associated symbols σ_{∂_i} and σ_{∂_j} satisfy a certain compatibility condition, namely, their respective restrictions to $\text{ff}\pi_i \cap \text{ff}\pi_j$ agree. From the point of view of suspended operators, this

means that the restriction of $\sigma_{\partial_j}(P) \in \Psi_{\mathbb{H}\pi_j}^m(X)$ to ${}^\pi N^*S_j|_{S_{j_i}}$ is the symbol of the suspended family $\sigma_{\partial_i}(P)$ associated to the face $H_j \cap H_i$.

8. COMPOSITION

To show that \mathcal{S} -operators compose nicely, various strategy could be used. One approach consists in defining pseudodifferential operators using Lie groupoids as in [39], in which case the fact the composition of operators in the calculus remains in the calculus follows directly from the definition. As indicated earlier, the disadvantage with such an approach is that the inverse of an invertible operator is not typically within the algebra. Another approach, developed by Melrose and collaborators (see for instance [23]), is to consider a triple space suitably blown up where composition can be represented by a pushforward map coming from a b -fibration. The result then follows from the description in [27] of general mapping properties that such pushforward maps satisfy. Such an approach is likely to work in our context, but might involves a rather complicated triple space. Instead, we will proceed by less geometric means and follow the approach of [18] by working locally and using a proof by induction on the dimension of the manifold with fibred corners.

Theorem 8.1. *Let E, F, G be smooth complex vector bundles on a manifold with fibred corners X . Then for $A \in \Psi_{\mathcal{S}}^m(X; F, G)$ and $B \in \Psi_{\mathcal{S}}^n(X; E, F)$, we have that*

$$A \circ B \in \Psi_{\mathcal{S}}^{m+n}(X; E, G), \quad \text{with } \sigma_{\partial_i}(A \circ B) = \sigma_{\partial_i}(A) \circ \sigma_{\partial_i}(B)$$

for each hypersurface $H_i \subset X$ of X . Moreover, the induced map

$$\Psi_{\mathcal{S}}^m(X; F, G) \times \Psi_{\mathcal{S}}^n(X; E, F) \rightarrow \Psi_{\mathcal{S}}^{m+n}(X; E, G)$$

is continuous with respect to the natural Fréchet topology on each space. A similar result holds for polyhomogeneous \mathcal{S} -operators.

To describe the inductive step in the proof of this theorem, consider, for $p \in \mathbb{N}$, the new manifold with corners $\overline{\mathbb{R}^p} \times X$ where $\overline{\mathbb{R}^p}$ is the radial compactification of \mathbb{R}^p as described in [31]. A natural boundary defining function for the boundary $\partial\overline{\mathbb{R}^p} \cong \mathbb{S}^{p-1}$ is given by $(1 + r^2)^{-\frac{1}{2}}$ where r is the Euclidean distance from the origin.

Notice that $\overline{\mathbb{R}^p} \times X$ has a natural structure of manifold with fibred corners induced from the one of X . Indeed, the fibration on the boundary hypersurface $Z_0 = \partial\overline{\mathbb{R}^p} \times X$ is given by the projection on $\partial\overline{\mathbb{R}^p}$, while on the boundary hypersurface $Z_i = \overline{\mathbb{R}^p} \times H_i$, where $H_i \subset X$ is a hypersurface of X with fibration $\pi_i : H_i \rightarrow S_i$, the fibration is given by

$$\text{Id} \times \pi_i : \overline{\mathbb{R}^p} \times H_i \rightarrow \overline{\mathbb{R}^p} \times S_i.$$

Lemma 8.2. *Suppose that the conclusion of Theorem 8.1 holds for the manifold with fibred corners X . Then for $A \in \Psi_{\mathcal{S}}^m(\overline{\mathbb{R}^p} \times X)$ and $B \in \Psi_{\mathcal{S}}^n(\overline{\mathbb{R}^p} \times X)$, we have*

$$A \circ B \in \Psi_{\mathcal{S}}^{m+n}(\overline{\mathbb{R}^p} \times X), \quad \text{with } \sigma_{\partial_j}(A \circ B) = \sigma_{\partial_j}(A) \circ \sigma_{\partial_j}(B)$$

for all boundary hypersurfaces $Z_j \subset \overline{\mathbb{R}^p} \times X$.

Proof. Using the Fourier transform on \mathbb{R}^p , we can describe the action of operators $A \in \Psi_{\mathbb{S}}^m(\overline{\mathbb{R}^p} \times X)$ and $B \in \Psi_{\mathbb{S}}^n(\overline{\mathbb{R}^p} \times X)$ on $u \in \dot{C}^\infty(\overline{\mathbb{R}^p} \times X)$ by

$$(8.1) \quad \begin{aligned} Au(t) &= \frac{1}{(2\pi)^p} \int e^{i(t-t') \cdot \tau} a(t; \tau) u(t') dt' d\tau, \\ Bu(t) &= \frac{1}{(2\pi)^p} \int e^{i(t-t') \cdot \tau} b(t; \tau) u(t') dt' d\tau. \end{aligned}$$

Here, a and b are operator-valued symbols,

$$(8.2) \quad a \in \mathcal{C}^\infty(\overline{\mathbb{R}^p}; \Psi_{\mathbb{S}-\text{sus}(p)}^m(X)), \quad b \in \mathcal{C}^\infty(\overline{\mathbb{R}^p}; \Psi_{\mathbb{S}-\text{sus}(p)}^n(X)),$$

where $\Psi_{\mathbb{S}-\text{sus}(p)}^\ell(X)$ is the space of \mathbb{R}^p -suspended \mathbb{S} -operators of order ℓ on X and the variable $\tau \in \mathbb{R}^p$ in (8.1) is seen as the suspension parameter.

If we forget that a and b are operator-valued, then their symbol class is the one introduced in [40] and [49] (see also [31]). In this setting, there are standard methods to study the composition of operators, see for instance the proof of Propositione 1.4 in [40], or in the context of the Weyl calculus, the proof of Theorem 29.1 in [50]. Since the operator-valued symbols are such that

$$(8.3) \quad a \in \mathcal{C}^\infty(\overline{\mathbb{R}_t^p}; \Psi_{\mathbb{S}-\text{sus}(p)}^m(X)) \implies D_t^\alpha D_\tau^\beta a \in (1+t^2)^{-\frac{|\alpha|}{2}} \mathcal{C}^\infty(\overline{\mathbb{R}_t^p}; \Psi_{\mathbb{S}-\text{sus}(p)}^{m-|\beta|}(X)),$$

where τ is the suspension parameter, these methods have a straightforward generalization.

Indeed, let $c(t, \tau)$ be the operator-valued symbol such that

$$(8.4) \quad ABu(t) = \frac{1}{(2\pi)^p} \int e^{i(t-t') \cdot \tau} c(t, \tau) u(t') dt' d\tau.$$

As in [40], for each $N \in \mathbb{N}$, we have

$$(8.5) \quad c(t, \tau) = \sum_{|\alpha| < N} \frac{1}{\alpha!} \partial_\tau^\alpha a(t, \tau) D_t^\alpha b(t, \tau) + R_N(t, \tau)$$

with remainder term $R_N(t, \tau)$ given by

$$(8.6) \quad \sum_{|\alpha|=N} \frac{N}{\alpha!} \int_0^1 (1-\lambda)^{N-1} \left(\frac{1}{(2\pi)^p} \int e^{-iz \cdot \zeta} \partial_\tau^\alpha a(t, \tau + \lambda \zeta) D_t^\alpha b(t+z, \tau) dz d\zeta \right) d\lambda$$

Using our inductive hypothesis on the composition of \mathbb{S} -operators on X as well as (8.3), we can essentially proceed as in [40] to estimate the remainder term $R_N(t, \tau)$. The only significant difference is that taking a τ derivative of the symbol a not only improves its decay as τ tends to infinity, but it also reduces its order as an operator on X . If we only focus on the decay behavior in τ , we obtain

$$R_N \in \mathcal{S}_{-N}^{m+n-N}(\mathbb{R}_t^p \times \mathbb{R}_\tau^p; \Psi_{\mathbb{S}}^{m+n}(X))$$

where $\mathcal{S}_\ell^k(\mathbb{R}_t^p \times \mathbb{R}_\tau^p; \Psi_{\mathbb{S}}^{m+n}(X))$ is the space of operator-valued symbols q such that for any Fréchet semi-norm $\|\cdot\|$ of $\Psi_{\mathbb{S}}^{m+n}(X)$,

$$\sup_{t, \tau} \left((1+|\tau|^2)^{\frac{|\beta|-k}{2}} (1+|t|^2)^{\frac{|\alpha|-\ell}{2}} \|D_t^\alpha D_\tau^\beta q\| \right) < \infty \quad \forall \alpha, \beta \in \mathbb{N}_0^p.$$

It is more useful however to control R_N as a symbol valued in a space of lower order \mathbb{S} -operators. Thus, if instead we use half of the τ derivative to reduce the

order of the operator, we obtain for N even

$$(8.7) \quad R_N \in \mathcal{S}_{-N}^{m+n-\frac{N}{2}}(\mathbb{R}_t^p \times \mathbb{R}_\tau^p; \Psi_S^{m+n-\frac{N}{2}}(X)).$$

By Borel's lemma, there is an operator-valued symbol $e \in \mathcal{C}^\infty(\overline{\mathbb{R}_t^p}; \Psi_{S-\text{sus}(p)}^{m+n}(X))$ such that

$$e(t, \tau) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_\tau^\alpha a(t, \tau) D_t^\alpha b(t, \tau),$$

where this notation means that for all $N \in \mathbb{N}$,

$$e(t, \tau) - \sum_{\alpha < N} \frac{1}{\alpha!} \partial_\tau^\alpha a(t, \tau) D_t^\alpha b(t, \tau) \in (1 + |t|^2)^{-\frac{N}{2}} \mathcal{C}^\infty(\overline{\mathbb{R}_t^p}; \Psi_{S-\text{sus}(p)}^{m+n-N}(X)),$$

we see from (8.7) that

$$c(t, \tau) - e(t, \tau) \in \dot{\mathcal{C}}^\infty(\overline{\mathbb{R}_t^p} \times \overline{\mathbb{R}_\tau^p}; \Psi_S^{-\infty}(X)) = \dot{\mathcal{C}}^\infty(\overline{\mathbb{R}_t^p}; \Psi_{S-\text{sus}(p)}^{-\infty}(X)).$$

Therefore, we have that

$$(8.8) \quad c \in \mathcal{C}^\infty(\overline{\mathbb{R}_t^p}; \Psi_{S-\text{sus}(p)}^{m+n}(X)) \quad \text{with } c \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_\tau^\alpha a(t, \tau) D_t^\alpha b(t, \tau),$$

which implies in particular that $AB \in \Psi_S^{m+n}(\overline{\mathbb{R}^p} \times X)$. Moreover, since this argument automatically provides uniform control on each of the Fréchet semi-norms of c in terms of the Fréchet semi-norms of a and b , we see that the induced map

$$\Psi_S^m(\overline{\mathbb{R}^p} \times X) \times \Psi_S^n(\overline{\mathbb{R}^p} \times X) \rightarrow \Psi_S^m(\overline{\mathbb{R}^p} \times X)$$

is continuous with respect to the natural Fréchet topology on each spaces.

By our assumptions on X , It follows directly that $\sigma_{\partial_j}(A \circ B) = \sigma_{\partial_j}(A) \circ \sigma_{\partial_j}(B)$ for hypersurfaces of the form $Z_j = \overline{\mathbb{R}^p} \times H_j$, while for the hypersurface $Z_0 = \partial \overline{\mathbb{R}^p} \times X$, it is a consequence of the asymptotic expansion of the operator-valued symbol c in (8.8), namely

$$\sigma_{\partial_0}(AB) = c|_{\partial \overline{\mathbb{R}_t^p}} = a|_{\partial \overline{\mathbb{R}_t^p}} b|_{\partial \overline{\mathbb{R}_t^p}} = \sigma_{\partial_0}(A) \sigma_{\partial_0}(B).$$

□

To proceed further, we need some notation. For each hypersurface H_i of X , let

$$(8.9) \quad \nu_i : H_i \times [0, \epsilon_i) \rightarrow \mathcal{U}_i \subset X$$

be a tubular neighborhood of H_i in X compatible with the boundary defining function x_i , that is, such that $x_i(\nu_i(h, t)) = t$ for $h \in H_i$ and $t \in [0, \epsilon_i)$. To show that \mathcal{S} -operators form an algebra, we will use the previous lemma to deal with \mathcal{S} -operators having their Schwartz kernels supported near the front face ff_{π_i} , but supported away from the front faces ff_{π_j} for $H_j < H_i$.

Lemma 8.3. *Suppose that the conclusions of Theorem 8.1 hold for all manifolds with fibred corners Y of dimension less than the one of X . Suppose that $A \in \Psi_S^m(X)$ and $B \in \Psi_S^n(X)$ are such that their Schwartz kernels are supported inside the set*

$$\beta_\pi^{-1}(\nu_i(\pi_i^{-1}(V_i) \times [0, \epsilon_i))^2) \subset X_\pi^2$$

where $V_i \subset S_i \setminus \partial S_i$ is some open set in the interior of the base S_i of the fibration $\pi_i : H_i \rightarrow S_i$. Then

$$A \circ B \in \Psi_S^{m+n}(X) \quad \text{with } \sigma_{\partial_j}(A \circ B) = \sigma_{\partial_j}(A) \circ \sigma_{\partial_j}(B)$$

for all hypersurfaces $H_j \subset X$.

Proof. Let F_i be the typical fibre of the fibration

$$(8.10) \quad \begin{array}{ccc} F_i & \longrightarrow & H_i \\ & & \downarrow \pi_i \\ & & S_i. \end{array}$$

Then as described in § 1, the fibre F_i is naturally a manifold with fibred corners. Since $\dim F_i < \dim X$, it is part of our assumptions that $\Psi_{\mathfrak{S}}^m(F_i) \circ \Psi_{\mathfrak{S}}^n(F_i) \subset \Psi_{\mathfrak{S}}^{m+n}(F_i)$. The strategy of the proof is to reduce composition of the operators A and B to Lemma 8.2.

Let $\{W_q\}_{q \in \mathcal{Q}}$ be a finite covering of the closure of V_i in $S_i \setminus \partial S_i$ by open sets in $S_i \setminus \partial S_i$ diffeomorphic to open balls and such that the fibration (8.10) restricts to a trivial fibration over each S_i . Let $\varphi_q \in \mathcal{C}_c^\infty(W_q)$ be functions which restricts to give a partition of unity on V_i and let $\tilde{\varphi}_q \in \mathcal{C}_c^\infty(W_q)$ be functions such that $\varphi_q \tilde{\varphi}_q = \varphi_q$. Let

$$\psi_q = (\nu_i)_* \text{pr}^* \pi_i^* \varphi_q, \quad \tilde{\psi}_q = (\nu_i)_* \text{pr}^* \pi_i^* \tilde{\varphi}_q,$$

be the corresponding pulled back functions on $\mathcal{U}_i = \nu_i(H_i \times [0, \epsilon_i])$ where $\text{pr} : H_i \times [0, \epsilon_i] \rightarrow H_i$ is the projection on the left factor. Then we can write the operator A as

$$(8.11) \quad A = \sum_q A\psi_q = \sum_q \left(\tilde{\psi}_q A\psi_q + (1 - \tilde{\psi}_q) A\psi_q \right).$$

Since $\tilde{\psi}_q \psi_q = \psi_q$, the Schwartz kernel of the second term is supported away from the diagonal in X^2 , which means it is an element of $\dot{\Psi}_{\mathfrak{S}}^{-\infty}(X)$. Thus, we have that

$$(8.12) \quad A \equiv \sum_q \tilde{\psi}_q A\psi_q \quad \text{mod } \dot{\Psi}_{\mathfrak{S}}^{-\infty}(X).$$

Similarly, we have that

$$(8.13) \quad \psi_q B = \psi_q B \tilde{\psi}_q + \psi_q B (1 - \tilde{\psi}_q) \equiv \psi_q B \tilde{\psi}_q \quad \text{mod } \dot{\Psi}_{\mathfrak{S}}^{-\infty}(X).$$

Thus, using Corollary 5.5, we see that

$$(8.14) \quad \begin{aligned} AB &\equiv \sum_q \tilde{\psi}_q A\psi_q B \quad \text{mod } \dot{\Psi}_{\mathfrak{S}}^{-\infty}(X) \\ &= \sum_q \tilde{\psi}_q A\psi_q \tilde{\psi}_q B = \sum_q (\tilde{\psi}_q A \tilde{\psi}_q) \psi_q B \\ &\equiv \sum_q (\tilde{\psi}_q A \tilde{\psi}_q) (\psi_q B \tilde{\psi}_q) \quad \text{mod } \dot{\Psi}_{\mathfrak{S}}^{-\infty}(X). \end{aligned}$$

This means we can assume both K_A and K_B are supported in the subset

$$\beta_\pi^{-1}(\nu_i(\pi_i^{-1}(W_q) \times [0, \epsilon])^2) \subset X_\pi^2.$$

Since we are assuming W_q is diffeomorphic to an open ball, this means there exists an embedding

$$(8.15) \quad \iota_q : W_q \hookrightarrow \mathbb{S}^{p_i-1},$$

where $p_i - 1 = \dim S_i = \dim W_q$. Since the fibration π_i is trivial when restricted to W_q , we can assume

$$\pi_i^{-1}(W_q) = F_i \times W_q$$

with π_i given by projecting on the right factor. The embedding (8.15) can be extended to an embedding

$$(8.16) \quad W_q \times [0, \epsilon_i) \xrightarrow{\iota_q \times \text{Id}} \mathbb{S}^{p_i-1} \times [0, \epsilon_i) \longrightarrow \overline{\mathbb{R}^{p_i}}$$

where the second map is the standard collar neighborhood of $\mathbb{S}^{p_i-1} = \partial \overline{\mathbb{R}^{p_i-1}}$ in the radial compactification $\overline{\mathbb{R}^{p_i-1}}$ of \mathbb{R}^{p_i} using the boundary defining function $\frac{1}{\sqrt{r^2+1}}$ where r is the distance from the origin.

