3

Index theory and groupoids

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Abstract

This chapter is mainly devoted to a proof, using groupoids and KK-theory, of Atiyah and Singer's index theorem on compact smooth manifolds. We first present an elementary introduction to groupoids, C^*-algebras, KK-theory and pseudodifferential calculus on groupoids. We then show how the point of view adopted here generalizes to the case of conical pseudomanifolds.

3.1 Introduction

This chapter is meant to give the tools involved in our approach to index theory for singular spaces. The global framework adopted here is noncommutative geometry, with a particular focus on groupoids, C^*-algebras and bivariant K-theory.

The idea of using C^*-algebras to study *spaces* may be understood with the help of the Gelfand theorem, which asserts that Hausdorff locally compact spaces are in one-to-one correspondence with commutative C^*-algebras. A starting point in noncommutative geometry is then to think of noncommutative C^*-algebras as corresponding to a wider class of spaces, more singular than Hausdorff locally compact spaces. As a first consequence, given a geometrical or topological object which is badly behaved with respect to classical tools, noncommutative geometry suggests defining a C^*-algebra encoding relevant information carried by the original object.

* We would like to thank Georges Skandalis, who allowed us to use several of his works, in particular the manuscript of one of his courses [48, 49]. We would like to warmly thank Jorge Plazas for having typewritten a part of this chapter during the summer school, and Jérôme Chabert, who carefully read the chapter and corrected several mistakes. We are grateful to all the organizers for their kind invitation to the extremely stimulating summer school held at Villa de Leyva in July 2007, and we particularly thank Sylvie Paycha, both as an organizer and for her valuable comments on this document.

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Refining this construction, one may try to define this C^*-algebra as the C^*-algebra of a groupoid [46, 47]. That is, one can try to build a groupoid directly, encoding the original object and regular enough to allow the construction of its C^*-algebra. In the ideal case where the groupoid is smooth, one gets much more than a C^*-algebra, which only reflects topological properties: the groupoid has a geometrical and analytical flavor enabling many applications.

An illuminating example is the study of the space of leaves of a foliated manifold $(M, \mathcal{F})$ [10, 11, 14]. Although this space $M/\mathcal{F}$ is usually very singular, the holonomy groupoid of the foliation leads to a C^*-algebra $C^*(M, \mathcal{F})$ replacing with great success the algebra of continuous functions on the space $M/\mathcal{F}$. Moreover, the holonomy groupoid is smooth and characterizes the original foliation.

Once a C^*-algebra is built for the study of a given problem, one can look for *invariants* attached to it. For ordinary spaces, basic invariants live in the homology or cohomology of the space. When dealing with C^*-algebras, the suitable homology theory is K-theory, or better the KK-theory developed by G. Kasparov [30, 31, 49] (when a smooth subalgebra of the C^*-algebra is specified, which for instance is the case if a smooth groupoid is available, one may also consider cyclic (co)homology, but this theory is beyond the scope of these notes).

There is a fundamental theory which links the previous ideas, namely index theory. In the 1960s, Atiyah and Singer [6] proved their famous index theorem. Roughly speaking, they showed that, given a closed manifold, one can associate to any elliptic operator an integer called the *index*, which can be described in two different ways: one purely analytic and the other purely topological. This result is stated with the help of K-theory of spaces. Hence, using the Swan–Serre theorem, it can be formulated with K-theory of (commutative) C^*-algebras. This point, and the fact that the index theorem can be proved in many ways using K-theoretic methods, leads to the attempt to generalize it to more *singular* situations where appropriate C^*-algebras are available. Noncommutative geometry therefore offers a general framework in which one can try to state and prove index theorems. The case of foliations illustrates this perfectly again: elliptic operators along the leaves, equivariant with respect to the holonomy groupoid, admit an analytical index living in the K-theory of the C^*-algebra $C^*(M, \mathcal{F})$. Moreover, this index can also be described in a topological way, and this is the content of the index theorem for foliations of Connes and Skandalis [14].

Connes [13] also observed the important role played by groupoids in the definition of the index map: in both cases of closed manifolds and foliations, the analytical index map can be described with the use of a groupoid, namely a *deformation groupoid*. This approach has been extended by the authors and Nistor [20], who showed that the topological index of Atiyah and Singer can also be described

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using deformation groupoids. This leads to a *geometrical proof* of the index theorem of Atiyah and Singer; moreover, this proof is easily applied to a class of singular spaces (namely, pseudomanifolds with isolated singularities).

The content of this chapter is divided into three parts. Let us briefly describe them:

Part I: Groupoids and their C^*-algebras. As mentioned earlier, the first problem in the study of a singular geometrical situation is to associate to it a mathematical object which carries the information one wants to study and which is regular enough to be analyzed in a reasonable way. In noncommutative geometry, answering this question amounts to looking for a good *groupoid* and constructing its *C^*-algebra*. These points will be the subject of Sections 3.2 and 3.4.

Part II: KK-theory. Once the situation is desingularized, say through the construction of a groupoid and its C^*-algebra, one may look for invariants which capture the basic properties. Roughly speaking, the *KK-theory* groups are convenient groups of invariants for C^*-algebras, and KK-theory comes with powerful tools to carry out computations. Kasparov's bivariant K-theory will be the main topic of Sections 3.4 to 3.6.

Part III: Index theorems. We first briefly explain in Section 3.7 the pseudodifferential calculus on groupoids. Then, in Section 3.8, we give a geometrical proof of the Atiyah–Singer index theorem for closed manifolds, using the language of groupoids and KK-theory. Finally we show in the last section how these results can be extended to *conical pseudomanifolds*.

Prerequisites. The reader interested in this course should have background in several domains. Familiarity with basic differential geometry (manifolds, tangent spaces) is needed. The notions of fiber bundle and of K-theory for locally compact spaces should be known. Basic functional analysis, including the analysis of linear operators on Hilbert spaces, should be familiar. The knowledge of pseudodifferential calculus (basic definitions, ellipticity) is necessary. Although it is not absolutely necessary, some familiarity with C^*-algebras is preferable.

I. Groupoids and their C^*-Algebras

This first part will be devoted to the notion of groupoid, specifically that of differentiable groupoid. We provide definitions and consider standard examples. The interested reader may look for example at [12, 35]. We then recall the definition of C^*-algebras and see how one can associate a C^*-algebra to a groupoid. The theory of C^*-algebras of groupoids was initiated by Jean Renault [46]. A good reference for the construction of groupoid C^*-algebras is [32], by which the end of Section 3.3.2 is inspired.

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3.2 Groupoids

3.2.1 Definitions and basic examples of groupoids

Definition 3.2.1 Let G and $G^{(0)}$ be two sets. A *groupoid* structure on G over $G^{(0)}$ is given by the following homomorphisms:

- An injective map $u : G^{(0)} \to G$. The map u is called the *unit map*. We often identify $G^{(0)}$ with its image in G. The set $G^{(0)}$ is called the *set of units* of the groupoid.
- Two surjective maps: $r, s : G \to G^{(0)}$, which are respectively the *range* and *source* maps. They are equal to the identity on the space of units.
- An involution

$$\begin{aligned} i : G &\to G, \\ \gamma &\mapsto \gamma^{-1}, \end{aligned}$$

  called the *inverse* map. It satisfies $s \circ i = r$.
- A map

$$\begin{aligned} p : G^{(2)} &\to G, \\ (\gamma_1, \gamma_2) &\mapsto \gamma_1 \cdot \gamma_2, \end{aligned}$$

  called the *product*, where the set

$$G^{(2)} := \{(\gamma_1, \gamma_2) \in G \times G \mid s(\gamma_1) = r(\gamma_2)\}$$

  is the set of *composable pairs*. Moreover, for $(\gamma_1, \gamma_2) \in G^{(2)}$ we have $r(\gamma_1 \cdot \gamma_2) = r(\gamma_1)$ and $s(\gamma_1 \cdot \gamma_2) = s(\gamma_2)$.

The following properties must be fulfilled:

- The product is associative: for any $\gamma_1,\ \gamma_2,\ \gamma_3$ in G such that $s(\gamma_1) = r(\gamma_2)$ and $s(\gamma_2) = r(\gamma_3)$ the following equality holds:

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3).$$

- For any γ in G, one has $r(\gamma) \cdot \gamma = \gamma \cdot s(\gamma) = \gamma$ and $\gamma \cdot \gamma^{-1} = r(\gamma)$.

A groupoid structure on G over $G^{(0)}$ is usually denoted by $G \rightrightarrows G^{(0)}$, where the arrows stand for the source and target maps.

We will often use the following notation:

$$G_A := s^{-1}(A), \qquad G^B = r^{-1}(B) \quad \text{and} \quad G_A^B = G_A \cap G^B.$$

If x belongs to $G^{(0)}$, the s-*fiber* (r-*fiber*) of G over x is $G_x = s^{-1}(x)$ ($G^x = r^{-1}(x)$).

The groupoid is *topological* when G and $G^{(0)}$ are topological spaces with $G^{(0)}$ Hausdorff, the structural homomorphisms are continuous, and i is a homeomorphism. We will often require that our topological groupoids be *locally compact*.

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This means that $G \rightrightarrows G^{(0)}$ is a topological groupoid, such that G is second countable, each point γ in G has a compact (Hausdorff) neighborhood, and the map s is open. In this situation the map r is open and the s-fibers of G are Hausdorff.

The groupoid is *smooth* when G and $G^{(0)}$ are second-countable smooth manifolds with $G^{(0)}$ Hausdorff, the structural homomorphisms are smooth, u is an embedding, s is a submersion, and i is a diffeomorphism.

When $G \rightrightarrows G^{(0)}$ is at least topological, we say that G is *s-connected* when for any $x \in G^{(0)}$ the s-fiber of G over x is connected. The s-connected component of a groupoid G is $\bigcup_{x\in G^{(0)}} CG_x$, where CG_x is the connected component of the s-fiber G_x which contains the unit $u(x)$.