Via these identifications, this means we can regard A and B as operators acting on functions of $\mathbb{R}^{p_i} \times F_i$, more precisely: $A \in \Psi_S^m(\overline{\mathbb{R}^{p_i}} \times F_i)$, $B \in \Psi_S^n(\overline{\mathbb{R}^{p_i}} \times F_i)$. The result then follows by applying Lemma 8.2. \square

We have now all the ingredients to prove the composition theorem.

Proof of Theorem 8.1. By using a partition of unity, we can work locally in open sets where the vector bundles E , F and G are trivial. Thus, without loss of generality, we can assume that $E = F = G = \underline{\mathbb{C}}$ and $A \in \Psi_S^m(X)$, $B \in \Psi_S^n(X)$.

Since the case where $\dim X = 0$ is trivial, we can assume by induction on the dimension that the theorem is true for manifolds with fibred corners of dimension less than the one of X . For each boundary hypersurface H_i of X , consider the tubular neighborhood $\nu_i : H_i \times [0, \epsilon_i)_{x_i} \rightarrow \mathcal{U}_i \subset X$ of (8.9). Let also $\chi_i, \tilde{\chi}_i, \hat{\chi}_i \in \mathcal{C}_c^\infty(\mathcal{U}_i) \subset \mathcal{C}^\infty(X)$ be non-negative cut-off functions such that $\hat{\chi}_i \equiv 1$ near H_i , $\chi_i \hat{\chi}_i = \hat{\chi}_i$ and $\tilde{\chi}_i \chi_i = \chi_i$. Using the cut-off functions $\chi_i, \tilde{\chi}_i$ and $\hat{\chi}_i$, we can rewrite the composition of A and B as

$$(8.17) \quad \begin{aligned} AB &= A\chi_i B + A(1 - \chi_i)B \\ &= \tilde{\chi}_i A\chi_i B + (1 - \tilde{\chi}_i)A\chi_i B + \hat{\chi}_i A(1 - \chi_i)B + (1 - \hat{\chi}_i)A(1 - \chi_i)B. \end{aligned}$$

Since $\chi_i \hat{\chi}_i = \hat{\chi}_i$ and $\tilde{\chi}_i \chi_i = \chi_i$, the Schwartz kernels of $(1 - \tilde{\chi}_i)A\chi_i$ and $\hat{\chi}_i A(1 - \chi_i)$ are both supported away from the diagonal in $X \times X$, which means the operators $(1 - \tilde{\chi}_i)A\chi_i$ and $\hat{\chi}_i A(1 - \chi_i)$ are both in $\dot{\Psi}_S^{-\infty}(X)$. Thus, using Corollary 5.5, we see that modulo operators in $\dot{\Psi}_S^{-\infty}(X)$, we have

$$(8.18) \quad AB \equiv \tilde{\chi}_i A\chi_i B + (1 - \hat{\chi}_i)A(1 - \chi_i)B \pmod{\dot{\Psi}_S^{-\infty}(X)}.$$

Similarly, if $\chi'_i \in \mathcal{C}_c^\infty(\mathcal{U}_i)$ is such that $\hat{\chi}_i \chi'_i = \chi'_i$ and $\chi'_i \equiv 1$ near H_i , then we can write the operator B as

$$(8.19) \quad \begin{aligned} B &= \hat{\chi}_i B\chi_i + \hat{\chi}_i B(1 - \chi_i) + (1 - \hat{\chi}_i)B\chi'_i + (1 - \hat{\chi}_i)B(1 - \chi'_i) \\ &\equiv \hat{\chi}_i B\chi_i + (1 - \hat{\chi}_i)B(1 - \chi'_i) \pmod{\dot{\Psi}_S^{-\infty}(X)}. \end{aligned}$$

If $\check{\chi}_i \in \mathcal{C}_c^\infty(\mathcal{U}_i)$ is another cut-off function such that $\tilde{\chi}_i \check{\chi}_i = \tilde{\chi}_i$, then we can also write B as

$$(8.20) \quad \begin{aligned} B &= \tilde{\chi}_i B\check{\chi}_i + \tilde{\chi}_i B(1 - \check{\chi}_i) + (1 - \tilde{\chi}_i)B\chi_i + (1 - \tilde{\chi}_i)B(1 - \chi_i) \\ &\equiv \tilde{\chi}_i B\check{\chi}_i + (1 - \tilde{\chi}_i)B(1 - \chi_i) \pmod{\dot{\Psi}_S^{-\infty}(X)}. \end{aligned}$$

Substituting (8.19) and (8.20) in (8.18), we see by Corollary 5.5 that

$$(8.21) \quad \begin{aligned} AB &\equiv (\tilde{\chi}_i A\chi_i)(\tilde{\chi}_i B\check{\chi}_i) + \tilde{\chi}_i A\chi_i(1 - \tilde{\chi}_i)B(1 - \chi_i) + (1 - \hat{\chi}_i)A(1 - \chi_i)\hat{\chi}_i B\chi_i \\ &\quad + (1 - \hat{\chi}_i)A(1 - \chi_i)(1 - \hat{\chi}_i)B(1 - \chi'_i) \pmod{\dot{\Psi}_S^{-\infty}(X)} \\ &\equiv (\tilde{\chi}_i A\chi_i)(\tilde{\chi}_i B\check{\chi}_i) + (1 - \hat{\chi}_i)A(1 - \chi_i)(1 - \hat{\chi}_i)B(1 - \chi'_i) \pmod{\dot{\Psi}_S^{-\infty}(X)}. \end{aligned}$$

Thus, from (8.21), we can reduce the problem of composition to two situations,

- K_A and K_B are supported near ff_{π_i} ;
- K_A and K_B are supported away from ff_{π_i} .

In particular, if H_i is a minimal hypersurface with respect to the partial order of hypersurfaces of X , then the first term on the right hand side of (8.21) can be taken care of by Lemma 8.3. In fact, starting with the minimal hypersurfaces H_i and proceeding recursively on the partial order of boundary hypersurfaces of X using (8.21) and Lemma 8.3 at each step, we can reduce to the case where K_A and K_B are supported away from ff_{π_i} for all i . Adding operators in $\dot{\Psi}_{\mathcal{S}}^{-\infty}(X)$ if necessary, we can even reduce to the case the Schwartz kernels of A and B have compact support in $(X \setminus \partial X)^2$. By doubling X to get a smooth closed manifold, this reduces to the standard result about composition of pseudodifferential operators on closed manifolds. It is straightforward to check that polyhomogeneity is preserved under composition. \square

9. MAPPING PROPERTIES

Let (X, π) be a manifold with fibred corners. Let H_1, \dots, H_k be its boundary hypersurfaces with choice of boundary defining functions x_1, \dots, x_k . As for the Φ -calculus of [23], an important ingredient in the study of mapping properties of \mathcal{S} -operators is the construction of a parametrix for fully elliptic operators. We will also need such a result for \mathcal{S} -suspended operators, in which case the notation

$$\begin{aligned} \dot{\Psi}_{\mathcal{S}\text{-sus}(V)}^{-\infty}(X; E_1, E_2) &= \{A \in \Psi_{\mathcal{S}\text{-sus}(V)}^{-\infty}(X; E_1, E_2); \\ &\quad \widehat{A}(\Upsilon) \in \dot{\Psi}_{\mathcal{S}}^{-\infty}(X; E_1, E_2) \forall \Upsilon \in V^*\}, \end{aligned}$$

for E_1 and E_2 complex vector bundles over X , is useful to describe the error term.

Proposition 9.1 (Parametrix). *If $P \in \Psi_{\mathcal{S}}^m(X; E, F)$ is fully elliptic, then there exists $Q \in \Psi_{\mathcal{S}}^{-m}(X; F, E)$ such that*

$$\text{Id} - QP \in \dot{\Psi}_{\mathcal{S}}^{-\infty}(X; E), \quad \text{Id} - PQ \in \dot{\Psi}_{\mathcal{S}}^{-\infty}(X; F).$$

Moreover, $\ker P \subset \dot{C}^\infty(X; E)$ and $\ker P^* \subset \dot{C}^\infty(X; F)$. Similarly, if V is an Euclidean vector space and $P \in \Psi_{\mathcal{S}\text{-sus}(V)}^m(X; E, F)$ is fully elliptic, then there exists $Q \in \Psi_{\mathcal{S}\text{-sus}(V)}^{-m}(X; F, E)$ such that

$$\text{Id} - QP \in \dot{\Psi}_{\mathcal{S}\text{-sus}(V)}^{-\infty}(X; E), \quad \text{Id} - PQ \in \dot{\Psi}_{\mathcal{S}\text{-sus}(V)}^{-\infty}(X; F).$$

Proof. Using this proposition and Corollary 9.2 below and proceeding by induction on the dimension of X , we can assume that $\sigma_{\partial_i}(P)^{-1} \in \Psi_{\text{ff}_{\pi_i}}^{-m}(H_i; F, E)$. This means we can choose $Q_0 \in \Psi_{\mathcal{S}}^{-m}(X; F, E)$ such that $\sigma_{-m}(Q_0) = \sigma_m(P)^{-1}$ and $\sigma_{\partial_j}(Q_0) = \sigma_{\partial_j}(P)^{-1}$. Then we have

$$(9.1) \quad \text{Id} - Q_0P \in x\Psi_{\mathcal{S}}^{-1}(X; E), \quad \text{Id} - PQ_0 \in x\Psi_{\mathcal{S}}^{-1}(X; F).$$

Suppose for a proof by induction that we have defined $Q_\ell \in x^\ell \Psi_{\mathcal{S}}^{-m-\ell}(X; F, E)$ for $\ell \leq n$ such that $\tilde{Q}_n = Q_1 + \dots + Q_n$ satisfies

$$(9.2) \quad \text{Id} - \tilde{Q}_n P \in x^{n+1} \Psi_{\mathcal{S}}^{-n-1}(X; E), \quad \text{Id} - P \tilde{Q}_n \in x^{n+1} \Psi_{\mathcal{S}}^{-n-1}(X; F).$$

Then, setting $\tilde{Q}_{n+1} = \tilde{Q}_n + Q_{n+1}$, we would like to find $Q_{n+1} \in x^{n+1}\Psi_S^{-m-n-1}(X; F, E)$ such that

$$(9.3) \quad \text{Id} - \tilde{Q}_{n+1}P = \text{Id} - \tilde{Q}_n P - Q_{n+1}P \in x^{n+2}\Psi_S^{-n-2}(X; E),$$

that is, such that

$$(9.4) \quad Q_{n+1}P = \text{Id} - \tilde{Q}_n P \quad \text{modulo} \quad x^{n+2}\Psi_S^{-n-2}(X; E).$$

Thus, taking $Q_{n+1} = (\text{Id} - \tilde{Q}_n P)\tilde{Q}_n$ will give

$$(9.5) \quad \text{Id} - \tilde{Q}_{n+1}P \in x^{n+2}\Psi_S^{-n-2}(X; E)$$

with $\tilde{Q}_{n+1} = \tilde{Q}_n + Q_{n+1}$. As one can check, we will also have that

$$(9.6) \quad \text{Id} - PQ_{n+1} \in x^{n+2}\Psi_S^{-n-2}(X; F).$$

We can then define Q to be the asymptotic sum of the Q_ℓ giving the desired parametrix. If $f \in \ker P$, then

$$(9.7) \quad \begin{aligned} Pf = 0 &\Rightarrow QPf = 0 \\ &\Rightarrow f = (\text{Id} - QP)f \in \dot{C}^\infty(X; E) \end{aligned}$$

since $\text{Id} - QP \in \dot{\Psi}_S^{-\infty}(X; E)$. There is a similar argument for the kernel of P^* . For fully elliptic V -suspended S -operators, the proof is similar and is left to the reader. \square

Corollary 9.2. *If V is an Euclidean vector space and $P \in \Psi_{S\text{-sus}(V)}^m(X; E, F)$ is a fully elliptic V -suspended operators which is invertible as a map $P : \mathcal{S}(V \times X; E) \rightarrow \mathcal{S}(V \times X; F)$, then it has an inverse in $\Psi_{S\text{-sus}(V)}^{-m}(X; F, E)$.*

Proof. Let $P \in \Psi_{S\text{-sus}(V)}^m(X; E, F)$ be as in the statement of the corollary and let $Q \in \Psi_{S\text{-sus}(V)}^{-m}(X; F, E)$ be the parametrix of Proposition 9.1. in particular, we have that

$$\hat{P}(\Upsilon)\hat{Q}(\Upsilon) = \text{Id} + \hat{R}(\Upsilon), \quad \forall \Upsilon \in V^*, \quad \text{where } R \in \dot{\Psi}_{S\text{-sus}(V)}^{-\infty}(X; F).$$

By (6.21) we see that $\hat{R}(\Upsilon)$ is small for $|\Upsilon|$ large, so that there exists $K > 0$ with the property that $\text{Id} + \hat{R}(\Upsilon)$ is invertible for $|\Upsilon| > K$ with inverse of the form $\text{Id} + \hat{S}(\Upsilon)$, where

$$\hat{S}(\Upsilon) = \sum_{k=1}^{\infty} (-1)^k \hat{R}(\Upsilon)^k \in \dot{\Psi}_S^{-\infty}(X; F)$$

satisfies (6.21). Thus, for $|\Upsilon| > K$, we have that

$$(9.8) \quad \hat{P}(\Upsilon)^{-1} = \hat{Q}(\Upsilon)(\text{Id} + \hat{S}(\Upsilon)).$$

Now, the invertibility of P clearly implies the invertibility of $\hat{P}(\Upsilon)$ for all $\Upsilon \in V^*$. Using the parametrix Q , we have

$$(9.9) \quad \hat{P}(\Upsilon)^{-1} = \hat{P}(\Upsilon)^{-1}(\hat{P}(\Upsilon)\hat{Q}(\Upsilon) - \hat{R}(\Upsilon)) = \hat{Q}(\Upsilon) - \hat{P}(\Upsilon)^{-1}\hat{R}(\Upsilon).$$

By Proposition 5.4, we must have $\hat{P}(\Upsilon)^{-1}\hat{R}(\Upsilon) \in \dot{\Psi}_S^{-\infty}(X; F, E)$ for all $\Upsilon \in V^*$. Thus, from (9.8) and (9.9), we see that

$$\hat{P}(\Upsilon)^{-1} = \hat{Q}(\Upsilon) + \hat{W}(\Upsilon),$$

where $W \in \dot{\Psi}_{\mathcal{S}\text{-sus}(V)}^{-\infty}(X; F, E)$ is such that $\widehat{W}(\Upsilon) = \widehat{Q}(\Upsilon)\widehat{S}(\Upsilon)$ for $|\Upsilon| > K$. Taking the inverse Fourier transform, we finally obtain that

$$P^{-1} = Q + W \in \Psi_{\mathcal{S}\text{-sus}(V)}^{-m}(X; F, E).$$

□

As we will see, this last corollary will be useful to study the action of \mathcal{S} -operators on square integrable functions. Precisely, let g_π be a choice of \mathcal{S} -metric and let $dg_\pi \in \mathcal{C}^\infty(X; \pi\Omega)$ be its volume form. Let $L_{g_\pi}^2(X)$ be the corresponding space of functions on $X \setminus \partial X$ that are square integrable with respect to the density dg_π . To establish the L^2 -boundedness of \mathcal{S} -pseudodifferential operators of order zero, we will, as in [23], follow the standard trick of Hörmander relying on the construction of an approximate square root.

Proposition 9.3. *If $B \in \Psi_{\mathcal{S}}^0(X)$ is formally self-adjoint with respect to a positive \mathcal{S} -density ν on X , then there exists $C > 0$ sufficiently large so that*

$$C + B = A^*A + R$$

for some $A \in \Psi_{\mathcal{S}}^0(X)$ and $R \in \dot{\Psi}_{\mathcal{S}}^{-\infty}(X)$.

Proof. The proof is by induction on the depth of X . The case where X is a closed manifold is well-known and the case where X is a manifold with boundary is proven by Mazzeo and Melrose in [23].

For $i \in \{1, \dots, k\}$, let ν_{S_i} be a positive section of $\Omega(\pi N S_i)$ and write $\nu|_{H_i} = \nu_{F_i} \otimes \nu_{S_i}$ where ν_{F_i} is a positive density in the fibres of the fibration $\pi_i : H_i \rightarrow S_i$. Then the suspended family of \mathcal{S} -pseudodifferential operators $\hat{\sigma}_{\partial_i}(B)(\eta)$ with $\eta \in \pi N^* S_i$ is formally self-adjoint with respect to the density ν_{F_i} . By our induction hypothesis (see Corollary 9.6), for $C > 0$ big enough, $(\hat{\sigma}_{\partial_i}(B)(\eta) + C) \in \Psi_{\mathcal{S}}^0(F_i)$ has a unique positive square root so that $(C + \sigma_{\partial_i}(B))$ also has a unique positive square root in $\Psi_{\text{ff}\pi_i}^0(X)$. Similarly, $(C + \sigma_0(B))$ has unique positive square root provided $C > 0$ is large enough. Thus, we can find $A_0 \in \Psi_{\mathcal{S}}^0(X)$ such that

$$(9.10) \quad \sigma_0(A_0) = (C + \sigma_0(B))^{\frac{1}{2}}, \quad \sigma_{\partial_i}(A_0) = (C + \sigma_{\partial_i}(B))^{\frac{1}{2}}, \quad i \in \{1, \dots, k\}.$$

Replacing A_0 by $\frac{1}{2}(A_0 + A_0^*)$ if necessary, we can assume that A_0 is formally self-adjoint with

$$(9.11) \quad C + B - A_0^2 \in x\Psi_{\mathcal{S}}^{-1}(X).$$

To get an error term in $\dot{\Psi}_{\mathcal{S}}^{-\infty}(X)$, we can proceed by induction. Thus, assume that we have found a formally self-adjoint operator $A_\ell \in \Psi_{\mathcal{S}}^0(X)$ such that

$$(9.12) \quad C + B - A_\ell^2 = R_{\ell+1} \in x^{\ell+1}\Psi_{\mathcal{S}}^{-\ell-1}(X).$$

Writing $A_{\ell+1} = A_\ell + Q_\ell$ where the formally self-adjoint operator $Q_\ell \in x^{\ell+1}\Psi_{\mathcal{S}}^{-\ell-1}(X)$ is to be found, we have

$$(9.13) \quad \begin{aligned} C + B - A_{\ell+1}^2 &= R_{\ell+1} - Q_\ell A_\ell - A_\ell Q_\ell - Q_\ell^2 \\ &= R_{\ell+1} - Q_\ell A_\ell - A_\ell Q_\ell \end{aligned}$$

modulo $x^{\ell+2}\Psi_{\mathcal{S}}^{-\ell-2}(X)$. First, this means we need to solve

$$(9.14) \quad \sigma_{-\ell-1}(R_{\ell+1}) = 2\sigma_0(A_\ell)\sigma_{-\ell-1}(Q_\ell),$$

which clearly has a formally self-adjoint solution $Q_{\ell,0} \in x^{\ell+1}\Psi_S^{-\ell-1}(X)$. Thus, replacing A_ℓ by $A_{\ell,0} = A_\ell + Q_{\ell,0}$ in (9.13), this means we have to solve (9.13) with $R_{\ell+1}$ replaced by $R_{\ell+1,0} \in x^{\ell+1}\Psi_S^{-\ell-2}(X)$.

Proceeding by induction on $i \in \{1, \dots, k\}$ and with the convention that $x_0 = 1$ and $w_i = \prod_{j=1}^i x_j$, assume more generally we have found formally self-adjoint operators $Q_{\ell,j} \in w_{j-1}x^{\ell+1}\Psi_S^{-\ell-2}(X)$ for $j \leq i-1$ such that $A_{\ell,i-1} = A_\ell + \sum_{j=0}^{i-1} Q_{\ell,j}$ satisfies

$$(9.15) \quad C + B - A_{\ell,i-1}^2 = R_{\ell+1,i-1} \in w_{i-1}x^{\ell+1}\Psi_S^{-\ell-2}(X).$$

To find $Q_{\ell,i}$, write $Q_{\ell,i} = x_i^{\ell+1}\tilde{Q}_{\ell,i}$ where $\tilde{Q}_{\ell,i} \in x_i^{-\ell-1}w_{i-1}x^{\ell+1}\Psi_S^{-\ell-2}(X)$. Using Remark 7.2, this means we need to solve

$$(9.16) \quad \hat{\sigma}_{\partial_i}(x_i^{-\ell-1}R_{\ell+1,i-1}) = \hat{\sigma}_{\partial_i}(\tilde{Q}_{\ell,i})\hat{\sigma}_{\partial_i}(A_0) + \hat{\sigma}_{\partial_i}(A_0)\hat{\sigma}_{\partial_i}(\tilde{Q}_{\ell,i}).$$

As pointed out in [23], this is solvable with $\tilde{Q}_{\ell,i}$ formally self-adjoint as $\hat{\sigma}_{\partial_i}(A_0)^2$ is positive and (9.16) is the linearization of the square root equation

$$(9.17) \quad (\hat{\sigma}_{\partial_i}(A_0) + \hat{\sigma}_{\partial_i}(\tilde{Q}_{\ell,i}))^2 = \hat{\sigma}_{\partial_i}(x_i^{-\ell-1}R_{\ell+1,i-1}) + \hat{\sigma}_{\partial_i}(A_0)^2.$$

Thus, we can find $Q_{\ell,i}$ such that (9.16) satisfied. Replacing $Q_{\ell,i}$ by $\frac{Q_{\ell,i}^* + Q_{\ell,i}}{2}$ if necessary, we can assume furthermore that $Q_{\ell,i}$ is formally self-adjoint. Thus, taking $A_{\ell+1} = A_\ell + \sum_{i=0}^k Q_{\ell,i}$ insures that $A_{\ell+1} = A_{\ell+1}^*$ and

$$(9.18) \quad C + B - A_{\ell+1}^2 \in x^{\ell+2}\Psi_S^{-\ell-2}(X).$$

We can then define A as an asymptotic sum specified by the A_ℓ . □

Theorem 9.4. *Any element $P \in \Psi_S^0(X; E_1, E_2)$ defines a bounded linear operator from $\mathcal{H}_1 = L^2(X; E_1)$ to $\mathcal{H}_2 = L^2(X; E_2)$ with L^2 -norms defined by a positive S -density on X and Hermitian metrics on E_1 and E_2 . Furthermore, the map $\Psi_S^0(X; E_1, E_2) \rightarrow \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is continuous.*

Proof. Considering a local trivialization if necessary, we can assume that $E_1 = E_2 = \underline{\mathbb{C}}$ and $\mathcal{H}_1 = \mathcal{H}_2 = L^2(X)$. Then $B = -P^*P \in \Psi_S^0(X)$ is formally self-adjoint. By the previous proposition, there exists $C > 0$ and $A \in \Psi_S^0(X)$ formally self-adjoint such that

$$(9.19) \quad C - P^*P = A^*A + R$$

for some $R \in x^\infty\Psi_S^{-\infty}(X)$. Thus, given $u \in \dot{C}^\infty(X)$, we have

$$(9.20) \quad \begin{aligned} \|Pu\|^2 &= C\|u\|^2 - \|Au\|^2 - \langle u, Ru \rangle \\ &\leq C\|u\|^2 + |\langle u, Ru \rangle| \leq C'\|u\|^2, \end{aligned}$$

where the fact elements of $\dot{\Psi}_S^{-\infty}(X)$ are in $\mathcal{L}(\mathcal{H})$ has been used. Thus, there is a well-defined linear map

$$(9.21) \quad \Psi_S^0(X) \rightarrow \mathcal{L}(\mathcal{H}).$$

Since the map

$$\Psi_S^0(X) \ni A \mapsto \langle u, Av \rangle_{\mathcal{H}} = K_A(\pi_L^*(u\nu_\pi) \otimes \pi_R^*(v))$$

is continuous for all $u, v \in \dot{C}^\infty(X)$, where ν_π is the S -density used to define the L^2 -norm, we see that the graph of the linear map (9.21) is closed with the respect to the topology induced by the norms $A \mapsto |\langle u, Av \rangle|$. Since this topology is weaker

than the norm topology, this means the graph of this map is also closed when we use the norm topology on $\mathcal{L}(\mathcal{H})$. The map (9.21) is therefore continuous by the closed graph theorem. \square

There is a similar result for suspended S-operators. Let V be a Euclidean vector space and let g_V be the corresponding Euclidean metric. On the manifold with fibred corners $\bar{V} \times X$, consider the ϖ -metric

$$g_\varpi = \text{pr}_1^* g_V + \text{pr}_2^* g_\pi$$

where $\text{pr}_1 : \bar{V} \times X \rightarrow \bar{V}$ and $\text{pr}_2 : \bar{V} \times X \rightarrow X$ are the projections on the first and second factors respectively.

Corollary 9.5. *Any element $P \in \Psi_{\text{S-sus}(V)}^0(X; E_1, E_2)$ defines a bounded linear operator from $\mathcal{H}_1 = L_{g_\varpi}^2(\bar{V} \times X; E_1)$ to $\mathcal{H}_2 = L_{g_\varpi}^2(\bar{V} \times X; E_2)$ with L^2 -norm defined by a volume form dg_ϖ and Hermitian metrics on E_1 and E_2 . Furthermore, the map*

$$\Psi_{\text{S-sus}(V)}^0(X; E_1, E_2) \rightarrow \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$$

is continuous.

Proof. Since our proof of Theorem 9.4 is by induction on the depth of X and since the inductive step is not yet completed, we cannot at this stage simply use the statement of Theorem 9.4 for the manifold with fibred corners $\bar{V} \times X$ to obtain the result. Instead, consider the Fourier transform of P ,

$$\Upsilon \mapsto \hat{P}(\Upsilon) \in \Psi_{\text{S}}^0(X; E_1, E_2), \quad \Upsilon \in V^*.$$

By Theorem 9.4, we know that for each $\Upsilon \in V^*$, the operator $\hat{P}(\Upsilon)$ induces a continuous linear map

$$\hat{P}(\Upsilon) : L_{g_\pi}^2(X; E_1) \rightarrow L_{g_\pi}^2(X; E_2).$$

Let g_{V^*} be the metric on V^* which is dual to g_V and let

$$g_{\varpi^*} = \text{pr}_1^* g_{V^*} + \text{pr}_2^* g_\pi$$

be the corresponding metric on $V^* \times X$. Since the Fourier transform induces an isomorphism of Hilbert spaces

$$\mathcal{F}_i : L_{g_\varpi}^2(V \times X; E_i) \rightarrow L_{g_{\varpi^*}}^2(V^* \times X; E_i),$$

we conclude from (6.22) and Theorem 9.4 that $P \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. The continuity of the map $\Psi_{\text{S-sus}(V)}^0(X; E_1, E_2) \rightarrow \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ can be proved in the same way as before. \square

As a family of suspended operators, the symbol $\sigma_{\partial_i}(P)$ of an operator $P \in \Psi_{\text{S}}^0(X; E, F)$ will act on the Banach space $L_{g_\pi}^2(\pi NH_i/S_i; E)$ obtained by taking the closure of the space of Schwartz sections $\mathcal{S}(\pi NH_i; E)$ with respect to the norm

$$(9.22) \quad \|f\|_{L_{g_\pi}^2(\pi NH_i/S_i; E)} = \sup_{s \in S_i} \|f|_{\phi_i^{-1}(s)}\|_{L_{g_\pi}^2(\phi_i^{-1}(s); E)}, \quad f \in \mathcal{S}(\pi NH_i; E),$$

where $\phi_i = \pi_i \circ \nu_i : \pi NH_i \rightarrow S_i$ and $\nu_i : \pi NH_i \rightarrow H_i$ is the vector bundle projection. On each fibre of ϕ_i , the L^2 -norm of a section of E is specified by a choice of Hermitian metric on E and the natural density induced by g_π . Thus,

from Corollary 9.5, we see that the symbol $\sigma_{\partial_i}(P)$ of an operator $P \in \Psi_{\mathcal{S}}^0(X; E, F)$ naturally induce a continuous linear map

$$(9.23) \quad \sigma_{\partial_i}(P) : L_{g_\pi}^2(\pi NH_i/S_i; E) \rightarrow L_{g_\pi}^2(\pi NH_i/S_i; F).$$

Notice that the Banach space $L_{g_\pi}^2(\pi NH_i/S_i; E)$ also has a natural structure of $\mathcal{C}^0(S_i)$ -Hilbert module.

To complete the inductive step necessary to the proof of Proposition 9.3, we can now use this fact with Corollary 9.2 to construct the unique positive square root of the operator $C + B$ in Proposition 9.3 and its suspended versions.

Corollary 9.6. *Given a formally self-adjoint operator $B \in \Psi_{\mathcal{S}-\text{sus}(V)}^0(X)$, there exists a positive constant C such that $C + B$ is invertible and has a well-defined formally self-adjoint positive definite square root in $\Psi_{\mathcal{S}-\text{sus}(V)}^0(X)$.*

Proof. From Corollary 9.5, we know that B gives a bounded operator

$$(9.24) \quad B : L_{g_\varpi}^2(V \times X) \rightarrow L_{g_\varpi}^2(V \times X).$$

Thus, taking C big enough, we can define the square root of $C + B$ as a bounded operator by

$$(9.25) \quad C^{\frac{1}{2}} \left(1 + \frac{B}{C}\right)^{\frac{1}{2}} = C^{\frac{1}{2}} \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} \left(\frac{B}{C}\right)^j$$

using the power series of $f(x) = (1 + x)^{\frac{1}{2}}$ at $x = 0$. To see it is an element

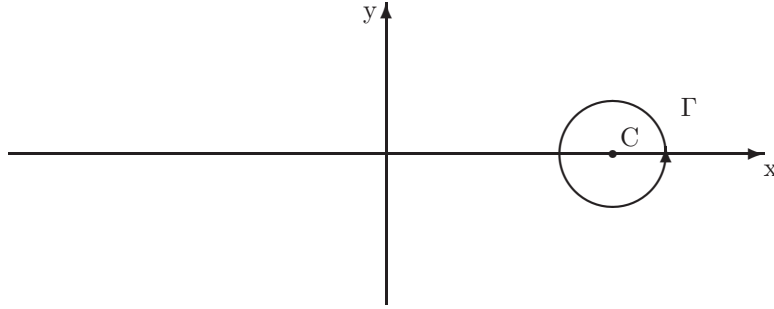


FIGURE 4.

of $\Psi_{\mathcal{S}-\text{sus}(V)}^0(X)$, we can use the alternative representation in terms of a contour integral

$$(9.26) \quad (C + B)^{\frac{1}{2}} = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{\frac{1}{2}} (\lambda - (C + B))^{-1} d\lambda$$

where Γ is an anti-clockwise circle centered at C and radius r such that $\|B\|_{\mathcal{L}(\mathcal{H})} < r < C$ (see Figure 4).

This way, the family $(\lambda - (C + B))$ is invertible for all $\lambda \in \Gamma$ and the square root of λ is well-defined along Γ . By Corollary 9.2, the family $(\lambda - (C + B))^{-1}$ is a smooth family in $\Psi_{\mathcal{S}-\text{sus}(V)}^0(X)$ for $\lambda \in \Gamma$. Thus, we see from (9.26) that $(C + B)^{\frac{1}{2}}$ is in $\Psi_{\mathcal{S}-\text{sus}(V)}^0(X)$ as well. \square

Theorem 9.7. *For $\delta > 0$, an operator $A \in \Psi_S^{-\delta}(X; E)$ is compact when acting on $\mathcal{H} = L_{g_\pi}^2(X; E)$ if and only if $\sigma_{\partial_j}(A) = 0$ for all $j \in \{1, \dots, k\}$. In particular, a polyhomogeneous S -operator $A \in \Psi_{S-\text{ph}}^0(X; E)$ of order zero is compact when acting on $L_{g_\pi}^2(X; E)$ if and only if $A \in x\Psi_{S-\text{ph}}^{-1}(X; E)$.*

Proof. Without loss of generality, we can assume $E = \underline{\mathbb{C}}$ is the trivial vector bundle. By definition, the space of compact operators $\mathcal{K}(\mathcal{H})$ is the closure in $\mathcal{L}(\mathcal{H})$ of operators of finite ranks. Clearly, since $\dot{\mathcal{C}}^\infty(X)$ is dense in $L_{g_\pi}^2(X)$, we can as well define $\mathcal{K}(\mathcal{H})$ as the closure of finite rank operators represented by an element of $\dot{\Psi}_S^{-\infty}(X)$. These operators of finite rank are certainly dense in $\dot{\Psi}_S^{-\infty}(X)$. Thus, $\mathcal{K}(\mathcal{H})$ is given by the closure of $\dot{\Psi}_S^{-\infty}(X)$ in $\mathcal{L}(\mathcal{H})$. Since the map $\Psi_S^0(X) \rightarrow \mathcal{L}(\mathcal{H})$ is continuous, we conclude that the closure of $\dot{\Psi}_S^{-\infty}(X)$ in $\Psi_S^{-\delta}(X)$, namely, $x\Psi_S^{-\delta}(X)$, is included in $\mathcal{K}(\mathcal{H})$.