Examples

1. A space X is a groupoid over itself with $s = r = u = \mathrm{Id}$.
2. A group $G \rightrightarrows \{e\}$ is a groupoid over its unit e, with the usual product and inverse map.
3. A group bundle : $\pi : E \to X$ is a groupoid $E \rightrightarrows X$ with $r = s = \pi$ and algebraic operations given by the group structure of each fiber E_x, $x \in X$.
4. If $\mathcal{R}$ is an equivalence relation on a space X, then the graph of $\mathcal{R}$,
$$G_{\mathcal{R}} := \{(x, y) \in X \times X \mid x\mathcal{R}y\},$$
admits a structure of groupoid over X, which is given by
$$u(x) = (x, x), \qquad s(x, y) = y, \qquad r(x, y) = x,$$
$$(x, y)^{-1} = (y, x), \qquad (x, y)\cdot(y, z) = (x, z)$$
for $x,\ y,\ z$ in X. When $x\mathcal{R}y$ for any x, y in X, $G_{\mathcal{R}} = X \times X \rightrightarrows X$ is called the *pair groupoid*.
5. If G is a group acting on a space X, the *groupoid of the action* is $G \times X \rightrightarrows X$ with the following structural homomorphisms:
$$u(x) = (e, x), \qquad s(g, x) = x, \qquad r(g, x) = g \cdot x,$$
$$(g, x)^{-1} = (g^{-1}, g \cdot x), \qquad (h, g \cdot x)\cdot(g, x) = (hg, x)$$
for x in X and $g,\ h$ in G.
6. Let X be a topological space. The *homotopy groupoid* of X is
$$\Pi(X) := \{\bar{c} \mid c : [0, 1] \to X \text{ a continuous path}\} \rightrightarrows X,$$
where $\bar{c}$ denotes the homotopy class (with fixed endpoints) of c. We let
$$u(x) = \overline{c_x},$$
where c_x is the constant path equal to x,
$$s(\overline{c}) = c(0), \qquad r(\overline{c}) = c(1), \qquad \overline{c}^{-1} = \overline{c^{-1}},$$

where $c^{-1}(t) = c(1-t)$,

$$\overline{c_1} \cdot \overline{c_2} = \overline{c_1 \cdot c_2},$$

where $c_1 \cdot c_2(t) = c_2(2t)$ for $t \in [0, \frac{1}{2}]$ and $c_1 \cdot c_2(t) = c_1(2t-1)$ for $t \in [\frac{1}{2}, 1]$. When X is a smooth manifold of dimension n, $\Pi(X)$ is naturally endowed with a smooth structure (of dimension $2n$). A neighborhood of $\bar{c}$ is of the form $\{\bar{c_1}\bar{c}\bar{c_0} \mid c_1(0) = c(1),\ c(0) = c_0(1),\ \mathrm{Im}\ c_i \subset U_i,\ i = 0, 1\}$ where U_i is a given neighborhood of $c(i)$ in X.

3.2.2 Homomorphisms and Morita equivalences

3.2.2.1 Homomorphisms

Let $G \rightrightarrows G^{(0)}$ be a groupoid of source s_G and range r_G, and $H \rightrightarrows H^{(0)}$ be a groupoid of source s_H and range r_H. A groupoid *homomorphism* from G to H is given by two maps:

$$f : G \to H \quad \text{and} \quad f^{(0)} : G^{(0)} \to H^{(0)}$$

such that

- $r_H \circ f = f^{(0)} \circ r_G$,
- $f(\gamma)^{-1} = f(\gamma^{-1})$ for any $\gamma \in G$,
- $f(\gamma_1 \cdot \gamma_2) = f(\gamma_1) \cdot f(\gamma_2)$ for $\gamma_1,\ \gamma_2$ in G such that $s_G(\gamma_1) = r_G(\gamma_2)$.

We say that f is a *homomorphism over* $f^{(0)}$. When $G^{(0)} = H^{(0)}$ and $f^{(0)} = \mathrm{Id}$, we say that f is a *homomorphism over the identity*.

The homomorphism f is an *isomorphism* when the maps f, $f^{(0)}$ are bijections and $f^{-1} : H \to G$ is a homomorphism over $(f^{(0)})^{-1}$.

As usual, when dealing with topological groupoids we require that f be continuous and, when dealing with smooth groupoids, that f be smooth.

3.2.2.2 Morita equivalence

In most situations, the right notion of isomorphism of locally compact groupoids is the weaker notion of Morita equivalence.

Definition 3.2.2 Two locally compact groupoids $G \rightrightarrows G^{(0)}$ and $H \rightrightarrows H^{(0)}$ are *Morita equivalent* if there exists a locally compact groupoid $P \rightrightarrows G^{(0)} \sqcup H^{(0)}$ such that

- the restrictions of P over $G^{(0)}$ and $H^{(0)}$ are respectively G and H:

$$P_{G^{(0)}}^{G^{(0)}} = G \quad \text{and} \quad P_{H^{(0)}}^{H^{(0)}} = H;$$

- for any $\gamma \in P$ there exists η in $P_{G^{(0)}}^{H^{(0)}} \cup P_{H^{(0)}}^{G^{(0)}}$ such that (γ, η) is a composable pair (i.e., $s(\gamma) = r(\eta)$).

Examples

1. Let $f : G \to H$ be an isomorphism of locally compact groupoid. Then the following groupoid defines a Morita equivalence between H and G:

$$P = G \sqcup \tilde{G} \sqcup \tilde{G}^{-1} \sqcup H \rightrightarrows G^{(0)} \sqcup H^{(0)},$$

where with the obvious notation we have

$$G = \tilde{G} = \tilde{G}^{-1},$$

$$s_P = \begin{cases} s_G & \text{on } G, \\ s_H \circ f & \text{on } \tilde{G}, \\ r_G & \text{on } \tilde{G}^{-1}, \\ s_H & \text{on } H, \end{cases} \qquad r_P = \begin{cases} r_G & \text{on } G \sqcup \tilde{G}, \\ s_H \circ f & \text{on } \tilde{G}^{-1}, \\ r_H & \text{on } H, \end{cases}$$

$$u_P = \begin{cases} u_G & \text{on } G^{(0)}, \\ u_H & \text{on } H^{(0)}, \end{cases} \qquad i_P(\gamma) = \begin{cases} i_G(\gamma) & \text{on } G, \\ i_H(\gamma) & \text{on } H, \\ \gamma \in \tilde{G}^{-1} & \text{on } \tilde{G}, \\ \gamma \in \tilde{G} & \text{on } \tilde{G}^{-1}, \end{cases}$$

$$p_P(\gamma_1, \gamma_2) = \begin{cases} p_G(\gamma_1, \gamma_2) & \text{on } G^{(2)}, \\ p_H(\gamma_1, \gamma_2) & \text{on } H^{(2)}, \\ p_G(\gamma_1, \gamma_2) \in \tilde{G} & \text{for } \gamma_1 \in G,\ \gamma_2 \in \tilde{G}, \\ p_G(\gamma_1, f^{-1}(\gamma_2)) \in \tilde{G} & \text{for } \gamma_1 \in \tilde{G},\ \gamma_2 \in H, \\ p_G(\gamma_1, \gamma_2) \in G & \text{for } \gamma_1 \in \tilde{G},\ \gamma_2 \in \tilde{G}^{-1}, \\ f \circ p_G(\gamma_1, \gamma_2) \in H & \text{for } \gamma_1 \in \tilde{G},\ \gamma_2 \in \tilde{G}^{-1}. \end{cases}$$

2. Suppose that $G \rightrightarrows G^{(0)}$ is a locally compact groupoid and $\varphi : X \to G^{(0)}$ is an open surjective map, where X is a locally compact space. The *pullback groupoid* is the groupoid

$${}^*\varphi^*(G) \rightrightarrows X,$$

where

$${}^*\varphi^*(G) = \{(x, \gamma, y) \in X \times G \times X \mid \varphi(x) = r(\gamma) \text{ and } \varphi(y) = s(\gamma)\}$$

with $s(x, \gamma, y) = y$, $r(x, \gamma, y) = x$, $(x, \gamma_1, y) \cdot (y, \gamma_2, z) = (x, \gamma_1 \cdot \gamma_2, z)$ and $(x, \gamma, y)^{-1} = (y, \gamma^{-1}, x)$. One can show that this endows ${}^*\varphi^*(G)$ with a locally compact groupoid structure. Moreover, the groupoids G and ${}^*\varphi^*(G)$ are Morita equivalent, but not isomorphic in general. To prove this last point, one can put a locally compact groupoid structure on $P = G \sqcup X \times_r G \sqcup G \times_s X \sqcup {}^*\varphi^*(G)$ over $X \sqcup G^{(0)}$, where $X \times_r G = \{(x, \gamma) \in X \times G \mid \varphi(x) = r(\gamma)\}$ and $G \times_s X = \{(\gamma, x) \in G \times X \mid \varphi(x) = s(\gamma)\}$.

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3.2.3 The orbits of a groupoid

Suppose that $G \rightrightarrows G^{(0)}$ is a groupoid of source s and range r.

Definition 3.2.3 The *orbit* of G passing trough x is the following subset of $G^{(0)}$:

$$Or_x = r(G_x) = s(G^x).$$

We let $G^{(0)}/G$ or $Or(G)$ be the *space of orbits*.

The *isotropy group* of G at x is G_x^x, which is naturally endowed with a group structure with x as unit. Notice that multiplication induces a free left (right) action of G_x^x on G^x (G_x). Moreover, the orbits space of this action is precisely Or_x and the restriction $s : G^x \to Or_x$ is the quotient map.

Examples and remarks

1. In example 4 in Section 3.2.1, the orbits of $G_{\mathcal{R}}$ correspond exactly to the orbits of the equivalence relation $\mathcal{R}$. In example 5, the orbits of the groupoid of the action are the orbits of the action.
2. The second assertion in the definition of Morita equivalence precisely means that both $G^{(0)}$ and $H^{(0)}$ meet all the orbits of P. Moreover, one can show that the map

$$Or(G) \to Or(H),$$
$$Or(G)_x \mapsto Or(P)_x \cap H^{(0)}$$

is a bijection. In other word, when two groupoids are Morita equivalent, they have the same orbit space.

Groupoids are often used in noncommutative geometry for the study of singular geometrical situations. In many geometrical situations, the topological space which arises is strongly non-Hausdorff, and the standard tools do not apply. Nevertheless, it is sometimes possible to associate to such a space X a relevant C^*-algebra as a substitute for $C_0(X)$. Usually we first associate a groupoid $G \rightrightarrows G^{(0)}$ such that its space of orbits $G^{(0)}/G$ is (equivalent to) X. If the groupoid is regular enough (smooth, for example), then we can associate natural C^*-algebras to G. This point will be discussed later. In other words, we desingularize a singular space by viewing it as coming from the action of a nice groupoid on its space of units. To illustrate this point let us consider two examples.

3.2.4 Groupoids associated to a foliation

Let M be a smooth manifold.

Definition 3.2.4 A (regular) smooth *foliation* $\mathcal{F}$ on M of dimension p is a partition $\{F_i\}_I$ of M where each F_i is an immersed submanifold of dimension p called a

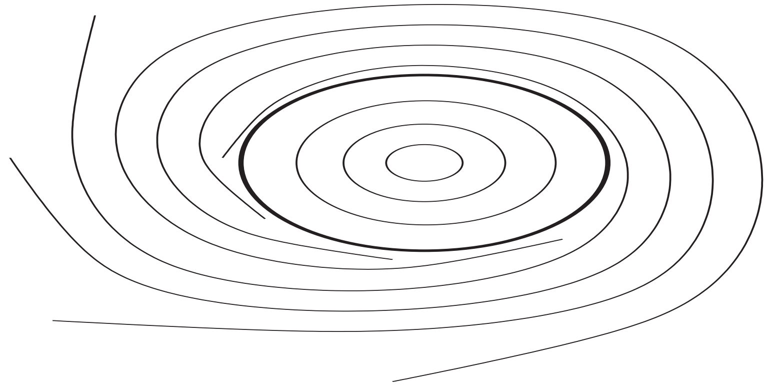

Fig. 3.1.

leaf. Moreover, the manifold M admits charts of the following type:

$$\varphi : U \to \mathbb{R}^p \times \mathbb{R}^q,$$

where U is open in M and such that for any connected component P of $F_i \cap U$ where $i \in I$, there is a $t \in \mathbb{R}^q$ such that $\varphi(P) = \mathbb{R}^p \times \{t\}$.

In this situation the *tangent space to the foliation*, $T\mathcal{F} := \bigcup_I TF_i$, is a subbundle of TM stable under Lie bracket.

The *space of leaves* $M/\mathcal{F}$ is the quotient of M by the equivalence relation of being on the same leaf.

A typical example: Take $M = P \times T$, where P and T are connected smooth manifolds with the partition into leaves given by $\{P \times \{t\}\}_{t\in T}$. Every foliation is locally of this type.

The space of leaves of a foliation is often difficult to study, as appears in the following two examples:

Examples

1. Let $\tilde{\mathcal{F}}_a$ be the foliation on the plane $\mathbb{R}^2$ by lines $\{y = ax + t\}_{t\in\mathbb{R}}$ where a belongs to $\mathbb{R}$. Take the torus $T = \mathbb{R}^2/\mathbb{Z}^2$ to be the quotient of $\mathbb{R}^2$ by translations of $\mathbb{Z}^2$. We denote by $\mathcal{F}_a$ the foliation induced by $\tilde{\mathcal{F}}_a$ on T. When a is rational the space of leaves is a circle, but when a is irrational it is topologically equivalent to a point (i.e., each point is in any neighborhood of any other point).
2. Let $\mathbb{C} \setminus \{(0)\}$ be foliated by

$$\{S_t\}_{t\in]0,1]} \cup \{D_t\}_{t\in]0,2\pi]},$$

where $S_t = \{z \in \mathbb{C} \mid |z| = t\}$ is the circle of radius t, and $D_t = \{z = e^{i(x+t)+x} \mid x \in \mathbb{R}^+_*\}$. (See Figure 3.1.)

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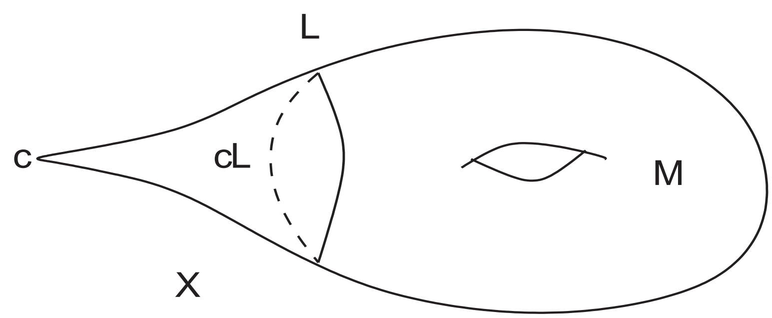

Fig. 3.2.

A *holonomy groupoid* is a smooth groupoid which desingularizes the space of leaves of a foliation. Precisely, if $\mathcal{F}$ is a smooth foliation on a manifold M, its *holonomy groupoid* is the smallest s-connected smooth groupoid $G \rightrightarrows M$ whose orbits are precisely the leaves of the foliation. Here, *smallest* means that if $H \rightrightarrows M$ is another s-connected smooth groupoid whose orbits are the leaves of the foliation, then there is a surjective groupoid homomorphism : $H \to G$ over identity.

The first naive attempt to define such a groupoid is to consider the graph of the equivalence relation defined by being on the same leaf. This does not work: you get a groupoid, but it may not be smooth. This fact can be observed in the preceding example 2. Another idea consists in looking at the *homotopy groupoid*. Let $\Pi(\mathcal{F})$ be the set of homotopy classes of smooth paths lying on leaves of the foliation. It is naturally endowed with a groupoid structure similarly to the homotopy groupoid of Section 3.2.1, example 6. Such a groupoid can be naturally equipped with a smooth structure (of dimension $2p + q$), and the holonomy groupoid is a quotient of this homotopy groupoid. In particular, when the leaves have no homotopy, the holonomy groupoid is the graph of the equivalence relation of being in the same leaf.

3.2.5 The noncommutative tangent space of a conical pseudomanifold

It may happen that the underlying topological space which is under study is a nice compact space which is "almost" smooth. This is the case of pseudomanifolds [24, 36, 53]; for a review on the subject see [9, 28]. In such a situation we can desingularize the tangent space [18, 19]. Let us see how this works in the case of a conical pseudomanifold with one singularity.

Let M be an m-dimensional compact manifold with a compact boundary L. We attach to L the cone $cL = L \times [0, 1]/L \times \{0\}$, using the obvious map $L \times \{1\} \to L \subset \partial M$. The new space $X = cL \cup M$ (see Figure 3.2) is a compact pseudomanifold with a singularity [24]. In general, there is no manifold structure around the vertex c of the cone.

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We will use the following notation: $X^\circ = X \setminus \{c\}$ is the *regular part*, X^+ denotes $M \setminus L = X \setminus cL$, $\overline{X_+} = M$ denotes its closure in X, and $X^- = L \times]0, 1[$. If y is a point of the cylindrical part of $X \setminus \{c\}$, we write $y = (y_L, k_y)$, where $y_L \in L$ and $k_y \in\ ]0, 1]$ are the tangential and radial coordinates. The map $y \to k_y$ is extended into a smooth defining function for the boundary of M. In particular, $k^{-1}(1) = L = \partial M$ and $k(M) \subset [1, +\infty[$.

Let us consider $T\overline{X^+}$, the restriction to $\overline{X^+}$ of the tangent bundle of X°. As a $\mathcal{C}^\infty$ vector bundle, it is a smooth groupoid with unit space $\overline{X^+}$. We define the groupoid $T^{\mathsf{S}}X$ as the disjoint union

$$T^{\mathsf{S}}X = X^- \times X^- \ \cup\ T\overline{X^+} \underset{r}{\overset{s}{\rightrightarrows}} X^\circ,$$

where $X^- \times X^- \rightrightarrows X^-$ is the pair groupoid.

In order to endow $T^{\mathsf{S}}X$ with a smooth structure, compatible with the usual smooth structure on $X^- \times X^-$ and on $T\overline{X^+}$, we have to take care of what happens around points of $T\overline{X^+}|_{\partial\overline{X^+}}$.

Let τ be a smooth positive function on $\mathbb{R}$ such that $\tau^{-1}(\{0\}) = [1, +\infty[$. We let $\tilde{\tau}$ be the smooth map from X° to $\mathbb{R}^+$ given by $\tilde{\tau}(y) = \tau \circ k(y)$.

Let (U, ϕ) be a local chart for X° around $z \in \partial\overline{X^+}$. Setting $U^- = U \cap X^-$ and $\overline{U^+} = U \cap \overline{X^+}$, we define a local chart of $T^{\mathsf{S}}X$ by

$$\tilde{\phi} : U^- \times U^- \cup\ T\overline{U^+} \longrightarrow \mathbb{R}^m \times \mathbb{R}^m,$$

$$\tilde{\phi}(x, y) = \left(\phi(x), \frac{\phi(y) - \phi(x)}{\tilde{\tau}(x)}\right) \qquad \text{if } (x, y) \in U^- \times U^-, \tag{3.1}$$

$$\tilde{\phi}(x, V) = (\phi(x), (\phi)_*(x, V)) \qquad \text{elsewhere.}$$

We define in this way a smooth groupoid structure on $T^{\mathsf{S}}X$. Note that at the topological level, the space of orbits of $T^{\mathsf{S}}X$ is equivalent to X: there is a canonical isomorphism between the algebras $C(X)$ and $C(X^\circ/T^{\mathsf{S}}X)$.

The smooth groupoid $T^{\mathsf{S}}X \rightrightarrows X^\circ$ is called the *noncommutative tangent space* of X.

3.2.6 Lie theory for smooth groupoids

Let us go into the more specific world of smooth groupoids. Similarly to Lie groups which admit Lie algebras, any smooth groupoid has a *Lie algebroid* [42, 43].

Definition 3.2.5 A *Lie algebroid* $\mathcal{A} = (p : \mathcal{A} \to TM, [\ ,\ ]_{\mathcal{A}})$ on a smooth manifold M is a vector bundle $\mathcal{A} \to M$ equipped with a bracket $[\ ,\ ]_{\mathcal{A}} : \Gamma(\mathcal{A}) \times \Gamma(\mathcal{A}) \to \Gamma(\mathcal{A})$ on the module of sections of $\mathcal{A}$ together with a homomorphism of fiber

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bundle $p : \mathcal{A} \to TM$ from $\mathcal{A}$ to the tangent bundle TM of M called the *anchor*, such that:

(i) the bracket $[\ ,\ ]_{\mathcal{A}}$ is $\mathbb{R}$-bilinear, is antisymmetric and satisfies the Jacobi identity,
(ii) $[X, fY]_{\mathcal{A}} = f[X, Y]_{\mathcal{A}} + p(X)(f)Y$ for all $X,\ Y \in \Gamma(\mathcal{A})$ and f a smooth function of M,
(iii) $p([X, Y]_{\mathcal{A}}) = [p(X), p(Y)]$ for all $X,\ Y \in \Gamma(\mathcal{A})$.

Each Lie groupoid admits a Lie algebroid. Let us recall this construction.

Let $G \underset{r}{\overset{s}{\rightrightarrows}} G^{(0)}$ be a Lie groupoid. We denote by T^sG the subbundle of TG of s-vertical tangent vectors. In other words, T^sG is the kernel of the differential Ts of s.

For any γ in G let $R_\gamma : G_{r(\gamma)} \to G_{s(\gamma)}$ be the right multiplication by γ. A tangent vector field Z on G is *right invariant* if it satisfies:

- Z is s-vertical: $Ts(Z) = 0$.
- For all (γ_1, γ_2) in $G^{(2)}$, $Z(\gamma_1 \cdot \gamma_2) = TR_{\gamma_2}(Z(\gamma_1))$.

Note that if Z is a right invariant vector field and h^t its flow, then for any t, the local diffeomorphism h^t is a *local left translation* of G, that is, $h^t(\gamma_1 \cdot \gamma_2) = h^t(\gamma_1) \cdot \gamma_2$ when it makes sense.

The Lie algebroid $\mathcal{A}G$ of G is defined as follows:

- The fiber bundle $\mathcal{A}G \to G^{(0)}$ is the restriction of T^sG to $G^{(0)}$. In other words: $\mathcal{A}G = \bigcup_{x \in G^{(0)}} T_x G_x$ is the union of the tangent spaces to the s-fiber at the corresponding unit.
- The *anchor* $p : \mathcal{A}G \to TG^{(0)}$ is the restriction of the differential Tr of r to $\mathcal{A}G$.
- If $Y : U \to \mathcal{A}G$ is a local section of $\mathcal{A}G$, where U is an open subset of $G^{(0)}$, we define the local *right invariant vector field* Z_Y *associated* with Y by

$$Z_Y(\gamma) = TR_\gamma(Y(r(\gamma))) \quad \text{for all} \quad \gamma \in G^U.$$

The Lie bracket is then defined by

$$\begin{aligned}[\ ,\ ] : \Gamma(\mathcal{A}G) \times \Gamma(\mathcal{A}G) &\longrightarrow \Gamma(\mathcal{A}G),\\ (Y_1, Y_2) &\mapsto [Z_{Y_1}, Z_{Y_2}]_{G^{(0)}},\end{aligned}$$

where $[Z_{Y_1}, Z_{Y_2}]$ denotes the s-vertical vector field obtained with the usual bracket, and $[Z_{Y_1}, Z_{Y_2}]_{G^{(0)}}$ is the restriction of $[Z_{Y_1}, Z_{Y_2}]$ to $G^{(0)}$.