Conversely, let $A \in \Psi_S^{-\delta}(X)$ be a compact operator. Suppose for a contradiction that $\sigma_{\partial_i}(A) \neq 0$ for some $i \in \{1, \dots, k\}$. This means that we can find $y_i \in S_i$ and a function $f \in \mathcal{C}_c^\infty(\pi N_{y_i} H_i)$ such that

$$(9.27) \quad \sigma_{\partial_i}(A)|_{y_i} f \neq 0.$$

Without loss of generality, we can assume in fact that $y_i \in S_i \setminus \partial S_i$. Let \mathcal{V} be a small neighborhood of $y_i \in S_i$ such that the fibration $\pi_i : H_i \rightarrow S_i$ is trivial over \mathcal{V} , namely, there is a diffeomorphism $\psi : \pi_i^{-1}(\mathcal{V}) \rightarrow F_i \times \mathcal{V}$ inducing a commutative diagram

$$(9.28) \quad \begin{array}{ccc} \pi_i^{-1}(\mathcal{V}) & \xrightarrow{\psi} & \mathcal{V} \times F_i \\ & \searrow \pi_i & \swarrow \text{pr}_L \\ & \mathcal{V} & \end{array}$$

where $\text{pr}_L : \mathcal{V} \times F_i \rightarrow \mathcal{V}$ is the projection on the left factor. Let $\iota_i : H_i \times [0, \epsilon)_{x_i} \rightarrow X$ be a tubular neighborhood of H_i in X compatible with the boundary defining function x_i . Using the diffeomorphism ψ , we can identify the open set $\iota_i(\pi_i^{-1}(\mathcal{V}) \times [0, \epsilon)_{x_i}) \subset X$ with the open set

$$(9.29) \quad \mathcal{V} \times F_i \times [0, \epsilon)_{x_i}.$$

Choosing \mathcal{V} to be smaller if needed, we can assume it is diffeomorphic to an open ball in the Euclidean space. Let y be a choice of coordinates on \mathcal{V} such that the point $y_i \in \mathcal{V}$ corresponds to $y = 0$. On the open set $\mathcal{V} \times (0, \epsilon)_{x_i}$, consider the coordinates

$$(9.30) \quad u = \frac{1}{x_i}, \quad v = \frac{y}{x_i}.$$

Considering alternatively v and u as linear coordinates on the vector space $\pi_i N_{y_i} S_i = T_{y_i} S_i \times \mathbb{R}_u$, we regard $\mathcal{V} \times (0, \epsilon)_{x_i}$ as an open subset in $\pi_i N_{y_i} S_i$, and consequently we can regard $\mathcal{U} = \mathcal{V} \times F_i \times (0, \epsilon)_{x_i}$ as a subset of $\pi_i N_{y_i} S_i \times F_i = \pi_i N_{y_i} H_i$. For $k \in \mathbb{N}_0$, consider the new function

$$(9.31) \quad f_k(u, v, z) = f(u - k, v, z), \quad z \in F_i,$$

obtained by translating f in the u variable. Since we assume that the support of f is compact, by taking k sufficiently large, we can insure that the support of f is contained in the open set \mathcal{U} . In fact, since the operator $\sigma_{\partial_i}(A)$ is translation

invariant, we will still have that (9.27) holds after translating f in the u variable, so without loss of generality, we can assume that the support of f_k is contained in \mathcal{U} for all $k \in \mathbb{N}_0$. Again, by translation invariance of $\sigma_{\partial_i}(A)$, we will have that

$$\sigma_{\partial_i}(A)|_{y_i} f_k \neq 0$$

for all $k \in \mathbb{N}_0$. Since the function f_k is supported in \mathcal{U} , we can also regard it as a function on X . Let $\chi \in \mathcal{C}_c^\infty(\mathcal{V} \times F_i \times [0, \epsilon]_{x_i})$ be a cut-off function such that $\chi \equiv 1$ in a neighborhood of $\{y_i\} \times F_i \times \{0\}$. Thus, if we consider the operator $P = \chi A \in \Psi_S^{-\delta}(X)$, P will also obviously be compact, and we will have that

$$\sigma_{\partial_i}(P)|_{y_i} = \sigma_{\partial_i}(A)|_{y_i}.$$

Now, thanks to the cut-off function χ , the action of P on $f \in \mathcal{C}_c^\infty(U) \subset \mathcal{C}^\infty(X)$ is given by:

$$P f_k(u, v, z) = \int_U K_P(u, v, u', v', z, z') f_k(u', v', z') du' dv' dz',$$

where the integral is in the sense of distributions. Similarly, the action of $\sigma_{\partial_i}(P)|_{y_i}$ can be described by

$$\sigma_{\partial_i}(P)|_{y_i} f_k(u, v, z) = \int_U K_{\sigma_{\partial_i}(P)|_{y_i}}(u, v, u', v', z, z') f_k(u', v', z') du' dv' dz'$$

Since as a function on $\mathcal{U} \subset X$, the support of the function f_k is uniformly approaching the fibre $\pi_i^{-1}(y_i) \subset H_i$ as $k \rightarrow +\infty$, we see from the definition of the normal operator that we must have that as k tends to infinity,

$$(9.32) \quad P f_k - \sigma_{\partial_i}(P) f_k \rightarrow 0$$

in the L^2 -norm defined by the S-metric

$$g + du^2 + dv^2 + g_{F_i},$$

where g_{F_i} is a choice of S-metric on F_i . By translation invariance of this metric and of $\sigma_{\partial_i}(P)|_{y_i}$, we have that, on $\pi_i N_{y_i} H_i$,

$$\| \sigma_{\partial_i}(P)|_{y_i} f_k \|_{L^2} = \| \sigma_{\partial_i}(P)|_{y_i} f \|_{L^2} \neq 0$$

If we restrict $\sigma_{\partial_i}(P)|_{y_i} f_k$ to \mathcal{U} , we still clearly have that

$$\lim_{k \rightarrow \infty} \| \sigma_{\partial_i}(P)|_{y_i} f_k \|_{L^2(\mathcal{U})} = \| \sigma_{\partial_i}(P)|_{y_i} f \|_{L^2(\pi_i N_{y_i} S_i)} \neq 0.$$

On the other hand, $\sigma_{\partial_i}(P)|_{y_i} f_k$ being moved to infinity as $k \rightarrow \infty$, we see that it converges pointwise to zero everywhere on \mathcal{U} , so that the sequence $\sigma_{\partial_i}(P) f_k$ cannot converge in L^2 . We conclude from (9.32) that the sequence $P f_k$ also fails to converge in L^2 . Since by translation invariance of the metric, the sequence f_k is bounded in L^2 , this contradicts the fact P is a compact operator. To avoid a contradiction, we must conclude that $\sigma_{\partial_i}(A) = 0$ for all $i \in \{1, \dots, k\}$, which completes the proof. \square

More generally, there are natural Sobolev spaces associated to S-operators. As before, let g_π be a S-metric on X and let $E \rightarrow X$ be a complex vector bundle with a Hermitian metric, so that we have a corresponding space $L_{g_\pi}^2(X; E)$ of square integrable sections. For $m > 0$, we define the associated S-Sobolev space by

$$(9.33) \quad H_S^m(X; E) = \{f \in \dot{\mathcal{C}}^{-\infty}(X; E) ; P f \in L_{g_\pi}^2(X; E) \quad \forall P \in \Psi_S^m(X; E)\},$$

while for $m < 0$, we define it by

$$(9.34) \quad H_S^m(X; E) = \{f \in \dot{C}^{-\infty}(X; E);$$

$$f = \sum_{i=1}^N P_i g_i, \quad g_i \in L_{g_\pi}^2(X; E), P_i \in \Psi^{-m}(X; E)\}.$$

If V is a Euclidean vector space, we define the corresponding V -suspended S-Sobolev space by

$$(9.35) \quad H_{S-\text{sus}(V)}^m(X; E) = H_S^m(\bar{V} \times X; E).$$

These spaces can be given the structure of a Hilbert space using fully elliptic operators. More precisely, for $m > 0$, let $A_m \in \Psi_S^{\frac{m}{2}}(X; E)$ be a choice of elliptic S-operator and consider the formally self-adjoint operator $D_m \in \Psi_S^m(X; E)$ defined by

$$(9.36) \quad D_m = A_m^* A_m + \text{Id}_E.$$

Lemma 9.8. *For $m > 0$, the operator D_m is fully elliptic and invertible. In particular, its inverse $D_{-m} := (D_m)^{-1}$ is an element of $\Psi_S^{-m}(X; E)$.*

Proof. Let H_1, \dots, H_k be the boundary hypersurfaces of X and suppose that they are labelled in such a way that

$$H_i < H_j \implies i < j.$$

We will first prove by induction on $i \in \{1, \dots, k\}$ starting with $i = k$ that $\sigma_{\partial_i}(D_m)$ is fully elliptic and invertible. For $i = k$, the fibres of the fibration $\pi_k : H_k \rightarrow S_k$ are closed manifolds, so that in this case, $\sigma_{\partial_k}(D_m)$ is automatically fully elliptic since it is elliptic. Thus, for $i \in \{1, \dots, k\}$, the inductive step we need to show is that if $\sigma_{\partial_i}(D_m)$ is fully elliptic, then it is invertible. To see this, fix $s \in S_i$ and consider the ${}^\pi N_s S_i$ -suspended operator $\sigma_{\partial_i}(A_m)_s$ above s . For a fixed $\Upsilon \in {}^\pi N_s^* S_i$, consider the operator

$$Q = \widehat{\sigma_{\partial_i}(D_m)}(\Upsilon) \in \Psi_S^m(\pi_i^{-1}(s); E).$$

Thus, if $B = \widehat{\sigma_{\partial_i}(A_m)}(\Upsilon)$, we have that $Q = B^* B + \text{Id}_E$. By Proposition 9.1, if $Qu = 0$, then $u \in \dot{C}^\infty(\pi_i^{-1}(s); E)$. Thus, we have in particular

$$(9.37) \quad \begin{aligned} Qu = 0 &\implies \langle u, B^* Bu + u \rangle_{L^2}, \\ &\implies \|Bu\|_{L^2}^2 + \|u\|_{L^2}^2 = 0, \\ &\implies u \equiv 0. \end{aligned}$$

Thus, since Q is formally self-adjoint, we have that $\ker Q = \ker Q^* = \{0\}$, so that Q is invertible. Since $\Upsilon \in N_s^* S_i$ was arbitrary, this means that $\sigma_{\partial_i}(D_m)_s$ is invertible. Thus, since $s \in S_i$ was arbitrary, this means that $\sigma_{\partial_i}(D_m)$ is invertible, which completes the inductive step.

With this argument, we see D_m is fully elliptic. In particular, by Proposition 9.1, if $D_m u = 0$, then $u \in \dot{C}^\infty(X; E)$. We can then show D_m is invertible using a similar argument as in (9.37), which completes the proof. \square

Using the operator D_m with $D_m = (D_{-m})^{-1}$ for $m < 0$ and $D_0 = \text{Id}_E$, we can then define an inner product on $H_S^m(X; E)$ by

$$(9.38) \quad \langle u, v \rangle_{H_S^m(X; E)} = \langle D_m u, D_m v \rangle_{L_{g_\pi}^2(X; E)},$$

with corresponding norm

$$(9.39) \quad \|u\|_{H_S^m(X; E)} = \|D_m u\|_{L_{g_\pi}^2(X; E)}.$$

Using Theorem 9.4, it is straightforward to check $H_S^m(X; E)$ is precisely the closure of $\dot{C}^\infty(X; E)$ with respect to this norm.

Proposition 9.9. *Any S -pseudodifferential operator $P \in \Psi_S^m(X; E, F)$ induces a bounded linear map*

$$P : x^\ell H_S^p(X; E) \rightarrow x^\ell H_S^{p-m}(X; F)$$

for $p, \ell \in \mathbb{R}$.

Proof. Thinking of E and F as subbundles of a bigger bundle H , we reduce to the case where $E = F$. The result then follows from Theorem 9.4 by noticing

$$P = x^\ell D_{m-p} \tilde{P} D_p x^{-\ell} \quad \text{with } \tilde{P} = D_{p-m} x^{-\ell} P x^\ell D_{-p} \in \Psi_S^0(X; E).$$

□

In particular, we conclude from Proposition 9.9 that for all $\ell \in \mathbb{R}$ and $p \in \mathbb{R}$, the operator D_m induces an isomorphism

$$(9.40) \quad D_m : x^\ell H_S^p(X; E) \rightarrow x^\ell H_S^{p-m}(X; F)$$

Proposition 9.10. *We have a continuous inclusion $x^\ell H_S^m(X; E) \subset x^{\ell'} H_S^{m'}(X; E)$ if and only if $\ell \geq \ell'$ and $m \geq m'$. The inclusion is compact if and only if $\ell > \ell'$ and $m > m'$.*

Proof. The fact that these are continuous inclusions follows from the isomorphism (9.40) and Proposition 9.9. The statement about compactness follows by using the isomorphism (9.40) and the fact that for $\epsilon > 0$, the operator $x^\epsilon D_{-\epsilon} \in x^\epsilon \Psi_S^{-\epsilon}(X; E)$ is a compact operator from $L_{g_\pi}^2(X; E)$ to itself. □

By the parametrix construction of Proposition 9.1 as well as Proposition 9.9 and Proposition 9.10, an operator $P \in \Psi_S^m(X; E, F)$ is Fredholm as an operator

$$(9.41) \quad P : x^\ell H_S^{p+m}(X; E) \rightarrow x^\ell H_S^p(X; F)$$

whenever it is fully elliptic. When P is polyhomogeneous, it is also possible to establish the converse.

Theorem 9.11. *An operator $P \in \Psi_{S-\text{ph}}^m(X; E, F)$ induces a Fredholm operator*

$$P : x^\ell H_S^{p+m}(X; E) \rightarrow x^\ell H_S^p(X; F)$$

if and only if it is fully elliptic.

Proof. We will follow the approach of [19, Theorem 4]. First, by considering instead the operator $\tilde{P} = x^{-\ell} D_p P D_{-p-m} x^\ell$, we can assume that P is of order 0 and is seen as a bounded operator

$$P : L_{g_\pi}^2(X; E) \rightarrow L_{g_\pi}^2(X; F).$$

Furthermore, by considering instead the operator

$$\begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix} : L_{g_\pi}^2(X; E \oplus F) \rightarrow L_{g_\pi}^2(X; E \oplus F),$$

we can reduce to the case $E = F$ with P self-adjoint. By Theorem 9.4, we have a continuous linear map

$$\iota : \Psi_{S\text{-ph}}^0(X; E) \rightarrow \mathcal{L}(\mathcal{H}, \mathcal{H}),$$

where $\mathcal{H} = L_{g_\pi}^2(X; E)$. Let $\mathcal{P}_{S\text{-ph}}^0(X; E)$ be the image of this map and $\overline{\mathcal{P}}_{S\text{-ph}}^0(X; E)$ its closure in $\mathcal{L}(\mathcal{H}, \mathcal{H})$. Now, the principal symbol induces a continuous linear map

$$\sigma_0 : \Psi_{S\text{-ph}}^0(X; E) \rightarrow \mathcal{C}^\infty(\pi S^*X; \text{hom}(E)).$$

Using instead the \mathcal{C}^0 -topology on $\mathcal{C}^\infty(\pi S^*X; \text{hom}(E))$, this extends to a homomorphism of C^* -algebras

$$\overline{\sigma}_0 : \overline{\mathcal{P}}_{S\text{-ph}}^0(X; E) \rightarrow \mathcal{C}^0(\pi S^*X; \text{hom}(E)).$$

Similarly, the symbol map σ_{∂_i} induces a continuous linear map

$$\overline{\sigma}_{\partial_i} : \overline{\mathcal{P}}_{S\text{-ph}}^0(X; E) \rightarrow \overline{\mathcal{P}}_{\text{ff}\pi_i\text{-ph}}^0(H_i; E),$$

where $\overline{\mathcal{P}}_{\text{ff}\pi_i\text{-ph}}^0(H_i; E)$ is the closure of $\Psi_{\text{ff}\pi_i\text{-ph}}^0(H_i; E)$ in $\mathcal{L}(\mathcal{H}_i, \mathcal{H}_i)$ with \mathcal{H}_i the Banach space $L_{g_\pi}^2(\pi N H_i / S_i; E)$ introduced in (9.23). By Theorem 9.7, this induces an injective map

(9.42)

$$(\overline{\sigma}_0, \bigoplus_{i=1}^k \overline{\sigma}_{\partial_i}) : \overline{\mathcal{P}}_{S\text{-ph}}^0(X; E) / \mathcal{K} \hookrightarrow \mathcal{C}^0(\pi S^*X; \text{hom}(E)) \oplus \left(\bigoplus_{i=1}^k \overline{\mathcal{P}}_{\text{ff}\pi_i\text{-ph}}^0(X; E) \right)$$

where $\mathcal{K} \subset \mathcal{L}(\mathcal{H}, \mathcal{H})$ is the subspace of compact operators. Since this is an injective map of C^* -algebras mapping the identity to the identity, it is a standard fact (see for instance Proposition 1.3.10 in [12]) that an element of $\overline{\mathcal{P}}_{S\text{-ph}}^0(X; E) / \mathcal{K}$ is invertible if and only if its image under the map (9.42) is invertible. Since a bounded operator in $\mathcal{L}(\mathcal{H}, \mathcal{H})$ is Fredholm if and only if it is invertible in $\mathcal{L}(\mathcal{H}, \mathcal{H}) / \mathcal{K}$, the result follows. \square

10. THE SEMICLASSICAL S-CALCULUS

Consider the manifold with corner $X_\pi^2 \times [0, 1]_\epsilon$ where ϵ should be considered as a semiclassical parameter. The **semiclassical π -double space** is obtained by blowing up the p -submanifold $\Delta_\pi \times \{0\}$,

$$(10.1) \quad X_{\pi\text{-sl}}^2 = [X_\pi^2 \times [0, 1]_\epsilon; \Delta_\pi \times \{0\}]$$

with blow-down map

$$(10.2) \quad \beta_{\text{sl}} : X_{\pi\text{-sl}}^2 \rightarrow X_\pi^2 \times [0, 1]_\epsilon.$$

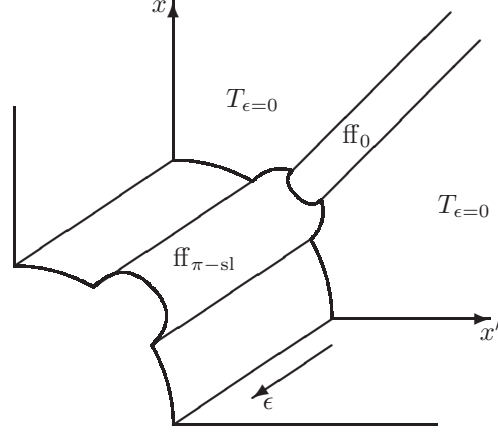
See Figure 5 for a picture of the semiclassical π -double space when X is a manifold with boundary. We denote the ‘new’ boundary face obtained via this blow-up by

$$(10.3) \quad \text{ff}_0 = \beta_{\text{sl}}^{-1}(\Delta_\pi \times \{0\}) \subset X_{\pi\text{-sl}}^2.$$

We also denote by

$$(10.4) \quad T_{\epsilon=0} = \overline{\beta_{\text{sl}}^{-1}(X_\pi^2 \times \{0\}) \setminus (\Delta_\pi \times \{0\})}$$

the lift of the ‘old’ face $X_\pi^2 \times \{0\}$ to $X_{\pi\text{-sl}}^2$.


 FIGURE 5. The semiclassical π -double space

Notice that $\text{ff}_0 \setminus (\text{ff}_0 \cap T_{\epsilon=0})$ is naturally diffeomorphic to $N\Delta_\pi \cong {}^\pi TX$ and that ff_0 is diffeomorphic to the radial compactification of ${}^\pi TX \rightarrow X$. We will also denote the lift of $\text{ff}_{\pi_i} \times [0, 1]$ to $X_{\pi-s1}^2$ by

$$(10.5) \quad \text{ff}_{\pi_i-s1} = \beta_{s1}^{-1}(\text{ff}_{\pi_i} \times [0, 1]) \cong [\text{ff}_{\pi_i} \times [0, 1]; (\Delta_\pi \cap \text{ff}_{\pi_i}) \times \{0\}].$$

It will be useful to consider the spaces

$$(10.6) \quad \text{ff}_{\pi-s1} = \bigcup_{i=1}^k \text{ff}_{\pi_i-s1}, \quad \text{ff}_{s1} = \text{ff}_{\pi-s1} \cup \text{ff}_0$$

as well as the lift of $\Delta_\pi \times [0, 1]$ to $X_{\pi-s1}^2$,

$$(10.7) \quad \Delta_{s1} = \overline{\beta_{s1}^{-1}(\Delta_\pi \times (0, 1])}.$$

Let also $\text{ff}_{\Delta_{s1}} = \text{ff}_{s1} \cup (X_\pi^2 \times \{1\})$ be the union of all the hypersurfaces of $\partial X_{\pi-s1}^2$ having a non-empty intersection with Δ_{s1} .

We can now define the space of **semiclassical S-pseudodifferential operators** of order m by

$$(10.8) \quad \Psi_{S-s1}^m(X; E, F) = \left\{ K \in I^m(X_{\pi-s1}^2, \Delta_{s1}; \beta_{s1}^* p_1^* \beta_\pi^*(\text{Hom}(E, F) \otimes \pi_R^*(\pi\Omega))) \right. \\ \left. K \equiv 0 \text{ at } \partial X_{\pi-s1}^2 \setminus \text{ff}_{\Delta_{s1}} \right\},$$

where $p_1 : X_\pi^2 \times [0, 1] \rightarrow X_\pi^2$ is the projection on the first factor. Polyhomogeneous semiclassical S-operators can be defined in a similar way.

As for S-pseudodifferential operators, there is a corresponding semiclassical Lie groupoid

$$(10.9) \quad \mathcal{G}_{\pi-s1}^{(0)} = \Delta_{s1}, \quad \mathcal{G}_{\pi-s1}^{(1)} = \overset{\circ}{X}_{\pi-s1}^2 \cup \overset{\circ}{\text{ff}}_{\Delta_{s1}},$$

where $\overset{\circ}{\text{ff}}_{\Delta_{s1}} = \text{ff}_{\Delta_{s1}} \setminus \partial \text{ff}_{\Delta_{s1}}$ is the interior of $\text{ff}_{\Delta_{s1}}$ as a subset of $\partial X_{\pi-s1}^2$. Clearly, Δ_{s1} is naturally identified with $X \times [0, 1]$. Under this identification, we define the domain

and range of $\alpha \in \mathcal{G}_{\pi\text{-sl}}^{(1)}$ with $p_2 \circ \beta_{\text{sl}}(\alpha) = \epsilon$ and $\beta_\pi \circ p_1 \circ \beta_{\text{sl}}(\alpha) = (x_1, x_2) \in X^2$ by

$$(10.10) \quad d(\alpha) = (x_2, \epsilon), \quad r(\alpha) = (x_1, \epsilon),$$

where $p_2 : X_\pi^2 \times [0, 1] \rightarrow [0, 1]$ is the projection on the second factor. Since $\mathcal{G}_{\pi\text{-sl}}$ is a Lie groupoid, any choice of a metric on ${}^\pi TX \times [0, 1]$ provides a (smooth) Haar system on $\mathcal{G}_{\pi\text{-sl}}$ [41], giving to it the structure of a measured groupoid. As in the proof of Lemma 4.6, observe that $\mathcal{G}_{\pi\text{-sl}}$ can be written as a disjoint union of measurewise amenable groupoids,

$$(10.11) \quad \mathcal{G}_{\pi\text{-sl}} = ({}^\pi TX) \bigsqcup (\overset{\circ}{X} \times \overset{\circ}{X}) \times (0, 1]_\epsilon \bigsqcup_{i=1}^k (H_i \times_{\pi_i} {}^\pi TS_i \times_{\pi_i} H_i)|_{G_i} \times (0, 1]_\epsilon \times \mathbb{R},$$

where $G_i = H_i \setminus (\cup_{j>i} H_j)$. Thus, by Lemma 4.5, we conclude that $\mathcal{G}_{\pi\text{-sl}}$ is measurewise amenable with $\mathcal{C}^*(\mathcal{G}_{\pi\text{-sl}})$ nuclear and equal to $\mathcal{C}_r^*(\mathcal{G}_{\pi\text{-sl}})$.

In the terminology of [9], $\mathcal{G}_{\pi\text{-sl}}$ is the tangent groupoid of \mathcal{G}_π . From [2] and [39], there is a calculus of pseudodifferential operators associated to this groupoid. It corresponds to operators in $\Psi_{\pi\text{-sl}}^*(X; E, F)$ with Schwartz kernel compactly supported in $\mathcal{G}_{\pi\text{-sl}}^{(1)}$. As for $\mathcal{G}_\pi^{(1)}$, the inverse map ι and the composition maps comes from the natural smooth extensions of the corresponding maps on the Lie groupoid $\overset{\circ}{X} \times \overset{\circ}{X} \times [0, 1]$ with domain and range given by $d(x_1, x_2, \epsilon) = (x_2, \epsilon)$ and $r(x_1, x_2, \epsilon) = (x_1, \epsilon)$.

There are many symbol maps associated to $\Psi_{\text{S-sl}}^m(X; E, F)$. There is the obvious one associated to conormal distributions. With the natural identification of $N^*\Delta_{\text{sl}}$ with ${}^\pi TX \times [0, 1]$, which has a canonical volume form, we can write it as

$$(10.12) \quad \Psi_{\text{S-sl}}^m(X; E, F) \xrightarrow{\sigma_m} S^{[m]}(N^*\Delta_{\text{sl}}; \phi^* \text{Hom}(E, F))$$

where ϕ is the composition of the natural maps $N^*\Delta_{\text{sl}} \rightarrow \Delta_{\text{sl}}$ and $\Delta_{\text{sl}} = X \times [0, 1] \rightarrow X$. This gives a short exact sequence

$$(10.13) \quad 0 \longrightarrow \Psi_{\text{S-sl}}^{m-1}(X; E, F) \longrightarrow \Psi_{\text{S-sl}}^m(X; E, F) \xrightarrow{\sigma_m} S^{[m]}(N^*\Delta_{\text{sl}}; \phi^* \text{Hom}(E, F)) \longrightarrow 0.$$

We say an operator $P \in \Psi_{\text{S-sl}}^m(X; E, F)$ is **elliptic** if its principal symbol $\sigma_m(P)$ is invertible.

Restriction to boundary hypersurfaces of $X_{\pi\text{-sl}}^2$ leads to other types of symbols. Restricting to the hypersurface ff_0 , we get the short exact sequence

$$(10.14) \quad 0 \longrightarrow \epsilon \Psi_{\text{S-sl}}^m(X; E, F) \longrightarrow \Psi_{\text{S-sl}}^m(X; E, F) \xrightarrow{\sigma_{\epsilon=0}} \Psi_{\text{ff}_0}^m(X; E, F) \longrightarrow 0.$$

On the other hand, restricting to the face $\text{ff}_{\pi_i\text{-sl}}$ for $i \in \{1, \dots, k\}$, we get the short exact sequence

$$(10.15) \quad 0 \longrightarrow x_i \Psi_{\text{S-sl}}^m(X; E, F) \longrightarrow \Psi_{\text{S-sl}}^m(X; E, F) \xrightarrow{\sigma_{\text{ff}_{\pi_i\text{-sl}}}} \Psi_{\text{ff}_{\pi_i\text{-sl}}}^m(X; E, F) \longrightarrow 0.$$

Combining the symbol maps $\sigma_{\text{ff}_{\pi_i}}$ and $\sigma_{\epsilon=0}$, that is, restricting to the hypersurface ff_{sl} , we also get the short exact sequence

$$(10.16) \quad 0 \longrightarrow \epsilon x \Psi_{\text{S-sl}}^m(X; E, F) \longrightarrow \Psi_{\text{S-sl}}^m(X; E, F) \xrightarrow{\sigma_{\text{ff}_{\text{sl}}}} \Psi_{\text{ff}_{\text{sl}}}^m(X; E, F) \longrightarrow 0.$$

Finally, a symbol of particular importance is obtained by restricting at the face $X_\pi^2 \times \{1\}$, giving the short exact sequence (10.17)

$$0 \longrightarrow (1 - \epsilon)\Psi_{S\text{-sl}}^m(X; E, F) \longrightarrow \Psi_{S\text{-sl}}^m(X; E, F) \xrightarrow{\sigma_{\epsilon=1}} \Psi_S^m(X; E, F) \longrightarrow 0.$$

In fact, more generally, for $\epsilon \in (0, 1]$, we can restrict $A \in \Psi_{S\text{-sl}}^m(X; E, F)$ to the hypersurface $X_\pi^2 \times \{\epsilon\}$ to get an operator $A_\epsilon \in \Psi_S^m(X; E, F)$. This gives us a way of composing semiclassical S-operators,

$$(A \circ B)_\epsilon := A_\epsilon \circ B_\epsilon.$$

Proposition 10.1. *If E, F and G are smooth vector bundles on X , then*

$$\Psi_{S\text{-sl}}^m(X; F, G) \circ \Psi_{S\text{-sl}}^n(X; E, F) \subset \Psi_{S\text{-sl}}^{m+n}(X; E, G)$$

and the induced map is continuous with respect to the natural Fréchet topology. Furthermore, the various symbol maps induce composition laws in such a way that they become algebra homomorphisms. A similar result holds for polyhomogeneous semiclassical S-operators.

Proof. We can employ the same strategy as in the proof of Theorem 8.1 and proceed by induction on the dimension of the manifold with fibred corners. Notice that the second part of the proof of Theorem 8.1 (starting with Lemma 8.3) mostly involve partitions of unity and has a direct generalization to semiclassical S-operators. This means the proposition follows from Lemma 10.2 below, which is an analog of Lemma 8.2 for semiclassical S-operators. \square

Lemma 10.2. *Suppose that the conclusion of Proposition 10.1 holds for the manifold with fibred corners X . Then it also holds for the manifold with fibred corner $\overline{\mathbb{R}^p} \times X$ defined just before Lemma 8.2.*

Proof. The proof is similar to the one of Lemma 8.2. To avoid repetition, we will focus on the parts that require changes. First, without loss of generality, we can assume $E = F = G = \underline{\mathbb{C}}$. Using the Fourier transform on \mathbb{R}^p , we can describe the action of operators $A \in \Psi_{S\text{-sl}}^m(\overline{\mathbb{R}^p} \times X)$ and $B \in \Psi_{S\text{-sl}}^n(\overline{\mathbb{R}^p} \times X)$ by

$$(10.18) \quad \begin{aligned} A_\epsilon u(t) &= \frac{1}{(2\pi\epsilon)^p} \int e^{i(t-t') \cdot \frac{\tau}{\epsilon}} a(t; \tau) u(t') dt' d\tau, \\ B_\epsilon u(t) &= \frac{1}{(2\pi\epsilon)^p} \int e^{i(t-t') \cdot \frac{\tau}{\epsilon}} b(t; \tau) u(t') dt' d\tau. \end{aligned}$$

Here, a and b are operator-valued symbols,

$$(10.19) \quad a \in \mathcal{C}^\infty(\overline{\mathbb{R}^p}; \Psi_{S\text{-sl} - \text{sus}(p)}^m(X)), \quad b \in \mathcal{C}^\infty(\overline{\mathbb{R}^p}; \Psi_{S\text{-sl} - \text{sus}(p)}^n(X)),$$

where $\Psi_{S\text{-sl} - \text{sus}(p)}^\ell(X)$ is the space of \mathbb{R}^p -suspended semiclassical S-operators of order ℓ on X , and the variable $\tau \in \mathbb{R}^p$ in (8.1) is seen as the suspension parameter. Precisely, as for suspended S-operators, the space $\Psi_{S\text{-sl} - \text{sus}(p)}^\ell(X)$ can be defined as the subspace of $\Psi_{S\text{-sl}}^\ell(\overline{\mathbb{R}^p} \times X)$ consisting of operators that are unchanged by translations in \mathbb{R}^p . These operator-valued symbols are such that (10.20)

$$(10.20) \quad a \in \mathcal{C}^\infty(\overline{\mathbb{R}_t^p}; \Psi_{S\text{-sl} - \text{sus}(p)}^m(X)) \implies D_t^\alpha D_\tau^\beta a \in (1+t^2)^{-\frac{|\alpha|}{2}} \mathcal{C}^\infty(\overline{\mathbb{R}_t^p}; \Psi_{S\text{-sl} - \text{sus}(p)}^{m-|\beta|}(X)),$$

so that the techniques of [40] can be applied. More precisely, using the change of variable $\xi = \frac{\tau}{\epsilon}$, we deduce from (8.4), (8.5) and (8.6) that

$$(10.21) \quad A_\epsilon \circ B_\epsilon u(t) = \frac{1}{(2\pi\epsilon)^p} \int e^{i(t-t') \cdot \frac{\tau}{\epsilon}} c(t, \tau) u(t') dt' d\tau,$$

where $c(t, \tau)$ is an operator-valued symbol which for $N \in \mathbb{N}$ can be written in the form

$$(10.22) \quad c(t, \tau) = \sum_{|\alpha| < N} \frac{\epsilon^{|\alpha|}}{\alpha!} \partial_\tau^\alpha a(t, \tau) D_t^\alpha b(t, \tau) + R_N(t, \tau)$$

with remainder term $R_N(t, \tau)$ given by

$$(10.23) \quad \sum_{|\alpha|=N} \frac{N\epsilon^{|\alpha|}}{\alpha!} \int_0^1 (1-\lambda)^{N-1} \left(\frac{1}{(2\pi)^p} \int e^{-iz \cdot \zeta} \partial_\tau^\alpha a(t, \tau + \epsilon\lambda\zeta) D_t^\alpha b(t+z, \tau) dz d\zeta \right) d\lambda.$$

Proceeding as in the proof of Lemma 8.2, we can then show that for N even,

$$R_N \in \epsilon^N \mathcal{S}_{-N}^{m+n-\frac{N}{2}}(\mathbb{R}_t^p \times \mathbb{R}_\tau^p; \Psi_{\mathcal{S}\text{-sl}}^{m+n-\frac{N}{2}}(X))$$

where $\mathcal{S}_\ell^k(\mathbb{R}_t^p \times \mathbb{R}_\tau^p; \Psi_{\mathcal{S}\text{-sl}}^{m+n}(X))$ is the space of operator-valued symbols q such that for any Fréchet semi-norm $\|\cdot\|$ of $\Psi_{\mathcal{S}\text{-sl}}^{m+n}(X)$,

$$\sup_{t, \tau} \left((1+|\tau|^2)^{\frac{|\beta|-k}{2}} (1+|t|^2)^{\frac{|\alpha|-\ell}{2}} \|D_t^\alpha D_\tau^\beta q\| \right) < \infty \quad \forall \alpha, \beta \in \mathbb{N}_0^p.$$

Therefore, taking an asymptotic sum as before we can show that

$$(10.24) \quad c \in \mathcal{C}^\infty(\overline{\mathbb{R}_t^p}; \Psi_{\mathcal{S}\text{-sl} - \text{sus}(p)}^{m+n}(X)) \quad \text{with } c \sim \sum_\alpha \frac{\epsilon^{|\alpha|}}{\alpha!} \partial_\tau^\alpha a(t, \tau) D_t^\alpha b(t, \tau),$$

where the notation \sim means that for all $N \in \mathbb{N}$,

$$c(t, \tau) - \sum_{\alpha < N} \frac{\epsilon^{|\alpha|}}{\alpha!} \partial_\tau^\alpha a(t, \tau) D_t^\alpha b(t, \tau) \in \epsilon^N (1+|t|^2)^{-\frac{N}{2}} \mathcal{C}^\infty(\overline{\mathbb{R}_t^p}; \Psi_{\mathcal{S}\text{-sl} - \text{sus}(p)}^{m+n-N}(X)).$$

In particular, this shows $A \circ B \in \Psi_{\mathcal{S}\text{-sl}}^{m+n}(\overline{\mathbb{R}^p} \times X)$. To show that the various symbols are compatible with composition, we can proceed as in the proof of Lemma 8.2. \square

Definition 10.3. An operator $P \in \Psi_{\mathcal{S}\text{-sl}}^m(X; E, F)$ is said to be **elliptic** if $\sigma_m(P)$ is invertible. It is said to be **fully elliptic** if it is elliptic and $\sigma_{\epsilon=1}(P)$ is a fully elliptic \mathcal{S} -operator.

A natural sub-groupoid of $\mathcal{G}_{\pi\text{-sl}}^{(1)}$ is obtained by considering the interior of ff_{sl} (as a subset of $\partial X_{\pi\text{-sl}}^2$),

$$(10.25) \quad T^{\text{FC}}X = \overset{\circ}{\text{ff}}_{\text{sl}} = \text{ff}_{\text{sl}} \setminus \partial \text{ff}_{\text{sl}}, \quad \text{where } \partial \text{ff}_{\text{sl}} := \text{ff}_{\text{sl}} \cap (\overline{\partial X_{\pi\text{-sl}}^2} \setminus \text{ff}_{\text{sl}}).$$

The groupoid $T^{\text{FC}}X$ also contains $\text{ff}_0 \setminus (\text{ff}_0 \cap T_{\epsilon=0})$ as a subgroupoid. It inherits from the Lie structure of $\mathcal{G}_{\pi\text{-sl}}$ the structure of a continuous family groupoid ([41]). There is also an induced continuous Haar system once a Haar system is fixed on $\mathcal{G}_{\pi\text{-sl}}$.