Example If $\Pi(\mathcal{F})$ is the homotopy groupoid (or the holonomy groupoid) of a smooth foliation, its Lie algebroid is the tangent space $T\mathcal{F}$ to the foliation. The anchor is the inclusion. In particular, the Lie algebroid of the pair groupoid $M \times M$ on a smooth manifold M is TM, the anchor being the identity map.

Lie theory for groupoids is much trickier than for groups. For a long time people thought that, as for Lie algebras, every Lie algebroid integrates into a Lie groupoid [44]. In fact this assertion, named *Lie's third theorem for Lie algebroids*, is false. This was pointed out by a counterexample given by Almeida and Molino in [1]. Since then, a lot of work has been done around this problem. A few years ago Crainic and Fernandes [15] completely solved it by giving a necessary and sufficient condition for the integrability of Lie algebroids.

3.2.7 Examples of groupoids involved in index theory

Index theory is a part of noncommutative geometry where groupoids may play a crucial role. Index theory will be discussed later in this chapter, but we want to present here some of the groupoids which will arise.

Definition 3.2.6 A smooth groupoid G is called a *deformation groupoid* if

$$G = G_1 \times \{0\} \cup G_2 \times]0,1] \rightrightarrows G^{(0)} = M \times [0,1],$$

where G_1 and G_2 are smooth groupoids with unit space M. That is, G is obtained by gluing $G_2 \times]0,1] \rightrightarrows M \times]0,1]$, which is the groupoid G_2 parametrized by $]0,1]$, with the groupoid $G_1 \times \{0\} \rightrightarrows M \times \{0\}$.

Example Let G be a smooth groupoid, and let $\mathcal{A}G$ be its Lie algebroid. The *adiabatic groupoid* of G [13, 38, 39] is a deformation of G on its Lie algebroid:

$$G_{ad} = \mathcal{A}G \times \{0\} \cup G \times]0,1] \rightrightarrows G^{(0)} \times [0,1],$$

where one can put a natural smooth structure on G_{ad}. Here, the vector bundle $\pi : \mathcal{A}G \to G^{(0)}$ is considered as a groupoid in the obvious way.

3.2.7.1 The tangent groupoid

A special example of adiabatic groupoid is the *tangent groupoid* of Connes [13]. Consider the pair groupoid $M \times M$ on a smooth manifold M. We saw that its Lie algebroid is TM. In this situation, the adiabatic groupoid is called the *tangent groupoid* and is given by

$$\mathcal{G}_M^t := TM \times \{0\} \sqcup M \times M \times]0,1] \rightrightarrows M \times [0,1].$$

The Lie algebroid is the bundle $\mathcal{A}(\mathcal{G}_M^t) := TM \times [0,1] \to M \times [0,1]$ with anchor $p : (x, V, t) \in TM \times [0,1] \mapsto (x, tV, t, 0) \in TM \times T[0,1]$.

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Choose a Riemannian metric on M. The smooth structure on $\mathcal{G}^t_M$ is such that the map

$$\mathcal{U} \subset TM \times [0,1] \to \mathcal{G}^t_M,$$
$$(x, V, t) \mapsto \begin{cases} (x, V, 0) & \text{if } t = 0, \\ (x, \exp_x(-tV), t) & \text{elsewhere} \end{cases}$$

is a smooth diffeomorphism on its range, where $\mathcal{U}$ is an open neighborhood of $TM \times \{0\}$.

The previous construction of the tangent groupoid of a compact manifold generalizes to the case of conical manifold. When X is a conical manifold, its tangent groupoid is a deformation of the pair groupoid over X° into the groupoid $T^{\mathsf{S}}X$. This deformation has a nice description at the level of Lie algebroids. Indeed, with the notation of Definition 3.2.5, the Lie algebroid of $\mathcal{G}^t_X$ is the (unique) Lie algebroid given by the fiber bundle $\mathcal{A}\mathcal{G}^t_X = [0,1] \times \mathcal{A}(T^{\mathsf{S}}X) = [0,1] \times TX^\circ \to [0,1] \times X^\circ$, with anchor map

$$p_{\mathcal{G}^t_X} : \mathcal{A}\mathcal{G}^t_X = [0,1] \times TX^\circ \longrightarrow T([0,1] \times X^\circ) = T[0,1] \times TX^\circ,$$
$$(\lambda, x, V) \ \mapsto \ (\lambda, 0, x, (\tilde{\tau}(x) + \lambda)V).$$

Such a Lie algebroid is almost injective; thus it is integrable [15, 17]. Moreover, it integrates into the *tangent groupoid*, which is defined by

$$\mathcal{G}^t_X = X^\circ \times X^\circ \times]0,1] \ \cup \ T^{\mathsf{S}}X \times \{0\} \rightrightarrows X^\circ \times [0,1].$$

Once again one can equip such a groupoid with a smooth structure compatible with the usual one on each piece: $X^\circ \times X^\circ \times]0,1]$ and $T^{\mathsf{S}}X \times \{0\}$ [19].

3.2.7.2 The Thom groupoid

Another important deformation groupoid for our purpose is the *Thom groupoid* [20].

Let $\pi : E \to X$ be a *conical vector* bundle. This means that X is a conical manifold (or a smooth manifold without vertices) and we have a smooth vector bundle $\pi^\circ : E^\circ \to X^\circ$ whose restriction to $X^- = L \times]0,1[$ is equal to $E_L \times]0,1[$, where $E_L \to L$ is a smooth vector bundle. If $E^+ \to X^+$ denotes the bundle E° restricted to X^+, then E is the conical manifold $E = cE_L \cup E^+$.

When X is a smooth manifold (with no conical point), this boils down to the usual notion of smooth vector bundle.

From the definition, π restricts to a smooth vector bundle map $\pi^\circ : E^\circ \to X^\circ$. We let $\pi_{[0,1]} = \pi^\circ \times \mathrm{Id} : E^\circ \times [0,1] \to X^\circ \times [0,1]$.

We consider the tangent groupoids $\mathcal{G}^t_X \rightrightarrows X^\circ \times [0,1]$ for X and $\mathcal{G}^t_E \rightrightarrows E^\circ \times [0,1]$ for E, equipped with a smooth structure constructed using the same gluing function τ (in particular $\tilde{\tau_X} \circ \pi = \tilde{\tau_E}$). We denote by ${}^*\pi^*_{[0,1]}(\mathcal{G}^t_X) \rightrightarrows E^\circ \times [0,1]$ the pullback of $\mathcal{G}^t_X$ by $\pi_{[0,1]}$.

We first associate to the conical vector bundle E a deformation groupoid $\mathcal{T}^t_E$ from ${}^*\pi^*_{[0,1]}(\mathcal{G}^t_X)$ to $\mathcal{G}^t_E$. More precisely, we define

$$\mathcal{T}^t_E := \mathcal{G}^t_E \times \{0\} \sqcup {}^*\pi^*_{[0,1]}(\mathcal{G}^t_X) \times]0,1] \rightrightarrows E^\circ \times [0,1] \times [0,1].$$

Once again, one can equip $\mathcal{T}^t_E$ with a smooth structure [20], and the restriction of $\mathcal{T}^t_E$ to $E^\circ \times \{0\} \times [0,1]$ leads to a smooth groupoid,

$$\mathcal{H}_E = T^{\mathsf{S}}E \times \{0\} \sqcup {}^*\pi^*(T^{\mathsf{S}}X) \times]0,1] \rightrightarrows E^\circ \times [0,1],$$

called a *Thom groupoid* associated to the conical vector bundle E over X.

The following example explains what these constructions become if there is no singularity.

Example Suppose that $p : E \to M$ is a smooth vector bundle over the smooth manifold M. Then we have the usual tangent groupoids $\mathcal{G}^t_E = TE \times \{0\} \sqcup E \times E \times]0,1] \rightrightarrows E \times [0,1]$ and $\mathcal{G}^t_M = TM \times \{0\} \sqcup M \times M \times]0,1] \rightrightarrows M \times [0,1]$. In this example the groupoid $\mathcal{T}^t_E$ will be given by

$$\begin{aligned}\mathcal{T}^t_E &= TE \times \{0\} \times \{0\} \sqcup {}^*p^*(TM) \times \{0\} \times]0,1] \sqcup E \times E \times]0,1] \times [0,1] \\ &\rightrightarrows E \times [0,1] \times [0,1]\end{aligned}$$

and is smooth. Similarly, the Thom groupoid will be given by $\mathcal{H}_E := TE \times \{0\} \sqcup {}^*p^*(TM) \times]0,1] \rightrightarrows E \times [0,1]$.

3.2.8 Haar systems

A locally compact groupoid $G \rightrightarrows G^{(0)}$ can be viewed as a family of locally compact spaces

$$G_x = \{\gamma \in G \mid s(\gamma) = x\}$$

parametrized by $x \in G^{(0)}$. Moreover, right translations act on these spaces. Precisely, to any $\gamma \in G$ one associates the homeomorphism

$$\begin{aligned} R_\gamma : G_y &\to G_x, \\ \eta &\mapsto \eta \cdot \gamma. \end{aligned}$$

This picture enables us to define the right analogue of Haar measure on locally compact groups to locally compact groupoids, namely *Haar systems*. The following definition is due to Renault [46].

Definition 3.2.7 A *Haar system* on G is a collection $\nu = \{\nu_x\}_{x \in G^{(0)}}$ of positive regular Borel measure on G satisfying the following conditions:

(i) *Support:* For every $x \in G^{(0)}$, the support of ν_x is contained in G_x.
(ii) *Invariance:* For any $\gamma \in G$, the right-translation operator $R_\gamma : G_y \to G_x$ is measure-preserving. That is, for all $f \in C_c(G)$,

$$\int f(\eta) d\nu_y(\eta) = \int f(\eta \cdot \gamma) d\nu_x(\eta).$$

(iii) *Continuity:* For all $f \in C_c(G)$, the map

$$G^{(0)} \to \mathbb{C}, \quad x \mapsto \int f(\gamma) d\nu_x(\gamma)$$

is continuous.

In contrast to the case of locally compact groups, Haar systems on groupoids may not exist. Moreover, when such a Haar system exists, it may not be unique. In the special case of a smooth groupoid, a Haar system always exists [40, 45], and any two Haar systems $\{\nu_x\}$ and $\{\mu_x\}$ differ by a continuous and positive function f on $G^{(0)}$: $\nu_x = f(x)\mu_x$ for all $x \in G^{(0)}$.

Example When the source and range maps are local homeomorphisms, a possible choice for ν_x is the counting measure on G_x.

3.3 C^*-algebras of groupoids

This second part starts with the definition of a C^*-algebra together with some results. Then we construct the maximal and minimal C^*-algebras associated to a groupoid and compute explicit examples.

3.3.1 C^*-algebras – Basic definitions

In this subsection we introduce the terminology and give some examples and properties of C^*-algebras. We refer the reader to [3, 21, 41] for a more complete overview on this subject.

Definition 3.3.1 A *C^*-algebra A* is a complex Banach algebra with an involution $x \mapsto x^*$ such that:

(i) $(\lambda x + \mu y)^* = \bar{\lambda} x^* + \bar{\mu} y^*$ for $\lambda, \mu \in \mathbb{C}$ and $x, y \in A$,
(ii) $(xy)^* = y^* x^*$ for $x, y \in A$, and
(iii) $\|x^* x\| = \|x\|^2$ for $x \in A$.

Note that it follows from the definition that $*$ is isometric.

The element x in A is *self-adjoint* if $x^* = x$, and *normal* if $xx^* = x^*x$. When 1 belongs to A, x is *unitary* if $xx^* = x^*x = 1$.

Given two C^*-algebras A, B, a homomorphism respecting the involution is a called a $*$-homomorphism.

Examples

1. Let $\mathcal{H}$ be a Hilbert space. The algebra $\mathcal{L}(\mathcal{H})$ of all continuous linear transformations of $\mathcal{H}$ is a C^*-algebra. The involution of $\mathcal{L}(\mathcal{H})$ is given by the usual adjunction of bounded linear operators.
2. Let $\mathcal{K}(\mathcal{H})$ be the norm closure of finite-rank operators on $\mathcal{H}$. It is the C^*-algebra of compact operators on $\mathcal{H}$.
3. The algebra $M_n(\mathbb{C})$ is a C^*-algebra. It is a special example of the previous kind, when $dim(\mathcal{H}) = n$.
4. Let X be a locally compact, Hausdorff, topological space. The algebra $C_0(X)$ of continuous functions vanishing at ∞, endowed with the supremum norm and the involution $f \mapsto \bar{f}$, is a commutative C^*-algebra. When X is compact, 1 belongs to $C(X) = C_0(X)$.

Conversely, Gelfand's theorem asserts that every commutative C^*-algebra A is isomorphic to $C_0(X)$ for some locally compact space X (and it is compact precisely when A is unital). Precisely, a *character* $\mathcal{X}$ of A is a continuous homomorphism of algebras $\mathcal{X} : A \to \mathbb{C}$. The set X of characters of A, called the *spectrum* of A, can be endowed with a locally compact space topology. The *Gelfand transform* $\mathcal{F} : A \to C_0(X)$ given by $\mathcal{F}(x)(\mathcal{X}) = \mathcal{X}(x)$ is the desired $*$-isomorphism.

Let A be a C^*-algebra and $\mathcal{H}$ a Hilbert space.

Definition 3.3.2 A *$*$-representation* of A in $\mathcal{H}$ is a $*$-homomorphism $\pi : A \to \mathcal{L}(\mathcal{H})$. The representation is *faithful* if π is injective.

Theorem 3.3.3 *(Gelfand–Naimark) If A is a C^*-algebra, there exists a Hilbert space $\mathcal{H}$ and a faithful representation $\pi : A \to \mathcal{L}(\mathcal{H})$.*

In other words, any C^*-algebra is isomorphic to a norm-closed involutive subalgebra of $\mathcal{L}(\mathcal{H})$. Moreover, when A is separable, $\mathcal{H}$ can be taken to be the (unique up to isometry) separable Hilbert space of infinite dimension.

3.3.1.1 Enveloping algebra

Given a Banach $*$-algebra A, consider the family π_α of all continuous $*$-representations for A. The Hausdorff completion of A for the seminorm $\|x\| = \sup_\alpha(\|\pi_\alpha(x)\|)$ is a C^*-algebra called the *enveloping C^*-algebra* of A.

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3.3.1.2 Units

A C^*-algebra may or may not have a unit, but it can always be embedded into a unital C^*-algebra $\tilde{A}$:

$$\tilde{A} := \{x + \lambda \mid x \in A,\ \lambda \in \mathbb{C}\}$$

with the obvious product and involution. The norm on $\tilde{A}$ is given for all $x \in \tilde{A}$ by $\|x\|^{\sim} = \sup\{\|xy\|,\ y \in A;\ \|y\| = 1\}$. On A we have $\|\cdot\| = \|\cdot\|^{\sim}$. The algebra A is a closed two-sided ideal in $\tilde{A}$ and $\tilde{A}/A = \mathbb{C}$.

3.3.1.3 Functional calculus

Let A be a C^*-algebra. If x belongs to A, the *spectrum* of x in A is the compact set

$$Sp(x) = \{\lambda \in \mathbb{C} \mid x - \lambda \text{ is not invertible in } \tilde{A}\}$$

The *spectral radius* of X is the number

$$\nu(x) = \sup\{|\lambda|;\ \lambda \in Sp(x)\}.$$

We have

$$\begin{aligned}
&Sp(x) \subset \mathbb{R} && \text{when } x \text{ is self-adjoint } (x^* = x),\\
&Sp(x) \subset \mathbb{R}_+ && \text{when } x \text{ is positive } (x = y^*y \text{ with } y \in A),\\
&Sp(x) \subset U(1) && \text{when } x \text{ is unitary } (x^*x = xx^* = 1).
\end{aligned}$$

When x is *normal* ($x^*x = xx^*$), these conditions on the spectrum are equivalent.

When x is normal, $\nu(x) = \|x\|$. From these, one infers that for any polynomial $P \in \mathbb{C}[x]$ one has $\|P(x)\| = \sup\{P(t) \mid t \in Sp(x)\}$ (using that $Sp(P(x)) = P(Sp(x))$). We can then define $f(x) \in A$ for every continuous function $f : Sp(x) \to \mathbb{C}$. Precisely, according to Weierstrass' theorem, there is a sequence (P_n) of polynomials which converges uniformly to f on $Sp(x)$. We simply define $f(x) = \lim P_n(x)$.

3.3.2 The reduced and maximal C^*-algebras of a groupoid

We restrict our study to the case of Hausdorff locally compact groupoids. For the non-Hausdorff case (which is also important and not exceptional), in particular when dealing with foliations, we refer the reader to [11, 13, 32].

From now on, $G \rightrightarrows G^{(0)}$ is a locally compact Hausdorff groupoid equipped with a fixed Haar system $\nu = \{\nu_x\}_{x \in G^{(0)}}$. We let $C_c(G)$ be the space of complex-valued functions with compact support on G. It is provided with a structure of involutive

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algebra as follows. If f and g belong to $C_c(G)$, we define the *involution* by

$$\text{for } \gamma \in G, \qquad f^*(\gamma) = \overline{f(\gamma^{-1})},$$

and the *convolution product* by

$$\text{for } \gamma \in G, \qquad f * g(\gamma) = \int_{\eta \in G_x} f(\gamma\eta^{-1}) g(\eta) d\nu_x(\eta),$$

where $x = s(\gamma)$. The 1-norm on $C_c(G)$ is defined by

$$\|f\|_1 = \sup_{x \in G^{(0)}} \max\left(\int_{G_x} |f(\gamma)| d\nu_x(\gamma), \int_{G_x} |f(\gamma^{-1})| d\nu_x(\gamma) \right).$$

The *groupoid full* C^*-algebra $C^*(G, \nu)$ is defined to be the enveloping C^*-algebra of the Banach $*$-algebra $\overline{C_c(G)}^{\|\cdot\|_1}$ obtained by completion of $C_c(G)$ with respect to the norm $\|\cdot\|_1$.

Given x in $G^{(0)}$, f in $C_c(G)$, ξ in $L^2(G_x, \nu_x)$, and γ in G_x, we set

$$\pi_x(f)(\xi)(\gamma) = \int_{\eta \in G_x} f(\gamma\eta^{-1}) \xi(\eta) d\nu_x(\eta).$$

One can show that π_x defines a $*$-representation of $C_c(G)$ on the Hilbert space $L^2(G_x, \nu_x)$. Moreover, for any $f \in C_c(G)$, the inequality $\|\pi_x(f)\| \leq \|f\|_1$ holds. The *reduced norm* on $C_c(G)$ is

$$\|f\|_r = \sup_{x \in G^{(0)}} \{\|\pi_x(f)\|\}$$

which defines a C^*-norm. The *reduced* C^*-algebra $C_r(G, \nu)$ is defined to be the C^*-algebra obtained by completion of A with respect to $\|\cdot\|_r$.

When G is smooth, the reduced and maximal C^*-algebras of the groupoid G do not depend up to isomorphism on the choice of the Haar system ν. In the general case they do not depend on ν up to Morita equivalence [46]. When there is no ambiguity on the Haar system, we write $C^*(G)$ and $C_r^*(G)$ for the maximal and reduced C^*-algebras.

The identity map on $C_c(G)$ induces a surjective homomorphism from $C^*(G)$ to $C_r^*(G)$. Thus $C_r^*(G)$ is a quotient of $C^*(G)$.

For a quite large class of groupoids, *amenable* groupoids [2], the reduced and maximal C^*-algebras are equal. This will be the case for all the groupoids we will meet in the last part of this course devoted to index theory.

Examples

1. When $X \rightrightarrows X$ is a locally compact space, $C^*(X) = C_r^*(X) = C_0(X)$.
2. When $G \rightrightarrows e$ is a group and ν a Haar measure on G, we recover the usual notion of reduced and maximal C^*-algebras of a group.

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3. Let M be a smooth manifold, and $TM \rightrightarrows M$ the tangent bundle. Let us equip the vector bundle TM with a Euclidean structure. The Fourier transformation

$$f \in C_c(TM),\ (x,w) \in T^*M, \quad \hat{f}(x,w) = \frac{1}{(2\pi)^{n/2}} \int_{X \in T_x M} e^{-iw(X)} f(X) dX$$

   gives rise to an isomorphism between $C^*(TM) = C^*_r(TM)$ and $C_0(T^*M)$. Here, n denotes the dimension of M, and T^*M the cotangent bundle of M.
4. Let X be a locally compact space, with μ a measure on X, and consider the pair groupoid $X \times X \rightrightarrows X$. If $f,\ g$ belongs to $C_c(X \times X)$, the convolution product is given by

$$f * g(x,y) = \int_{z \in X} f(x,z) g(z,y) d\mu(z),$$

   and a representation of $C_c(X \times X)$ by

$$\pi : C_c(X \times X) \to \mathcal{L}(L^2(X,\mu)); \qquad \pi(f)(\xi)(x) = \int_{z \in X} f(x,z)\xi(z) d\mu(z)$$

   when $f \in C_c(X \times X),\ \xi \in L^2(X,\mu)$ and $x \in X$. It turns out that $C^*(X \times X) = C^*_r(X \times X) \simeq \mathcal{K}(L^2(X,\mu))$.
5. Let M be a compact smooth manifold, and $\mathcal{G}^t_M \rightrightarrows M \times [0,1]$ its tangent groupoid. In this situation $C^*(\mathcal{G}^t_M) = C^*_r(\mathcal{G}^t_M)$ is a continuous field $(A_t)_{t \in [0,1]}$ of C^*-algebras [21] with $A_0 \simeq C_0(T^*M)$ a commutative C^*-algebra, and for any $t \in]0,1]$, $A_t \simeq \mathcal{K}(L^2(M))$ [13].