As for $\mathcal{G}_{\pi\text{-sl}}$, the groupoid $T^{\text{FC}}X$ can be written as a disjoint union of topologically amenable groupoids,

$$(10.26) \quad T^{\text{FC}}X = \pi T X \sqcup_{i=1}^k (H_i \times_{\pi_i} \pi T S_i \times_{\pi_i} H_i)|_{G_i} \times (0, 1)_\epsilon \times \mathbb{R},$$

where $G_i = H_i \setminus (\cup_{j>i} H_j)$. Thus, we conclude from Lemma 4.5 that $T^{\text{FC}}X$ is measurewise amenable with $\mathcal{C}^*(T^{\text{FC}}X)$ nuclear and equal to $\mathcal{C}_r^*(T^{\text{FC}}X)$.

As we will now describe, the K -theory of $T^{\text{FC}}X$ corresponds to the stable homotopy classes of fully elliptic polyhomogeneous \mathcal{S} -operators. For this purpose, we will restrict our attention to fully elliptic polyhomogeneous \mathcal{S} -operators of order zero. This is not a serious restriction. if $P \in \Psi_{\mathcal{S}\text{-ph}}^m(X; E, F)$ is fully elliptic, we can replace it by the fully elliptic operator $P(\Delta_E + 1)^{-\frac{m}{2}} \in \Psi_{\mathcal{S}\text{-ph}}^0(X; E, F)$, where $\Delta_E \in \Psi_{\mathcal{S}\text{-ph}}^2(X; E)$ is some (positive) Laplacian associated to a choice of \mathcal{S} -metric on X and a choice of Hermitian metric on E .

Definition 10.4. Two fully elliptic operators $P_0 \in \Psi_{\mathcal{S}\text{-ph}}^0(X; E_0, E_1)$ and $P_1 \in \Psi_{\mathcal{S}\text{-ph}}^0(X; E_1, F_1)$ are **homotopic** if they can be connected by a continuous family of fully elliptic polyhomogeneous \mathcal{S} -operators

$$P_t \in \Psi_{\mathcal{S}\text{-ph}}^0(X; E_t, F_t), \quad t \in [0, 1].$$

We say instead that P_0 and P_1 are **stably homotopic** if they become homotopic after the addition to each of them of the identity operator Id_H acting on the sections of some complex vector bundle $H \rightarrow X$.

Stable homotopies induce an equivalence relation and we denote by $\text{FE}_{\mathcal{S}}(X)$ the set of fully elliptic operators modulo stable homotopies. This set is in fact an abelian group with addition given by direct sum and inverse given by the parametrix construction of Proposition 9.1. It can be identified with the K -theory of a mapping cone. To see this, let us use the notation of the proof of Theorem 9.11 and denote by $\mathcal{A} = \overline{\mathcal{P}}_{\mathcal{S}\text{-ph}}^0(X)$ the closure of $\Psi_{\mathcal{S}\text{-ph}}^0(X)$ in $\mathcal{L}(\mathcal{H}, \mathcal{H})$, where $\mathcal{H} = L_{g_\pi}^2(X)$. The algebra \mathcal{A} contains the subalgebra $\mathcal{K} \subset \mathcal{L}(\mathcal{H}, \mathcal{H})$ of compact operators so that we can consider the quotient map

$$(10.27) \quad q : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{K}.$$

The algebra $\mathcal{A}_0 = \mathcal{C}(X)$ of continuous functions on X is another subalgebra of \mathcal{A} . Denote also by $q : \mathcal{A}_0 \rightarrow \mathcal{A}/\mathcal{K}$ the restriction of the quotient map to \mathcal{A}_0 . Let

$$(10.28) \quad \mathcal{C}_q = \{(a_0, a) \in \mathcal{A}_0 \oplus \mathcal{C}([0, 1]; \mathcal{A}/\mathcal{K}); q(a_0) = a(0)\}$$

be the mapping cone of the map $q : \mathcal{A}_0 \rightarrow \mathcal{A}/\mathcal{K}$. Consider also the mapping cylinder

$$(10.29) \quad \mathcal{C}_q^+ = \{(a_0, a) \in \mathcal{A}_0 \oplus \mathcal{C}([0, 1]; \mathcal{A}/\mathcal{K}); q(a_0) = a(0)\}.$$

By Theorem 9.11, a fully elliptic operator $P \in \Psi_{\mathcal{S}\text{-ph}}^0(X; E, F)$, defines a K -class in

$$K_0(\mathcal{C}_q^+, \mathcal{C}_q) \cong K_0(\mathcal{C}_q).$$

This K -class only depends on the stable homotopy class of P so that there is a well-defined group homomorphism

$$(10.30) \quad \sigma_{\mathcal{C}_q} : \text{FE}_{\mathcal{S}}(X) \rightarrow K_0(\mathcal{C}_q).$$

Proposition 10.5. *The map $\sigma_{\mathcal{C}_q}$ is a group isomorphism.*

Proof. This can be seen as a particular case of a result of Savin [46, Theorem 4]. Alternatively, since Theorem 9.11 identifies $\text{FE}_{\mathcal{S}}(X)$ with the relative K -group $K(q)$ associated to the homomorphism $q : \mathcal{A}_0 \rightarrow \mathcal{A}/\mathcal{K}$ (see for instance [4] or [16] for a definition of $K(q)$), we can follow instead the approach in [16, Theorem 3.29]. This

consists in noticing that the map σ_{C_q} naturally fits into a commutative diagram of exact sequences,

$$(10.31) \quad \begin{array}{ccccccccc} K_1(\mathcal{A}_0) & \longrightarrow & K_1(\mathcal{A}/\mathcal{K}) & \longrightarrow & \mathrm{FE}_S(X) & \longrightarrow & K_0(\mathcal{A}_0) & \longrightarrow & K_0(\mathcal{A}/\mathcal{K}) \\ & & \downarrow s & & \downarrow s & & \downarrow \sigma_{C_q} & & \downarrow \mathrm{Id} & & \downarrow \mathrm{Id} \\ & & K_0(\mathcal{S}\mathcal{A}_0) & \longrightarrow & K_0(\mathcal{S}(\mathcal{A}/\mathcal{A})) & \longrightarrow & K_0(C_q) & \longrightarrow & K_0(\mathcal{A}_0) & \longrightarrow & K_0(\mathcal{A} \setminus \mathcal{K}), \end{array}$$

where the bottom row is the Puppe sequence associated to $q : \mathcal{A}_0 \rightarrow \mathcal{A}/\mathcal{K}$ and s denotes the suspension isomorphism. The result then follows by applying the five-lemma to this diagram. \square

The group $\mathrm{FE}_S(X)$ can also be related with the K -theory of the groupoid $T^{\mathrm{FC}}X$. Indeed, given a fully elliptic S -operator $P \in \Psi_{S-\mathrm{ph}}^0(X; E, F)$, let $\mathcal{P} \in \Psi_{S-\mathrm{sl}-\mathrm{ph}}^0(X; E, F)$ be a corresponding fully elliptic semiclassical S -operator such that $\sigma_{\epsilon=1}(\mathcal{P}) = P$. The full ellipticity insures that $\sigma_{\mathrm{ff}_{\mathrm{sl}}}(\mathcal{P}) \in \Psi_{\mathrm{ff}_{\mathrm{sl}}-\mathrm{ph}}^0(X; E, F)$ defines a K -class in

$$K_0(\overline{\mathcal{P}}_{\mathrm{ff}_{\mathrm{sl}}-\mathrm{ph}}^0(X), \mathcal{C}^*(T^{\mathrm{FC}}X)) \cong K_0(\mathcal{C}^*(T^{\mathrm{FC}}X)),$$

where $\mathcal{C}^*(T^{\mathrm{FC}}X) = \mathcal{C}_r^*(T^{\mathrm{FC}}X)$ is the reduced C^* -algebra of the groupoid $T^{\mathrm{FC}}X$ and $\overline{\mathcal{P}}_{\mathrm{ff}_{\mathrm{sl}}-\mathrm{ph}}^0(X)$ is the C^* -algebra obtained by taking the closure $\Psi_{\mathrm{ff}_{\mathrm{sl}}-\mathrm{ph}}^0(X)$ with respect to the reduced norm of the groupoid $T^{\mathrm{FC}}X$, see for instance [19, p.641].

This K -class only depends on the stable homotopy class of P , so that there is in fact a well-defined group homomorphism

$$(10.32) \quad \sigma_{\mathrm{nc}} : \mathrm{FE}_S(X) \rightarrow K_0(\mathcal{C}^*(T^{\mathrm{FC}}X)).$$

Theorem 10.6. *The map σ_{nc} in (10.32) is an isomorphism of abelian groups.*

Proof. By Proposition 10.5, it suffices to construct a natural identification between $K_0(\mathcal{C}^*(T^{\mathrm{FC}}X))$ and $K_0(C_q)$ inducing a commutative diagram

$$(10.33) \quad \begin{array}{ccc} & & K_0(\mathcal{C}^*(T^{\mathrm{FC}}X)) \\ & \nearrow \sigma_{\mathrm{nc}} & \downarrow \cong \\ \mathrm{FE}_S(X) & & \\ & \searrow \sigma_{C_q} & \\ & & K_0(C_q). \end{array}$$

To construct this natural identification, consider the algebra

$$(10.34) \quad (\sigma_0 \oplus \sigma_{\mathrm{ff}_{\mathrm{sl}}})(\Psi_{S-\mathrm{sl}-\mathrm{ph}}^0(X)) \subset \mathcal{C}^\infty(S(N^*\Delta_{\mathrm{sl}})) \oplus \Psi_{\mathrm{ff}_{\mathrm{sl}}-\mathrm{ph}}^0(X)$$

and let \mathcal{B} be its C^* -closure in $\mathcal{C}(S(N^*\Delta_{\mathrm{sl}})) \oplus \overline{\mathcal{P}}_{\mathrm{ff}_{\mathrm{sl}}-\mathrm{ph}}^0(X)$. The symbol $\sigma_{\epsilon=1}$ restricts to give a map

$$(10.35) \quad \sigma_{\epsilon=1} : \mathcal{B} \rightarrow \mathcal{C}^0(S(\pi T^*X)) \oplus \left(\bigoplus_{i=1}^k \overline{\mathcal{P}}_{\mathrm{ff}_{\pi_i}-\mathrm{ph}}^0(H_i) \right),$$

where H_1, \dots, H_k is an exhaustive list of the boundary hypersurfaces of X and $\overline{\mathcal{P}}_{\mathrm{ff}_{\pi_i}-\mathrm{ph}}^0(H_i)$ is the C^* -closure of $\Psi_{\mathrm{ff}_{\pi_i}-\mathrm{ph}}^0(H_i)$ with respect to the reduced norm

(see [19, p.641]) for the groupoid $\text{ff}_{\pi_i} \overset{\circ}{\cap} \text{ff}_{\pi}$. There is a natural inclusion $\iota : \mathcal{C}(X) \hookrightarrow \mathcal{B}$. Let \mathcal{B}_0 be the kernel of the map (10.35) and consider the subalgebra

$$\hat{\mathcal{B}}_0 = \{b \in \mathcal{B}_0 \mid \sigma_{\text{ff}_0}(b) \in \mathcal{C}(X)\}.$$

Clearly, there is a natural identification $K_0(\hat{\mathcal{B}}_0) \cong K_0(\mathcal{C}_\iota)$, where \mathcal{C}_ι is the mapping cone of the natural inclusion $\iota : \mathcal{C}(X) \rightarrow \sigma_{\epsilon=1}(\mathcal{B})$. On the other hand, the commutative diagram of short exact sequences

$$(10.36) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{S}(\sigma_{\epsilon=1}(\mathcal{B})) & \longrightarrow & \hat{\mathcal{B}}_0 & \longrightarrow & \mathcal{C}(X) \longrightarrow 0 \\ & & \downarrow \text{Id} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{S}(\sigma_{\epsilon=1}(\mathcal{B})) & \longrightarrow & \mathcal{B}_0 & \longrightarrow & \overline{\mathcal{P}}_{\text{ff}_0 - \text{ph}}^0(X) \longrightarrow 0 \end{array}$$

induces a corresponding commutative diagram of six-term exact sequences in K -theory. Since the inclusion $\mathcal{C}(X) \subset \overline{\mathcal{P}}_{\text{ff}_0 - \text{ph}}^0(X)$ induces isomorphisms in K -theory, we conclude by the five-lemma that the inclusion $\hat{\mathcal{B}}_0 \subset \mathcal{B}_0$ also induces isomorphisms in K -theory. This means there are natural identifications

$$(10.37) \quad \begin{aligned} K_0(\mathcal{B}_0) &\cong K_0(\hat{\mathcal{B}}_0) \\ &\cong K_0(\mathcal{C}_\iota) \\ &\cong K_0(\mathcal{C}_q), \end{aligned}$$

where we have used Theorem 9.11 in the last step. On the other hand, the principal symbol induces a short exact sequence

$$(10.38) \quad 0 \longrightarrow \mathcal{C}^*(T^{\text{FC}}X) \longrightarrow \mathcal{B}_0 \xrightarrow{\sigma_0} \mathcal{C}(S(\pi T^*X) \times [0, 1]) \longrightarrow 0.$$

Since the quotient is contractible, this induces a natural identification

$$K_0(\mathcal{C}^*(T^{\text{FC}}X)) \cong K_0(\mathcal{B}_0),$$

so that we obtain the desired identification by combining this with (10.37). Thanks to the naturality of our construction, one can readily check it induces a commutative diagram as in (10.33). \square

11. POINCARÉ DUALITY

This last section will involve some Kasparov bivariant K -theory. The unfamiliar reader may for instance have a look at [51, 5, 11]. We are using the notations of [5] and [11].

Let $P \in \Psi_{\text{S-ph}}^0(X; E, F)$ be a fully elliptic operator and let Q be a parametrix for P as constructed in Proposition 9.1. Set $\mathbf{H} = L_{g_\pi}^2(X, E) \oplus L_{g_\pi}^2(X, F)$ and $\mathbf{P} = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix}$. By Theorem 9.11, the operator \mathbf{P} is bounded and Fredholm on \mathbf{H} . Let $\mathcal{C}_\pi^\infty(X) \subset \mathcal{C}^\infty(X)$ be the subalgebra of smooth functions on X which are constants along the fibres of the fibration π_i for each boundary hypersurface H_i of X . Clearly, we have a dense inclusion $\mathcal{C}_\pi^\infty(X) \subset \mathcal{C}(S^1X)$. Denote by $\mathbf{m} : \mathcal{C}(S^1X) \rightarrow \mathcal{L}(\mathbf{H})$ the representation given by multiplication.

For $f \in \mathcal{C}_\pi^\infty(X)$, $\mathbf{m}(f)$ is naturally a S -operator of order 0. The commutator $[\mathbf{m}(f), \mathbf{P}]$ is a S -operator of order -1 such that $\sigma_{\partial_i}([\mathbf{m}(f), \mathbf{P}]) = 0$ for all i . Hence,

by Theorem 9.7, the commutator $[\mathbf{m}(f), \mathbf{P}]$ is a compact operator. By the density of $\mathcal{C}_\pi^\infty(X)$ in $\mathcal{C}(\mathcal{S}X)$, we conclude more generally that $[\mathbf{m}(f), \mathbf{P}] \in \mathcal{K}(\mathbf{H})$ for all $f \in \mathcal{C}(\mathcal{S}X)$. Since $\mathbf{P}^2 - \text{Id} \in \mathcal{K}(\mathbf{H})$, this means $(\mathbf{H}, \mathbf{m}, \mathbf{P})$ is a Kasparov $(\mathcal{C}(\mathcal{S}X), \mathbb{C})$ -module. We denote by

$$(11.1) \quad [P] = [(\mathbf{H}, \mathbf{m}, \mathbf{P})] \in KK(\mathcal{C}(\mathcal{S}X), \mathbb{C}) = K_0(\mathcal{S}X).$$

the corresponding Kasparov $(\mathcal{C}(\mathcal{S}X), \mathbb{C})$ -cycle.

It is straightforward to check that this Kasparov cycle only depends on the stable homotopy class of P . This means this procedure defines a homomorphism of abelian groups

$$(11.2) \quad \text{quan} : \text{FE}_5(X) \rightarrow K_0(\mathcal{S}X).$$

Using the identification of Theorem 10.6, this can be seen as defining a homomorphism of abelian groups

$$(11.3) \quad \text{PD} := \text{quan} \circ \sigma_{\text{nc}}^{-1} : K_0(\mathcal{C}^*(T^{\text{FC}}X)) \rightarrow K_0(\mathcal{S}X).$$

This map establishes a Poincaré duality between $T^{\text{FC}}X$ and $\mathcal{S}X$. This can be described in a systematic way using Kasparov bivariant K-theory.