In the sequel we will need the two following properties of C^*-algebras of groupoids.

1. Let G_1 and G_2 be two locally compact groupoids equipped with Haar systems, and suppose for instance that G_1 is amenable. Then according to [2], $C^*(G_1) = C^*_r(G_1)$ is *nuclear* – which implies that for any C^*-algebra B there is only one tensor product C^*-algebra $C^*(G_1) \otimes B$. The groupoid $G_1 \times G_2$ is locally compact, and

$$C^*(G_1 \times G_2) \simeq C^*(G_1) \otimes C^*(G_2) \text{ and } C^*_r(G_1 \times G_2) \simeq C^*(G_1) \otimes C^*_r(G_2).$$

2. Let $G \rightrightarrows G^{(0)}$ be a locally compact groupoid with a Haar system ν. An open subset $U \subset G^{(0)}$ is *saturated* if U is a union of orbits of G, in other words, if $U = s(r^{-1}(U)) = r(s^{-1}(U))$. The set $F = G^{(0)} \setminus U$ is then a closed saturated subset of $G^{(0)}$. The Haar system ν can be restricted to the restrictions $G|_U := G^U_U$ and $G|_F := G^F_F$, and we have the following exact sequence of C^*-algebras [27,45]:

$$0 \to C^*(G|_U) \xrightarrow{i} C^*(G) \xrightarrow{r} C^*(G|_F) \to 0$$

   where $i : C_c(G|_U) \to C_c(G)$ is the extension of functions by 0, and $r : C_c(G) \to C_c(G|_F)$ is the restriction of functions.

II. *KK*-Theory

This part on KK-theory starts with a historical introduction. In order to motivate our purpose we list most of the properties of the KK-functor. Sections 3.5 and 3.6 are devoted to a detailed description of the ingredients involved in KK-theory. In order to write this review we have made intensive use of the references [26, 48, 49, 54]. Moreover, a significant part of this chapter was written by Jorge Plazas from the lectures held in Villa de Leyva, and we would like to thank him for his great help.

3.4 Introduction to *KK*-theory

3.4.1 Historical comments

The story begins with several studies by Atiyah [4, 5].

Firstly, recall that if X is a compact space, the K-*theory* of X is constructed in the following way: let $\mathcal{E}v$ be the set of isomorphism classes of continuous vector bundles over X. Thanks to the direct sum of bundles, the set $\mathcal{E}v$ is naturally endowed with the structure of an abelian semigroup. One can then symmetrize $\mathcal{E}v$ in order to get a group; this gives the K-theory group of X:

$$K^0(X) = \{[E] - [F];\ [E], [F] \in \mathcal{E}v\}.$$

For example, the K-theory of a point is $\mathbb{Z}$, for a vector bundle on a point is just a vector space, and vector spaces are classified, up to isomorphism, by their dimension.

A first step towards KK-theory is the discovery, made by Atiyah [4] and independently by Jänich [29], that K-theory of a compact space X can be described with Fredholm operators.

When $\mathcal{H}$ is an infinite-dimensional separable Hilbert space, the set $\mathcal{F}(\mathcal{H})$ of *Fredholm operators* on $\mathcal{H}$ is the open subset of $\mathcal{L}(\mathcal{H})$ made of bounded operators T on $\mathcal{H}$ such that the dimensions of the kernel and cokernel of T are finite. The set $\mathcal{F}(\mathcal{H})$ is stable under composition. We set

$$[X, \mathcal{F}(\mathcal{H})] = \{\text{homotopy classes of continuous maps: } X \to \mathcal{F}(\mathcal{H})\}.$$

The set $[X, \mathcal{F}(\mathcal{H})]$ is naturally endowed with a semigroup structure. Atiyah and Jänich showed that $[X, \mathcal{F}(\mathcal{H})]$ is actually (a group) isomorphic to $K^0(X)$ [4]. The idea of the proof is the following. If $f : X \to \mathcal{F}(\mathcal{H})$ is a continuous map, one can choose a subspace V of $\mathcal{H}$ of finite codimension such that

$$\forall x \in X,\ V \cap \ker f_x = \{0\} \quad \text{and} \quad \bigcup_{x \in X} \mathcal{H}/f_x(V) \text{ is a vector bundle.} \qquad (3.2)$$

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Denoting by $\mathcal{H}/f(V)$ the vector bundle arising in (3.2) and by $\mathcal{H}/V$ the product bundle $X \times \mathcal{H}/V$, the Atiyah–Janich isomorphism is then given by

$$\begin{aligned} [X, \mathcal{F}(\mathcal{H})] &\to K^0(X), \\ [f] &\mapsto [\mathcal{H}/V] - [\mathcal{H}/f(V)]. \end{aligned} \tag{3.3}$$

Note that choosing V amounts to modifying f inside its homotopy class into $\tilde{f}$ (defined to be equal to f on V and to 0 on a supplement of V) such that

$$\mathrm{Ker}\tilde{f} := \bigcup_{x\in X} \mathrm{Ker}(\tilde{f}_x) \text{ and } \mathrm{CoKer}\tilde{f} := \bigcup_{x\in X} \mathcal{H}/\tilde{f}_x(\mathcal{H}) \tag{3.4}$$

are vector bundles over X. These constructions contain relevant information for the sequel: the map f arises as a *generalized Fredholm* operator on the *Hilbert* $C(X)$*-module* $C(X, \mathcal{H})$.

Later, Atiyah tried to describe the dual functor $K_0(X)$, the K-homology of X, with the help of Fredholm operators. This gave rise to Ell(X), whose cycles are triples (H, π, F) where:

- $H = H_0 \oplus H_1$ is a $\mathbb{Z}_2$ graded Hilbert space.
- $\pi : C(X) \to \mathcal{L}(H)$ is a representation by operators of degree 0, which means that

$$\pi(f) = \begin{pmatrix} \pi_0(f) & 0 \\ 0 & \pi_1(f) \end{pmatrix}.$$

- F belongs to $\mathcal{L}(H)$, is of degree 1 and thus is of the form

$$F = \begin{pmatrix} 0 & G \\ T & 0 \end{pmatrix},$$

and satisfies

$$F^2 - 1 \in \mathcal{K}(H) \quad \text{and} \quad [\pi, F] \in \mathcal{K}(H)\cdot$$

In particular, G is an inverse of T modulo compact operators.

Elliptic operators on closed manifolds produce natural examples of such cycles. Moreover, there exists a natural pairing between Ell(X) and $K^0(X)$, justifying the choice of Ell(X) as a candidate for the cycles of the K-homology of X:

$$\begin{aligned} K^0(X) \times \mathrm{Ell(X)} &\to \mathbb{Z}, \\ ([E], (H, \pi, F)) &\mapsto \mathrm{Index}(F_E), \end{aligned} \tag{3.5}$$

where $\mathrm{Index}(F_E) = \dim(\mathrm{Ker}(F_E)) - \dim(\mathrm{CoKer}(F_E))$ is the *index* of a Fredholm operator associated to a vector bundle E on X and a cycle (H, π, F), as follows. Let E' be a vector bundle on X such that $E \oplus E' \simeq \mathbb{C}^N \times X$, and let e be the projection of $\mathbb{C}^N \times X$ onto E. We can identify $C(X, \mathbb{C}^N) \underset{C(X)}{\otimes} H$ with H^N. Let $\tilde{e}$

be the image of $e \otimes 1$ under this identification. We define $F_E := \tilde{e} F^N|_{\tilde{e}(H^N)}$, where F^N is the diagonal operator with F in each diagonal entry. The operator F_E is the desired Fredholm operator on $\tilde{e}(H^N)$.

Now, we should recall that to any C^*-algebra A (actually, to any ring) is associated a group $K_0(A)$. When A is unital, it can be defined as follows:

$$K_0(A) = \{[\mathcal{E}] - [\mathcal{F}];\ [\mathcal{E}], [\mathcal{F}] \text{ are isomorphism classes of finitely generated projective } A\text{-modules}\}.$$

Recall that an A-module $\mathcal{E}$ is finitely generated and projective if there exists another A-module $\mathcal{G}$ such that $\mathcal{E} \oplus \mathcal{G} \simeq A^N$ for some integer N.

The Swan–Serre theorem asserts that for any compact space X, the category of (complex) vector bundles over X is equivalent to the category of finitely generated projective modules over $C(X)$; in particular, $K^0(X) \simeq K_0(C(X))$. This fact and the (C^*-)algebraic flavor of the preceding constructions lead to the natural attempt to generalize them for noncommutative C^*-algebras.

During 1979 and the 1980s G. Kasparov defined with great success, for any pair of C^*-algebras, a bivariant theory, the KK-theory. This theory generalizes both K-theory and K-homology and carries a product generalizing the pairing (3.5). Moreover, in many cases $KK(A, B)$ contains all the morphisms from $K_0(A)$ to $K_0(B)$. To understand this bifunctor, we will study the notions of Hilbert modules, of adjointable operators acting on them and of generalized Fredholm operators which generalize to arbitrary C^*-algebras the notions already encountered for $C(X)$. Before going to this functional-analytic part, we end this introduction by listing most of the properties of the bifunctor KK.

3.4.2 Abstract properties of $KK(A, B)$

Let A and B be two C^*-algebras. In order to simplify our presentation, we assume that A and B are separable. Here is a list of the most important properties of the KK functor:

$KK(A, B)$ is an abelian group.

Functorial properties. The functor KK is covariant in B and contravariant in A: if $f : B \to C$ and $g : A \to D$ are two homomorphisms of C^*-algebras, there exist two homomorphisms of groups,

$$f_* : KK(A, B) \to KK(A, C) \quad \text{and} \quad g^* : KK(D, B) \to KK(A, B)\cdot$$

In particular $\mathrm{Id}_* = \mathrm{Id}$ and $\mathrm{Id}^* = \mathrm{Id}$.

Each $*$-morphism $f : A \to B$ defines an element, denoted by $[f]$, in $KK(A, B)$. We set $1_A := [\mathrm{Id}_A] \in KK(A, A)$.

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Homotopy invariance. $KK(A, B)$ is homotopy invariant. Recall that the C^*-algebras A and B are *homotopic* if there exist two $*$-morphisms $f : A \to B$ and $g : B \to A$ such that $f \circ g$ is homotopic to Id_B and $g \circ f$ is homotopic to Id_A. Two homomorphisms $F, G : A \to B$ are homotopic when there exists a $*$-morphism $H : A \to C([0, 1], B)$ such that $H(a)(0) = F(a)$ and $H(a)(1) = G(a)$ for any $a \in A$.

Stability. If $\mathcal{K}$ is the algebra of compact operators on a Hilbert space, there are isomorphisms

$$KK(A, B \otimes \mathcal{K}) \simeq KK(A \otimes \mathcal{K}, B) \simeq KK(A, B).$$

More generally, the bifunctor KK is invariant under *Morita equivalence*.

Suspension. If E is a C^*-algebra, there exists a homomorphism

$$\tau_E : KK(A, B) \to KK(A \otimes E, B \otimes E)$$

which satisfies $\tau_E \circ \tau_D = \tau_{E \otimes D}$ for any C^*-algebra D.

Kasparov product. There is a well-defined bilinear coupling

$$KK(A, D) \times KK(D, B) \to KK(A, B),$$
$$(x, y) \mapsto x \otimes y,$$

called the *Kasparov product*. It is associative, covariant in B and contravariant in A: if $f : C \to A$ and $g : B \to E$ are two homomorphisms of C^*-algebras, then

$$f^*(x \otimes y) = f^*(x) \otimes y \quad \text{and} \quad g_*(x \otimes y) = x \otimes g_*(y).$$

If $g : D \to C$ is another $*$-morphism, $x \in KK(A, D)$ and $z \in KK(C, B)$, then

$$h_*(x) \otimes z = x \otimes h^*(z).$$

Moreover, the following equalities hold:

$$f^*(x) = [f] \otimes x, \qquad g_*(z) = z \otimes [g] \quad \text{and} \quad [f \circ h] = [h] \otimes [f].$$

In particular

$$x \otimes 1_D = 1_A \otimes x = x.$$

The Kasparov product behaves well with respect to suspensions. If E is a C^*-algebra,

$$\tau_E(x \otimes y) = \tau_E(x) \otimes \tau_E(y).$$

This enables us to extend the Kasparov product:

$$\underset{D}{\otimes} : KK(A, B \otimes D) \times KK(D \otimes C, E) \to KK(A \otimes C, B \otimes E),$$
$$(x, y) \mapsto x \underset{D}{\otimes} y := \tau_C(x) \otimes \tau_B(y).$$

The Kasparov product $\underset{\mathbb{C}}{\otimes}$ is commutative.

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Higher groups. For any $n \in \mathbb{N}$, let

$$KK_n(A, B) := KK(A, C_0(\mathbb{R}^n) \otimes B).$$

An alternative definition, leading to isomorphic groups, is

$$KK_n(A, B) := KK(A, C_n \otimes B),$$

where C_n is the Clifford algebra of $\mathbb{C}^n$. This will be explained later. The functor KK satisfies *Bott periodicity*: there is an isomorphism

$$KK_2(A, B) \simeq KK(A, B).$$

Exact sequences. Consider the following exact sequence of C^*-algebras:

$$0 \to J \stackrel{i}{\to} A \stackrel{p}{\to} Q \to 0,$$

and let B be another C^*-algebra. Under a few more assumptions (for example, all the C^*-algebras are nuclear or K-nuclear, or the preceding exact sequence admits a completely positive norm-decreasing cross section [50]), we have the following two periodic exact sequences:

$$\begin{array}{ccccc}
KK(B, J) & \xrightarrow{i_*} & KK(B, A) & \xrightarrow{p_*} & KK(B, Q) \\
{\scriptstyle \delta}\uparrow & & & & \downarrow{\scriptstyle \delta} \\
KK_1(B, Q) & \xleftarrow[p_*]{} & KK_1(B, A) & \xleftarrow[i_*]{} & KK_1(B, J)
\end{array}$$

$$\begin{array}{ccccc}
KK(Q, B) & \xrightarrow{p^*} & KK(A, B) & \xrightarrow{i^*} & KK(J, B) \\
{\scriptstyle \delta}\uparrow & & & & \downarrow{\scriptstyle \delta} \\
KK_1(J, B) & \xleftarrow[i^*]{} & KK_1(A, B) & \xleftarrow[p^*]{} & KK_1(Q, B)
\end{array}$$

where the connecting homomorphisms δ are given by Kasparov products.

Final remarks. Let us go back to the end of the introduction in order to make it more precise.

The usual K-theory groups appears as special cases of KK-groups:

$$KK(\mathbb{C}, B) \simeq K_0(B),$$

and the *K-homology* of a C^*-algebra A is defined by

$$K^0(A) = KK(A, \mathbb{C}).$$

Any $x \in KK(A, B)$ induces a homomorphism of groups:

$$KK(\mathbb{C}, A) \simeq K_0(A) \to K_0(B) \simeq KK(\mathbb{C}, B),$$

$$\alpha \mapsto \alpha \otimes x.$$

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In most situations, the induced homomorphism

$$KK(A, B) \to \mathrm{Mor}(K_0(A), K_0(B))$$

is surjective. Thus one can think of KK-elements as homomorphisms between K-groups.

When X is a compact space, one has $K^0(X) \simeq K_0(C(X)) \simeq KK(\mathbb{C}, C(X))$ and, as we will see shortly, $K^0(C(X)) = KK(C(X), \mathbb{C})$ is a quotient of the set $\mathrm{Ell}(X)$ introduced by Atiyah. Moreover the pairing $K^0(X) \times \mathrm{Ell}(X) \to \mathbb{Z}$ coincides with the Kasparov product $KK(\mathbb{C}, C(X)) \times KK(C(X), \mathbb{C}) \to KK(\mathbb{C}, \mathbb{C}) \simeq \mathbb{Z}$.

3.5 Hilbert modules

We review the main properties of Hilbert modules over C^*-algebras, necessary for a correct understanding of bivariant K-theory. We closely follow the presentation given by Skandalis [48]. Most of the proofs are taken from his lectures on the subject. We are indebted to him for allowing us to use his lecture notes. Some of the following material can also be found in [54], where the reader will find a guide to the literature and a more detailed presentation.

3.5.1 Basic definitions and examples

Let A be a C^*-algebra and E be a A-right module.

A sesquilinear form $(\cdot, \cdot) : E \times E \to A$ is *positive* if for all $x \in E$ one has $(x, x) \in A_+$. Here A_+ denotes the set of positive elements in A. It is *positive definite* if moreover $(x, x) = 0$ if and only if $x = 0$.

Let $(\cdot, \cdot) : E \times E \to A$ be a positive sesquilinear form, and set $Q(x) = (x, x)$. By the polarization identity

$$\forall x, y \in E, \quad (x, y) = \frac{1}{4}\left(Q(x + y) - i\,Q(x + iy) - Q(x - y) + i\,Q(x - iy)\right),$$

we get

$$\forall x, y \in E, \quad (x, y) = (y, x)^*.$$

Definition 3.5.1 A *pre-Hilbert A-module* is a right A-module E with a positive definite sesquilinear map $(\cdot, \cdot) : E \times E \to A$ such that $y \mapsto (x, y)$ is A-linear.

Proposition 3.5.2 *Let $(E, (\cdot, \cdot))$ be a pre-Hilbert A-module. Then*

$$\forall x \in E, \quad \|x\| = \sqrt{\|(x, x)\|} \tag{3.6}$$

defines a norm on E.

The only nontrivial fact is the triangle inequality, which results from

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Lemma 3.5.3 *(Cauchy–Schwarz inequality)*

$$\forall x, y \in E, \quad (x, y)^*(x, y) \leq \|x\|^2(y, y).$$

In particular, $\|(x, y)\| \leq \|x\|\|y\|$.

Set $a = (x, y)$. We have for all $t \in \mathbb{R}$ that $(xa + ty, xa + ty) \geq 0$; thus

$$2ta^*a \leq a^*(x, x)a + t^2(y, y). \tag{3.7}$$

Because $(x, x) \geq 0$, we have $a^*(x, x)a \leq \|x\|^2 a^*a$ (use the equivalence $z^*z \leq w^*w$ if and only if $\|zx\| \leq \|wx\|$ for all $x \in A$), and choosing $t = \|x\|^2$ in (3.7) gives the result.

Definition 3.5.4 A *Hilbert A-module* is a pre-Hilbert A-module which is complete for the norm defined in (3.6).

A *Hilbert A-submodule* of a Hilbert A-module is a closed A-submodule provided with the restriction of the A-valued scalar product.

When there is no ambiguity about the base C^*-algebra A, we simply say pre-Hilbert module and Hilbert module.

Let $(E, (\cdot, \cdot))$ be a pre-Hilbert A-module. From the continuity of the sesquilinear form $(\cdot, \cdot) : E \times E \to A$ and of the right multiplication $E \to E,\ x \mapsto xa$ for any $a \in A$, we infer that the completion of E for the norm (3.6) is a Hilbert A-module.

Remark 3.5.5 In the definition of a pre-Hilbert A-module, one can remove the hypothesis that $(\cdot, \cdot)$ is *definite*. In that case, (3.6) defines a seminorm, and one checks that the Hausdorff completion of a pre-Hilbert A-module, in this extended sense, is a Hilbert A-module.

We continue this subsection with classical examples.

1. The algebra A is a Hilbert A-module with its obvious right A-module structure and

$$(a, b) := a^*b.$$

2. For any positive integer n, A^n is a Hilbert A-module with its obvious right A-module structure and

$$((a_i), (b_i)) := \sum_{i=1}^{n} a_i^* b_i.$$

Observe that $\sum_{i=1}^{n} a_i^* a_i$ is a sum of positive elements in A, which implies that

$$\|(a_i)\| = \sqrt{\left\| \sum_{i=1}^{n} a_i^* a_i \right\|} \geq \|a_k\|$$

for all k. It follows that if $(a_1^m, \ldots, a_n^m)_m$ is a Cauchy sequence in A^n and that the sequences $(a_k^m)_m$ are Cauchy in A and thus convergent, and we conclude that A^n is complete.

3. Example 2 can be extended to the direct sum of n Hilbert A-modules $E_1, \ldots, E_n$ with the Hilbertian product:

$$((x_i), (y_i)) := \sum_{i=1}^{n} (x_i, y_i)_{E_i}.$$

4. If F is a closed A-submodule of a Hilbert A-module E, then F is a Hilbert A-module. For instance, a closed right ideal in A is a Hilbert A-module.
5. The *standard Hilbert A-module* is defined by

$$\mathcal{H}_A = \left\{ x = (x_k)_{k\in\mathbb{N}} \in A^{\mathbb{N}} \mid \sum_{k\in\mathbb{N}} x_k^* x_k \text{ converges} \right\}. \tag{3.8}$$

The right A-module structure is given by $(x_k)a = (x_k a)$, and the Hilbertian A-valued product is

$$((x_k), (y_k)) = \sum_{k=0}^{+\infty} x_k^* y_k. \tag{3.9}$$

This sum converges for elements of $\mathcal{H}_A$; indeed, for all $q > p \in \mathbb{N}$ we have

$$\begin{aligned}
\left\| \sum_{k=p}^{q} x_k^* y_k \right\| &= \left\| \left((x_k)_p^q, (y_k)_p^q \right)_{A^{q-p}} \right\| \\
&\le \left\| (x_k)_p^q \right\|_{A^{q-p}} \left\| (y_k)_p^q \right\|_{A^{q-p}} \qquad \text{(Cauchy–Schwarz inequality in } A^{q-p}) \\
&= \sqrt{\left\| \sum_{k=p}^{q} x_k^* x_k \right\|} \sqrt{\left\| \sum_{k=p}^{q} y_k^* y_k \right\|}.
\end{aligned}$$

This implies that $\sum_{k\ge 0} x_k^* y_k$ satisfies the Cauchy criterion, and therefore converges, so that (3.9) makes sense. Because for all $(x_k), (y_k)$ in $\mathcal{H}_A$

$$\sum_{k\ge 0} (x_k + y_k)^*(x_k + y_k) = \sum_{k\ge 0} x_k^* x_k + \sum_{k\ge 0} y_k^* x_k + \sum_{k\ge 0} x_k^* y_k + \sum_{k\ge 0} y_k^* y_k$$

is the sum of four convergent series, we find that $(x_k) + (y_k) = (x_k + y_k)$ is in $\mathcal{H}_A$. We also have, as before, that for all $a \in A$ and $(x_k) \in \mathcal{H}_A$,

$$\left\| \sum_{k=0}^{+\infty} (x_k a)^*(x_k a) \right\| \le \|a\|^2 \left\| \sum_{k=0}^{+\infty} x_k^* x_k \right\|.$$

Hence, $\mathcal{H}_A$ is a pre-Hilbert A-module, and we need to check that it is complete. Let $(u_n)_n = ((u_i^n))_n$ be a Cauchy sequence in $\mathcal{H}_A$. We get, as in Example 2, that for all $i \in \mathbb{N}$ the sequence $(u_i^n)_n$ is Cauchy in A and thus converges to an element denoted v_i. Let us check that (v_i) belongs to $\mathcal{H}_A$.