We first recall that two separable C^* -algebras A and B are Poincaré dual in K-theory if there exist $\alpha \in KK(A \otimes B, \mathbb{C})$ and $\beta \in KK(\mathbb{C}, A \otimes B)$ (minimal tensor products are understood) such that $\beta \otimes_A \alpha = 1_B$ and $\beta \otimes_B \alpha = 1_A$. Once such an α is given, the element β completing the Poincaré duality is unique. The element α (resp. β) is called the Dirac (resp. dual-Dirac) element of the Poincaré duality. For any C^* -algebras C, D , they provide isomorphisms

$$\cdot \otimes_A \alpha : KK(C, A \otimes D) \longrightarrow KK(B \otimes C, D),$$

with inverses given by

$$\beta \otimes_B \cdot : KK(B \otimes C, D) \longrightarrow KK(C, A \otimes D).$$

We are interested in the special case where $A = \mathcal{C}^*(T^{\text{FC}}X)$ and $B = \mathcal{C}(\mathcal{S}X)$. To construct a Dirac element, consider the groupoid $\mathcal{G}'_{\pi\text{-sl}} = \mathcal{G}_{\pi\text{-sl}} \setminus \text{ff}_\pi \times \{\epsilon = 1\}$. It enters in the short exact sequence

$$(11.4) \quad 0 \longrightarrow \mathcal{C}^*(\overset{\circ}{X} \times \overset{\circ}{X} \times (0, 1]) \longrightarrow \mathcal{C}^*(\mathcal{G}'_{\pi\text{-sl}}) \xrightarrow{\text{ev}_{\text{FC}}} \mathcal{C}^*(T^{\text{FC}}X) \longrightarrow 0,$$

where ev_{FC} is the obvious evaluation map induced by the inclusion $T^{\text{FC}}X \subset \mathcal{G}'_{\pi\text{-sl}}$. The ideal is contractible so by classical arguments $[\text{ev}_{\text{FC}}]$ is invertible in KK -theory and we set

$$(11.5) \quad \partial_X^{\text{FC}} = [\text{ev}_{\text{FC}}]^{-1} \otimes [\text{ev}_{\epsilon=1}] \otimes [\overset{\circ}{\mu}]^{-1} \in KK(\mathcal{C}^*(T^{\text{FC}}X), \mathbb{C}).$$

Here, $\text{ev}_{\epsilon=1} : \mathcal{C}^*(\mathcal{G}'_{\pi\text{-sl}}) \rightarrow \mathcal{C}^*(\overset{\circ}{X} \times \overset{\circ}{X})$ is the obvious evaluation map at $\epsilon = 1$ and the homomorphism $\overset{\circ}{\mu}$ is defined by $\lambda \mapsto \lambda q$ where q is a rank one self-adjoint projection and $[\overset{\circ}{\mu}]^{-1}$ is thus the Morita equivalence $\mathcal{C}^*(\overset{\circ}{X} \times \overset{\circ}{X}) \sim \mathbb{C}$.

The natural inclusion $\mathcal{C}_\pi^\infty(X) \subset \Psi_{\text{S-sl-ph}}^0(X)$ extends to an inclusion $\iota : \mathcal{C}(\mathcal{S}X) \hookrightarrow \overline{\mathcal{P}}_{\text{S-sl-ph}}^0(X)$, where $\overline{\mathcal{P}}_{\text{S-sl-ph}}^0(X)$ is the \mathcal{C}^* -closure of $\Psi_{\text{S-sl-ph}}^0(X)$ with respect

to the reduced norm for the groupoid $\mathcal{G}_{\pi\text{-sl}}$. This can be used to define a ‘zero sections’ homomorphism

$$(11.6) \quad \begin{array}{ccc} \Psi_{\pi\text{-sl}}^{\text{FC}} : \mathcal{C}({}^S X) \otimes \mathcal{C}^*(\mathcal{G}_{\pi\text{-sl}}) & \longrightarrow & \mathcal{C}^*(\mathcal{G}_{\pi\text{-sl}}) \\ g \otimes a & \longmapsto & \iota(g)a. \end{array}$$

By restriction to $T^{\text{FC}}X$, we also get a map

$$(11.7) \quad \Psi_X^{\text{FC}} : \mathcal{C}({}^S X) \otimes \mathcal{C}^*(T^{\text{FC}}X) \longrightarrow \mathcal{C}^*(T^{\text{FC}}X).$$

Consider then the following Kasparov cycle,

$$(11.8) \quad D_X^{\text{FC}} = [\Psi_X^{\text{FC}}] \otimes \partial_X^{\text{FC}} \in KK(\mathcal{C}({}^S X) \otimes \mathcal{C}^*(T^{\text{FC}}X), \mathbb{C}).$$

Theorem 11.1. *The Kasparov cycle D_X^{FC} is the Dirac element of a Poincaré duality between $\mathcal{C}^*(T^{\text{FC}}X)$ and $\mathcal{C}({}^S X)$.*

Proof. The groupoid $T^{\text{FC}}X$ is slightly different, but nevertheless intimately related to the noncommutative tangent space of [10] (see Corollary 11.3 below). At the cost of clarifying this relationship, it is therefore possible to transfer the Poincaré duality result of [10] to our context. To have instead a more self-contained approach, we will adapt the proof of [10] to our context. Really, this should be thought as a hybrid of the groupoid approach of [10] and the operator theoretic approach of [36] (see also [33]).

Let H_1, \dots, H_k be an exhaustive list of the boundary hypersurfaces of X such that

$$i < j, H_i \cap H_j \neq 0 \implies H_i < H_j.$$

Set $X_0 = X$ and consider the non-compact manifolds with fibred corners

$$(11.9) \quad X_j := X \setminus \left(\bigcup_{i=1}^j H_i \right), \quad \text{for } j \in \{1, \dots, k\}.$$

Let

$$\mathcal{C}({}^S X_j) = \{f \in \mathcal{C}({}^S X); f|_{q(H_i)} = 0 \text{ for } i \leq j\}$$

be the corresponding space of continuous functions on the associated stratified pseudomanifold, where $q : X \rightarrow {}^S X$ is the natural quotient map. Finally, set $T_0^{\text{FC}}X = T^{\text{FC}}X$ and consider the subgroupoid

$$T^{\text{FC}}X_j := T^{\text{FC}}X \setminus \left(\bigcup_{i=1}^j \overset{\circ}{\text{ff}}_{\pi_i\text{-sl}} \right),$$

where the interior is taken as a subset of $\partial X_{\pi\text{-sl}}^2$. Clearly, the morphism Ψ_X^{FC} restricts to give a morphism

$$\Psi_{X_j}^{\text{FC}} : \mathcal{C}({}^S X_j) \otimes \mathcal{C}^*(T^{\text{FC}}X_j) \rightarrow \mathcal{C}^*(T^{\text{FC}}X),$$

allowing us to define the following Kasparov cycle,

$$D_{X_j}^{\text{FC}} = [\Psi_{X_j}^{\text{FC}}] \otimes \partial_{X_j}^{\text{FC}} \in KK(\mathcal{C}({}^S X_j) \otimes \mathcal{C}^*(T^{\text{FC}}X_j), \mathbb{C}).$$

Now, for $j \in \{1, \dots, k\}$, we have two natural short exact sequences of \mathcal{C}^* -algebras,

$$(11.10) \quad 0 \longrightarrow \mathcal{C}({}^S X_j) \longrightarrow \mathcal{C}({}^S X_{j-1}) \xrightarrow{\alpha} \mathcal{C}(S_j \setminus \partial S_j) \longrightarrow 0,$$

$$(11.11) \quad 0 \longrightarrow \mathcal{C}^*(\mathcal{H}_j) \longrightarrow \mathcal{C}^*(T^{\text{FC}}X_{j-1}) \xrightarrow{\beta} \mathcal{C}^*(T^{\text{FC}}X_j) \longrightarrow 0,$$

where $\mathcal{H}_j \subset \text{ff}_{\pi_j - \text{sl}}$ is the subgroupoid given by

$$\mathcal{H}_j = T^{\text{FC}}X_{j-1} \setminus T^{\text{FC}}X_j.$$

It is naturally Morita equivalent to the groupoid πTS_j . For this latter groupoid, we have a natural Kasparov cycle given by

$$D_{S_j}^{\text{MC}} = [\Psi_{S_j}^{\text{MC}}] \otimes \partial_{S_j}^{\text{FC}} \in KK(\mathcal{C}(S_j \setminus \partial S_j) \otimes \mathcal{C}^*(\pi TS_j), \mathbb{C}),$$

where

$$\Psi_{S_j}^{\text{MC}} : \mathcal{C}(S_j \setminus \partial S_j) \otimes \mathcal{C}^*(\pi TS_j) \rightarrow \mathcal{C}^*(T^{\text{FC}}S_j)$$

is the morphism obtained by restriction of $\Psi_{S_j}^{\text{FC}}$. Using the Morita equivalence between \mathcal{H}_j and πTS_j , this gives a corresponding Kasparov cycle $D_{\mathcal{H}_j}^{\text{MC}} \in KK(\mathcal{C}(S_j \setminus \partial S_j) \otimes \mathcal{C}^*(\mathcal{H}_j), \mathbb{C})$. This cycle can be defined alternatively by $D_{\mathcal{H}_j}^{\text{MC}} = [\Psi_{\mathcal{H}_j}^{\text{MC}}] \otimes \partial_X^{\text{FC}}$, where

$$\Psi_{\mathcal{H}_j}^{\text{MC}} : \mathcal{C}(S_j \setminus \partial S_j) \otimes \mathcal{C}^*(\mathcal{H}_j) \rightarrow \mathcal{C}^*(T^{\text{FC}}X)$$

is the morphism obtained by restriction of Ψ_X^{FC} .

Now, the cycle D_{j-1}^{FC} , D_j^{FC} and $D_{\mathcal{H}_j}^{\text{MC}}$ can be used to obtain a diagram intertwining the six-term exact sequences in KK-theory associated to the short exact sequences (11.10) and (11.11),

$$(11.12) \quad \begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ KK_q(A, B \otimes \mathcal{C}(S X_j)) & \xrightarrow{c(S X_j) \otimes D_{X_j}^{\text{FC}}} & KK_q(A \otimes \mathcal{C}^*(T^{\text{FC}}X_j), B) \\ \downarrow & & \downarrow \\ KK_q(A, B \otimes \mathcal{C}(S X_{j-1})) & \xrightarrow{c(S X_{j-1}) \otimes D_{X_{j-1}}^{\text{FC}}} & KK_q(A \otimes \mathcal{C}^*(T^{\text{FC}}X_{j-1}), B) \\ \downarrow & & \downarrow \\ KK_q(A, B \otimes \mathcal{C}(S_j \setminus \partial S_j)) & \xrightarrow{c(S_j \setminus \partial S_j) \otimes D_{\mathcal{H}_j}^{\text{MC}}} & KK_q(A \otimes \mathcal{C}^*(\mathcal{H}_j), B) \\ \downarrow \partial_\alpha & & \downarrow \partial_\beta \\ \vdots & & \vdots \end{array}$$

where A and B are C^* -algebras.

The result then follows from the following two claims.

Claim 1. *The diagram (11.12) is commutative up to sign.*

Claim 2. *The Kasparov cycles $D_{X_k}^{\text{FC}} = D_X^{\text{MC}}$ and $D_{S_j}^{\text{MC}}$ for $j \in \{1, \dots, k\}$ are Dirac elements.*

Indeed, using the Morita equivalence between πTS_j and \mathcal{H}_j , we see that $D_{\mathcal{H}_j}^{\text{MC}}$ is also a Dirac element. Thus, starting with $j = k$ and applying the five-lemma to (11.12), we find that the map

$$KK_q(A, B \otimes \mathcal{C}(S X_{k-1})) \xrightarrow{c(S X_{k-1}) \otimes D_{X_{k-1}}^{\text{FC}}} KK_q(A \otimes \mathcal{C}^*(T^{\text{FC}}X_{k-1}), B)$$

is an isomorphism. By [10, Lemma 2], this implies $D_{X_{k-1}}^{\text{FC}}$ is a Dirac element. Repeating this argument for $j = k-1, k-2, \dots, 1$, we find more generally that $D_{X_j}^{\text{FC}}$ is a Dirac element for all $j \in \{0, 1, \dots, k\}$. In particular, $D_X^{\text{FC}} = D_{X_0}^{\text{FC}}$ is a Dirac element.

Thus, it remains to prove the two claims, which we do below. \square

Proof of Claim 1. The proof of the commutativity of the squares not involving boundary homomorphisms is straightforward and left to the reader. To obtain the commutativity of the remaining squares, we need to show that

$$(11.13) \quad \partial_\alpha \otimes_{\mathcal{C}(\mathcal{S}X_j)} D_{X_j}^{\text{FC}} = \partial_\beta \otimes_{\mathcal{C}^*(\mathcal{H}_j)} D_{\mathcal{H}_j}^{\text{MC}},$$

where $\partial_\alpha \in KK_1(\mathcal{C}(S_j \setminus \partial S_j), \mathcal{C}(\mathcal{S}X_j))$ and $\partial_\beta \in KK_1(\mathcal{C}^*(T^{\text{FC}}X_j), \mathcal{C}^*(\mathcal{H}_j))$ are the boundary homomorphisms associated to the short exact sequences (11.10) and (11.11). From the definition of D_j^{FC} and $D_{\mathcal{H}_j}^{\text{MC}}$, this means we need to show that

$$(11.14) \quad \partial_\alpha \otimes_{\mathcal{C}(\mathcal{S}X_j)} [\Psi_{X_j}^{\text{FC}}] = \partial_\beta \otimes_{\mathcal{C}^*(\mathcal{H}_j)} [\Psi_{\mathcal{H}_j}^{\text{MC}}]$$

in $KK_1(\mathcal{C}(S_j \setminus \partial S_j) \otimes \mathcal{C}^*(T^{\text{FC}}X_j), \mathcal{C}^*(T^{\text{FC}}X))$. To see this, consider the subgroupoid $L_j := T^{\text{FC}}X_j \cap \text{ff}_0 \cap \text{ff}_{\pi_j - \text{sl}} \subset \pi TX|_{H_j}$. Thus, there is a natural restriction homomorphism $\mathcal{C}^*(T^{\text{FC}}X_j) \rightarrow \mathcal{C}^*(L_j)$. There is also an obvious multiplication homomorphism

$$\mathcal{C}(S_j \setminus \partial S_j) \otimes \mathcal{C}(L_j) \rightarrow \mathcal{C}(\pi TX|_{\overset{\circ}{H}_j}).$$

Let also \mathcal{N}_{H_j} be a tubular neighborhood of H_j coming from an iterated fibred tube system and set $W = \overset{\circ}{\mathcal{N}}_{H_j}$. The tube system of H_j induces an identification

$$(11.15) \quad \mathcal{C}(\mathbb{R}) \otimes \mathcal{C}^*(\pi TX|_{\overset{\circ}{H}_j}) \rightarrow \mathcal{C}^*(TW).$$

On the other hand, the short exact sequence

$$0 \longrightarrow \mathcal{C}^*(\overset{\circ}{\text{ff}}_{\pi_j - \text{sl}}) \longrightarrow \mathcal{C}^*(\overset{\circ}{\text{ff}}_{\pi_j - \text{sl}} \cup \pi TX|_{\overset{\circ}{H}_j}) \longrightarrow \mathcal{C}^*(\pi TX|_{\overset{\circ}{H}_j}) \longrightarrow 0,$$

induces a boundary homomorphism in $KK_1(\mathcal{C}^*(\pi TX|_{\overset{\circ}{H}_j}), \mathcal{C}^*(\overset{\circ}{\text{ff}}_{\pi_j - \text{sl}}))$. By com-

posing with the inclusion $\mathcal{C}^*(\overset{\circ}{\text{ff}}_{\pi_j - \text{sl}}) \subset \mathcal{C}^*(\mathcal{H}_j)$, this induces a morphism $\partial \in KK_0(\mathcal{C}(\mathbb{R}) \otimes \mathcal{C}^*(\pi TX|_{\overset{\circ}{H}_j}), \mathcal{C}^*(\mathcal{H}_j))$. Using the identification (11.15), this gives a corresponding element in $\partial' \in KK_0(\mathcal{C}^*(TW), \mathcal{C}^*(\mathcal{H}_j))$ inducing a commutative diagram of Kasparov cycles

$$\begin{array}{ccc} \mathcal{C}(\mathbb{R}) \otimes \mathcal{C}(\pi TX|_{\overset{\circ}{H}_j}) & \longrightarrow & \mathcal{C}^*(TW) \\ & \searrow \partial & \downarrow \partial' \\ & & \mathcal{C}^*(\mathcal{H}_j). \end{array}$$

The result then follows by noticing this fits into a bigger diagram commutative up to sign involving the Kasparov cycles of (11.14), (11.16)

$$\begin{array}{ccc}
 & \mathcal{C}(\mathbb{R}) \otimes \mathcal{C}(S_j \setminus \partial S_j) \otimes \mathcal{C}(T^{\text{FC}}X_j) & \\
 & \downarrow & \\
 & \mathcal{C}(\mathbb{R}) \otimes \mathcal{C}^*(\pi TX|_{\overset{\circ}{H}_j}) & \\
 \swarrow & & \searrow \\
 \partial_\alpha \otimes_{\mathcal{C}(S_j)} [\Psi_{X_j}^{\text{FC}}] & & \partial_\beta \otimes_{\mathcal{C}^*(\mathcal{H}_j)} [\Psi_{\mathcal{H}_j}^{\text{MC}}] \\
 \swarrow & \mathcal{C}^*(TW) \xrightarrow{\partial'} \mathcal{C}^*(\mathcal{H}_j) & \searrow \\
 & \downarrow & \\
 & \mathcal{C}^*(T^{\text{FC}}X) &
 \end{array}$$

□

For Claim 2, this is the Poincaré duality for manifolds with corners obtained in [32]. The result of [32] is not formulated in terms of Dirac elements, but this can be remedied easily by using the semiclassical b -double space (or the semiclassical cusp double space). For the convenience of the reader, we will provide a brief outline. First, the semiclassical b -double space is defined by

$$X_{b\text{-sl}}^2 = [X_b^2 \times [0, 1]_\varepsilon; \Delta_b \times \{0\}],$$

where $\Delta_b \subset X_b^2$ is the lifted diagonal. Denote the new face obtained by this blow-up by $\text{ff}_{0,b}$. Notice that the b -tangent bundle is naturally included in $\text{ff}_{0,b}$. If $\text{ff}_{b\text{-sl}}$ is the union of all the boundary hypersurfaces intersecting the lift of $\Delta_b \times [0, 1]$ in $X_{b\text{-sl}}^2$, we get a corresponding groupoid

$$T^b X := \overset{\circ}{\text{ff}}_{b\text{-sl}} \setminus (\overset{\circ}{\text{ff}}_{b\text{-sl}} \cap X_b^2 \times \{1\}).$$

Using evaluation maps as in the fibred corners case, one can define a natural Kasparov cycle $\partial_X^b \in KK(\mathcal{C}^*(T^b X), \mathbb{C})$. There is also a ‘zero sections’ morphism $\Psi_X^b : \mathcal{C}(X) \otimes \mathcal{C}^*(T^b X) \rightarrow \mathcal{C}^*(T^b X)$, and so a corresponding Kasparov cycle $D_X^b = [\Psi_X^b] \otimes \partial_X^b$ in $KK(\mathcal{C}(X) \otimes \mathcal{C}^*(T^b X), \mathbb{C})$.

Let H_1, \dots, H_k be an exhaustive list of boundary hypersurfaces of X and set

$$X_j = X \setminus \bigcup_{i=1}^j H_i, \quad X'_j = X \setminus \bigcup_{i=j+1}^k H_i,$$

with the convention that $X_0 = X = X'_k$. Then, by restriction of D_X^b , we obtain corresponding cycles

$$D_{X_j}^b \in KK(\mathcal{C}(X_j) \otimes \mathcal{C}^*({}^b T X'_j), \mathbb{C}), \quad \text{where } {}^b T X'_j = {}^b T X|_{X'_j}.$$

Since $D_X^{\text{MC}} = D_{X_k}^b$, Claim 2 is a consequence of the following proposition.

Proposition 11.2. *If X is a compact manifold with corners and H_1, \dots, H_k is an exhaustive list of its boundary hypersurfaces, then the Kasparov cycle $D_{X_j}^b$ is a Dirac element for all $j \in \{0, 1, \dots, k\}$.*

Proof. From [17], we know that $D_{X_0}^b$ is a Dirac element. This suggests to proceed by induction on the depth of X . Thus, assume the proposition is true for all manifolds with corners of depth less than the one of X . If $N_{H_j} = H_j \times [0, 1]$ is a collar neighborhood of H_j in X , then, after making obvious identifications, the inclusion $N_{H_j} \subset X$ induces two natural short exact sequences of C^* -algebras,

$$(11.17) \quad 0 \longrightarrow \mathcal{C}(X_j) \longrightarrow \mathcal{C}(X_{j-1}) \longrightarrow \mathcal{C}(\hat{H}_j \times [0, 1]) \longrightarrow 0,$$

$$(11.18) \quad 0 \longrightarrow \mathcal{C}^*({}^bT\check{H}_j \times T(0, 1)) \longrightarrow \mathcal{C}^*({}^bTX'_{j-1}) \longrightarrow \mathcal{C}^*({}^bTX'_j) \longrightarrow 0,$$

where

$$\hat{H}_j = H_j \setminus \left(\bigcup_{i=1}^{j-1} (H_i \cap H_j) \right), \quad \check{H}_j = H_j \setminus \left(\bigcup_{i=j+1}^k (H_i \cap H_j) \right).$$

By our inductive assumption, the cycle $D_{\hat{H}_j}^b \in KK(\mathcal{C}(\hat{H}_j) \otimes \mathcal{C}^*({}^bT\check{H}_j), \mathbb{C})$ is a Dirac element. On the other hand, $D_{[0,1]}^b \in KK(\mathcal{C}([0, 1]) \otimes \mathcal{C}^*(T(0, 1)), \mathbb{C})$ is a Dirac element by the result of [17]. This means the corresponding cycle

$$D_{\hat{H}_j \times [0,1]}^b = D_{\hat{H}_j}^b \otimes D_{[0,1]}^b \in KK(\mathcal{C}(\hat{H}_j \times [0, 1]) \otimes \mathcal{C}^*({}^bT\check{H}_j \times T(0, 1)), \mathbb{C})$$

is a Dirac element. Now, this Dirac element combines with $D_{X_j}^b$ and $D_{X_{j-1}}^b$ to give a diagram intertwining the six-term exact sequences in KK -theory associated to (11.17) and (11.18),

$$(11.19) \quad \begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ KK_q(A, B \otimes \mathcal{C}(X_j)) & \xrightarrow{\mathcal{C}(X_j) \otimes D_{X_j}^b} & KK_q(A \otimes \mathcal{C}^*({}^bTX'_j), B) \\ \downarrow & & \downarrow \\ KK_q(A, B \otimes \mathcal{C}(X_{j-1})) & \xrightarrow{\mathcal{C}(X_{j-1}) \otimes D_{X_{j-1}}^b} & KK_q(A \otimes \mathcal{C}^*({}^bTX'_{j-1}), B) \\ \downarrow & & \downarrow \\ KK_q(A, B \otimes \mathcal{C}(\hat{H}_j \times [0, 1])) & \xrightarrow{\mathcal{C}(\hat{H}_j \times [0,1]) \otimes D_{\hat{H}_j \times [0,1]}^b} & KK_q(A \otimes \mathcal{C}^*({}^bT\check{H}_j \times T(0, 1)), B) \\ \downarrow & & \downarrow \\ \vdots & & \vdots \end{array}$$

where A and B are C^* -algebras. Using a similar method as for (11.12), it can be shown that this diagram is commutative up to sign. Thus, starting with $j = 0$ and applying the five-lemma recursively to (11.19) as well as [10, Lemma 2], we conclude that $D_{X_j}^b$ is a Dirac element for all $j \in \{0, 1, \dots, k\}$. \square

Since the noncommutative tangent space $T^S X$ of [10] is also Poincaré dual to the stratified pseudomanifold ${}^S X$, Theorem 11.1 has the following consequence.

Corollary 11.3. *The C^* -algebras $\mathcal{C}^*(T^S X)$ and $\mathcal{C}^*(T^{\text{FC}} X)$ are KK -equivalent.*

Proof. Let $D_X^S \in KK(\mathcal{C}^*(T^S X) \otimes \mathcal{C}^S(X), \mathbb{C})$ be the Dirac element of [10] that provides the Poincaré duality between $\mathcal{C}^*(T^S X)$ and $\mathcal{C}^S(X)$. Denote by

$$(D_X^{\text{FC}})^{-1} \in KK(\mathbb{C}, \mathcal{C}^*(T^{\text{FC}} X) \otimes \mathcal{C}^S(X)), \quad (D_X^S)^{-1} \in KK(\mathbb{C}, \mathcal{C}^*(T^S X) \otimes \mathcal{C}^S(X)),$$

the dual-Dirac elements of D_X^{FC} and D_X^S respectively. Then the element

$$\alpha = (D_X^S)^{-1} \otimes_{\mathcal{C}^S(X)} D_X^{\text{FC}} \in KK(\mathcal{C}^*(T^{\text{FC}} X), \mathcal{C}^*(T^S X))$$

is a KK -equivalence between $\mathcal{C}^*(T^{\text{FC}} X)$ and $\mathcal{C}^*(T^S X)$ with inverse

$$\alpha^{-1} = (D_X^{\text{FC}})^{-1} \otimes_{\mathcal{C}^S(X)} D_X^S \in KK(\mathcal{C}^*(T^S X), \mathcal{C}^*(T^{\text{FC}} X)).$$

□

The map PD in (11.3) can be described in terms of the Dirac element D_X^{FC} .

Theorem 11.4. *If $P \in \Psi_{\text{S-ph}}^0(X; E, F)$ is a fully elliptic operator, then*

$$(11.20) \quad \sigma_{\text{nc}}(P) \otimes_{\mathcal{C}^*(T^{\text{FC}} X)} D_X^{\text{FC}} = [P].$$

In particular, the map PD in (11.3) is an isomorphism of abelian groups.

Proof. Let $P \in \Psi_{\text{S-ph}}^0(Y; E, F)$ be a fully elliptic operator and let Q be a parametrix for P as constructed in Proposition 9.1. Let $\mathcal{P} \in \Psi_{\text{S-sl-ph}}^0(X; E, F)$ and $\mathcal{Q} \in \Psi_{\text{S-sl-ph}}^0(X; F, E)$ be fully elliptic semiclassical S -operators such that $\sigma_{\epsilon=1}(\mathcal{P}) = P$ and $\sigma_{\epsilon=1}(\mathcal{Q}) = Q$. Without loss of generality, we can choose \mathcal{Q} such that

$$(11.21) \quad \mathcal{P}\mathcal{Q} - 1 \in \Psi_{\text{S-sl}}^{-\infty}(X; F), \quad \mathcal{Q}\mathcal{P} - 1 \in \Psi_{\text{S-sl}}^{-\infty}(X; E).$$

By construction, $a := \mathcal{P}|_{T^{\text{FC}} X}$ is a pseudodifferential operator on the groupoid $T^{\text{FC}} X$ of order 0, so it gives a (bounded) morphism between the $\mathcal{C}^*(T^{\text{FC}} X)$ -Hilbert modules $\mathcal{C}^*(T^{\text{FC}} X, E)$ and $\mathcal{C}^*(T^{\text{FC}} X, F)$. Reverting E and F , the same is true for $b := \mathcal{Q}|_{T^{\text{FC}} X}$ so we get a bounded morphism

$$\mathbf{a} = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{C}^*(T^{\text{FC}} X, E \oplus F)).$$

Since $\sigma_{\epsilon=1}(\mathcal{P}) = P$ and $\sigma_{\epsilon=1}(\mathcal{Q}) = Q$, we have that $a|_{\epsilon=1}$ is invertible with inverse $b|_{\epsilon=1}$ so that $\mathbf{a}^2 - \text{Id} \in \mathcal{K}(\mathcal{C}^*(T^{\text{FC}} X, E \oplus F))$. This means

$$(11.22) \quad (\mathcal{C}^*(T^{\text{FC}} X, E \oplus F), \mathbf{a})$$

is a Kasparov $(\mathbb{C}, \mathcal{C}^*(T^{\text{FC}} X))$ -cycle. Its class in $K_0(\mathcal{C}^*(T^{\text{FC}} X))$ is the element $\sigma_{\text{nc}}(P)$ defined in (10.32).

Similarly, we get a K -theory class associated with \mathcal{P} . As before,

$$\mathcal{T} := \begin{pmatrix} 0 & \mathcal{Q} \\ \mathcal{P} & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{C}^*(\mathcal{G}'_{\pi\text{-sl}}, E \oplus F))$$

and $\mathcal{T}^2 - 1 \in \mathcal{K}(\mathcal{C}^*(\mathcal{G}'_{\pi\text{-sl}}, E \oplus F))$, so that

$$[\mathcal{T}] = (\mathcal{C}^*(\mathcal{G}'_{\pi\text{-sl}}, E \oplus F), \mathcal{T}) \in K_0(\mathcal{C}^*(\mathcal{G}'_{\pi\text{-sl}})).$$

The cycle $[\mathcal{T}]$ is such that

$$(11.23) \quad [\mathcal{T}] \otimes [\text{ev}_{\text{FC}}] = (\text{ev}_{\text{FC}})_*[\mathcal{T}] = \sigma_{\text{nc}}(P).$$

In order to achieve the computation proving (11.20), we observe that the homomorphism (11.6) naturally induces a map

$$(11.24) \quad \Psi_{\pi-sl'}^{\text{FC}} : \mathcal{C}(\mathcal{S}X) \otimes \mathcal{C}^*(\mathcal{G}'_{\pi-sl}) \longrightarrow \mathcal{C}^*(\mathcal{G}'_{\pi-sl})$$

leading to the equality of homomorphisms

$$(11.25) \quad \text{ev}_{\text{FC}} \circ \Psi_{\pi-sl'}^{\text{FC}} = \Psi_X^{\text{FC}} \circ (\text{Id}_{\mathcal{C}(\mathcal{S}X)} \otimes \text{ev}_{\text{FC}}).$$

Now, using the basic properties of the Kasparov product, we have,

$$\begin{aligned} \sigma_{nc}(P) \otimes_{\mathcal{C}^*(T^{\text{FC}}X)} D_X^{\text{FC}} &= ([\mathcal{T}] \otimes [\text{ev}_{\text{FC}}]) \otimes_{\mathcal{C}^*(T^{\text{FC}}X)} D_X^{\text{FC}}, && \text{by (11.23),} \\ &= \tau_{\mathcal{C}(\mathcal{S}X)}([\mathcal{T}] \otimes [\text{ev}_{\text{FC}}]) \otimes D_X^{\text{FC}} \\ &= \tau_{\mathcal{C}(\mathcal{S}X)}([\mathcal{T}]) \otimes [\text{Id}_{\mathcal{C}(\mathcal{S}X)} \otimes \text{ev}_{\text{FC}}] \otimes D_X^{\text{FC}} \\ &= \tau_{\mathcal{C}(\mathcal{S}X)}([\mathcal{T}]) \otimes [\Psi_{\pi-sl'}^{\text{FC}}] \otimes [\text{ev}_{\epsilon=1}] \otimes [\mu^\circ]^{-1}, && \text{by (11.8), (11.25).} \end{aligned}$$

The next step requires some details. We have

$$\tau_{\mathcal{C}(\mathcal{S}X)}(\mathcal{T}) = \left(\mathcal{C}(\mathcal{S}X) \otimes \mathcal{C}^*(\mathcal{G}'_{\pi-sl}, E \oplus F), l, \text{Id} \otimes \begin{pmatrix} 0 & \mathcal{Q} \\ \mathcal{P} & 0 \end{pmatrix} \right)$$

where $\mathcal{C}(\mathcal{S}X) \otimes \mathcal{C}^*(\mathcal{G}'_{\pi-sl}, E \oplus F)$ has the obvious right $\mathcal{C}(\mathcal{S}X) \otimes \mathcal{C}^*(\mathcal{G}'_{\pi-sl})$ -module structure and the representation l is defined by: $l(f)(g \otimes \xi) = (fg) \otimes \xi$. We then have,

$$(11.26) \quad \begin{aligned} \tau_{\mathcal{C}(\mathcal{S}X)}(\mathcal{T}) \otimes \Psi_{\pi-sl'}^{\text{FC}} &= \\ & \left([\mathcal{C}(\mathcal{S}X) \otimes \mathcal{C}^*(\mathcal{G}'_{\pi-sl}, E \oplus F)] \otimes_{\Psi_{\pi-sl'}^{\text{FC}}} \mathcal{C}^*(\mathcal{G}'_{\pi-sl}), l \otimes \text{Id}, (\text{Id} \otimes \begin{pmatrix} 0 & \mathcal{Q} \\ \mathcal{P} & 0 \end{pmatrix}) \otimes \text{Id} \right). \end{aligned}$$

By construction, $\mathcal{C}^*(\mathcal{G}'_{\pi-sl}, E \oplus F)$ is a finitely generated projective Hilbert $\mathcal{C}^*(\mathcal{G}'_{\pi-sl})$ -module, so we can choose a self-adjoint idempotent $e \in M_r(\mathcal{C}^*(\mathcal{G}'_{\pi-sl}))$ such that $\mathcal{C}^*(\mathcal{G}'_{\pi-sl}, E \oplus F) = e\mathcal{C}^*(\mathcal{G}'_{\pi-sl})^r$. This choice provides a Hilbert $\mathcal{C}^*(\mathcal{G}'_{\pi-sl})$ -module isomorphism

$$[\mathcal{C}(\mathcal{S}X) \otimes \mathcal{C}^*(\mathcal{G}'_{\pi-sl}, E \oplus F)] \otimes_{\Psi_{Y-sl'}^{\text{FC}}} \mathcal{C}^*(\mathcal{G}'_{\pi-sl}) \simeq \mathcal{C}^*(\mathcal{G}'_{\pi-sl}, E \oplus F)$$

under which the representation $l \otimes \text{Id}$ corresponds to $\nu : \mathcal{C}(\mathcal{S}X) \rightarrow \mathcal{L}(\mathcal{C}^*(\mathcal{G}'_{\pi-sl}, E \oplus F))$ defined by

$$\nu(f)(e(b_1, \dots, b_r)) := e(\Psi_{\pi-sl'}^{\text{FC}}(f, b_1), \dots, \Psi_{\pi-sl'}^{\text{FC}}(f, b_r)),$$

and the operator $(\text{Id} \otimes \begin{pmatrix} 0 & \mathcal{Q} \\ \mathcal{P} & 0 \end{pmatrix}) \otimes \text{Id}$ simply corresponds to $\begin{pmatrix} 0 & \mathcal{Q} \\ \mathcal{P} & 0 \end{pmatrix}$. In other words, we have the equality

$$(11.27) \quad \tau_{\mathcal{C}(\mathcal{S}X)}([\mathcal{T}]) \otimes [\Psi_{Y-sl'}^{\text{FC}}] = \left[\left(\mathcal{C}^*(\mathcal{G}'_{\pi-sl}, E \oplus F), \nu, \begin{pmatrix} 0 & \mathcal{Q} \\ \mathcal{P} & 0 \end{pmatrix} \right) \right]$$

in $KK(\mathcal{C}(\overset{\circ}{S}X), \mathcal{C}^*(\mathcal{G}'_{\pi\text{-sl}}))$. It follows that,

$$\begin{aligned}
 (11.28) \quad & \sigma_{nc}(P) \otimes_{\mathcal{C}^*(T^{\text{FC}}X)} D_X^{\text{FC}} \\
 &= \left[\left(\mathcal{C}^*(\mathcal{G}'_{\pi\text{-sl}}, E \oplus F), \nu, \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix} \right) \right] \otimes [\text{ev}_{\epsilon=1}] \otimes [\overset{\circ}{\mu}]^{-1} \\
 &= \left[\left(\mathcal{C}^*(\overset{\circ}{X} \times \overset{\circ}{X}, E \oplus F), \nu_{\epsilon=1}, \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix} \right) \right] \otimes [\overset{\circ}{\mu}]^{-1} \\
 &= \left[\left(L_{\pi}^2(X, E \oplus F), \mathbf{m}, \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix} \right) \right] = [P].
 \end{aligned}$$

□

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