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Let $\varepsilon > 0$. Choose n_0 such that

$$\forall p > q \geq n_0, \qquad \|u_q - u_p\|_{\mathcal{H}_A} \leq \varepsilon/2.$$

Choose i_0 such that

$$\forall k > j \geq i_0, \qquad \left\| \sum_{i=j}^{k} u_i^{n_0 *} u_i^{n_0} \right\|^{1/2} \leq \varepsilon/2.$$

Then thanks to the triangle inequality in A^{k-j}, we get for all $p, q \geq n_0$ and $j, k \geq i_0$

$$\left\| \sum_{i=j}^{k} u_i^{p*} u_i^{p} \right\|^{1/2} \leq \left\| \sum_{i=j}^{k} (u_i^p - u_i^{n_0})^* (u_i^p - u_i^{n_0}) \right\|^{1/2} + \left\| \sum_{i=j}^{k} u_i^{n_0 *} u_i^{n_0} \right\|^{1/2} \leq \varepsilon.$$

Taking the limit $p \to +\infty$, we get $\| \sum_{i=j}^{k} v_i^* v_i \|^{1/2} \leq \varepsilon$ for all $j, k \geq i_0$, which implies that $(v_i) \in \mathcal{H}_A$. It remains to check that $(u_n)_n$ converges to $v = (v_i)$ in $\mathcal{H}_A$. With the notation just defined,

$$\forall p, q \geq n_0,\ \forall I \in \mathbb{N}, \qquad \left\| \sum_{i=0}^{I} (u_i^p - u_i^q)^* (u_i^p - u_i^q) \right\|^{1/2} \leq \varepsilon.$$

Taking the limit $p \to +\infty$, we have

$$\forall q \geq n_0,\ \forall I \in \mathbb{N}, \qquad \left\| \sum_{i=0}^{I} (v_i - u_i^q)^* (v_i - u_i^q) \right\|^{1/2} \leq \varepsilon,$$

and taking the limit $I \to +\infty$,

$$\forall q \geq n_0, \quad \|v - u_q\| \leq \varepsilon,$$

which ends the proof.

The standard Hilbert module $\mathcal{H}_A$ is maybe the most important Hilbert module. Indeed, Kasparov proved:

Theorem 3.5.6 *Let E be a countably generated Hilbert A-module. Then $\mathcal{H}_A$ and $E \oplus \mathcal{H}_A$ are isomorphic.*

The proof can be found in [54]. This means that there exists an A-linear unitary map $U : E \oplus \mathcal{H}_A \to \mathcal{H}_A$. The notion of unitary uses the notion of adjoint, which will be explained later.

Remark 3.5.7

1. The algebraic sum $\bigoplus_{\mathbb{N}} A$ is dense in $\mathcal{H}_A$.
2. In $\mathcal{H}_A$ we can replace the summand A by any sequence of Hilbert A-modules $(E_i)_{i \in \mathbb{N}}$, and the Hilbertian A-valued product by

$$((x_k), (y_k)) = \sum_{k=0}^{+\infty} (x_k, y_k)_{E_k}.$$

If $E_i = E$ for all $i \in \mathbb{N}$, the resulting Hilbert A-module is denoted by $l^2(\mathbb{N}, E)$.

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3. We can generalize the construction to any family $(E_i)_{i\in I}$ using summable families instead of convergent series.

We end this subsection with two concrete examples.

a. Let X be a locally compact space and E a Hermitian vector bundle. The space $C_0(X, E)$ of continuous sections of E vanishing at infinity is a Hilbert $C_0(X)$-module with the module structure given by

$$\xi \cdot a(x) = \xi(x)a(x), \qquad \xi \in C_0(X, E), \quad a \in C_0(X),$$

and the $C_0(X)$-valued product given by

$$(\xi, \eta)(x) = (\xi(x), \eta(x))_{E_x}.$$

b. Let G be a locally compact groupoid with a Haar system, λ, and E a Hermitian vector bundle over $G^{(0)}$. Then

$$f, g \in C_c(G, r^*E), \quad (f, g)(\gamma) = \int_{G_{s(\gamma)}} (f(\eta\gamma^{-1}), g(\eta))_{E_{r(\eta)}} d\lambda^{s(\gamma)}(\eta) \tag{3.10}$$

gives a positive definite sesquilinear $C_c(G)$-valued form which has the correct behavior with respect to the right action of $C_c(G)$ on $C_c(G, r^*E)$. This leads to two norms $\|f\| = \|(f, f)\|^{1/2}_{C^*(G)}$ and $\|f\|_r = \|(f, f)\|^{1/2}_{C^*_r(G)}$ and two completions of $C_c(G, r^*E)$, denoted $C^*(G, r^*E)$ and $C^*_r(G, r^*E)$, which are Hilbert modules, respectively, over $C^*(G)$ and $C^*_r(G)$.

3.5.2 Homomorphisms of Hilbert A-modules

Let E, F be Hilbert A-modules. We will need the orthogonality in Hilbert modules:

Lemma 3.5.8 *Let S be a subset of E. The orthogonal of S,*

$$S^\perp = \{x \in E \mid \forall y \in S,\ (y, x) = 0\},$$

is a Hilbert A-submodule of E.

3.5.2.1 Adjoints

Let $T : E \to F$ be a map. T is *adjointable* if there exists a map $S : F \to E$ such that

$$\forall (x, y) \in E \times F, \quad (Tx, y) = (x, Sy). \tag{3.11}$$

Definition 3.5.9 Adjointable maps are called *homomorphisms of Hilbert A-modules*. The set of adjointable maps from E to F is denoted by $\mathrm{Mor}(E, F)$, and $\mathrm{Mor}(E) = \mathrm{Mor}(E, E)$. The space of linear continuous maps from E to F is denoted by $\mathcal{L}(E, F)$ and $\mathcal{L}(E) = \mathcal{L}(E, E)$.

The terminology will become clear after the next proposition.

Proposition 3.5.10 *Let* $T \in \mathrm{Mor}(E, F)$.

(a) *The operator satisfying (3.11) is unique. It is denoted by* T^* *and called the adjoint of* T. *One has* $T^* \in \mathrm{Mor}(F, E)$ *and* $(T^*)^* = T$.
(b) T *is linear, A-linear and continuous.*
(c) $\|T\| = \|T^*\|$, $\|T^*T\| = \|T\|^2$, *and* $\mathrm{Mor}(E, F)$ *is a closed subspace of* $\mathcal{L}(E, F)$. *In particular* $\mathrm{Mor}(E)$ *is a* C^**-algebra.*
(d) *If* $S \in \mathrm{Mor}(E, F)$ *and* $T \in \mathrm{Mor}(F, G)$, *then* $TS \in \mathrm{Mor}(E, G)$ *and* $(TS)^* = S^*T^*$.

Proof (a) Let R, S be two maps satisfying (3.11) for T. Then

$$\forall x \in E, y \in F, \quad (x, Ry - Sy) = 0,$$

and taking $x = Ry - Sy$ yields $Ry - Sy = 0$. The remaining part of the assertion is obvious.

(b) $\forall x, y \in E, z \in F, \lambda \in \mathbb{C}$,

$$(T(x + \lambda y), z) = (x + \lambda y, T^*z) = (x, T^*z) + \overline{\lambda}(y, T^*z) = (Tx, z)(\lambda Ty, z);$$

thus $T(x + \lambda y) = Tx + \lambda Ty$, and T is linear. Moreover,

$$\forall x \in E, y \in F, a \in A, \quad (T(xa), y) = (xa, T^*y) = a^*(x, T^*y) = ((Tx)a, y),$$

which gives the A-linearity. Consider the set

$$S = \{(-T^*y, y) \in E \times F \mid y \in F\}.$$

Then

$$\begin{aligned}(x_0, y_0) \in S^\perp &\Leftrightarrow \forall y \in F,\ (x_0, -T^*y) + (y_0, y) = 0\\ &\Leftrightarrow \forall y \in F,\ (y_0 - Tx_0, y) = 0.\end{aligned}$$

Thus $G(T) = \{(x, y) \in E \times F \mid y = Tx\} = S^\perp$ is closed, and the closed-graph theorem implies that T is continuous.

(c) We have

$$\|T\|^2 = \sup_{\|x\|\le 1} \|Tx\|^2 = \sup_{\|x\|\le 1} (x, T^*Tx) \le \|T^*T\| \le \|T^*\|\|T\|.$$

Thus $\|T\| \le \|T^*\|$, and switching T and T^* gives the equality. We have also proved

$$\|T\|^2 \le \|T^*T\| \le \|T^*\|\|T\| = \|T\|^2;$$

thus $\|T^*T\| = \|T\|^2$, and the norm of $\mathrm{Mor}(E)$ satisfies the C^*-algebraic equation.

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Let $(T_n)_n$ be a sequence in $\mathrm{Mor}(E, F)$ which converges to $T \in \mathcal{L}(E, F)$. Because $\|T\| = \|T^*\|$ and because $T \to T^*$ is (anti)linear, the sequence $(T_n^*)_n$ is a Cauchy sequence, and therefore converges to an operator $S \in \mathcal{L}(F, E)$. It then immediately follows that S is the adjoint of T. This proves that $\mathrm{Mor}(E, F)$ is closed; in particular, $\mathrm{Mor}(E)$ is a C^*-algebra.

(d) Easy. □

Remark 3.5.11 There exist continuous linear and A-linear maps $T : E \to F$ which do not have an adjoint. For instance, take $A = C([0, 1])$, $J = C_0(]0, 1])$ and $T : J \hookrightarrow A$ the inclusion. Assuming that T is adjointable, a one-line computation proves that $T^*1 = 1$. But 1 does not belong to J. Thus $J \hookrightarrow A$ has no adjoint.

One can also take $E = C([0, 1]) \oplus C_0(]0, 1])$ and $T : E \to E, x + y \mapsto y + 0$ to produce an example of $T \in \mathcal{L}(E)$ and $T \notin \mathrm{Mor}(E)$.

One can characterize self-adjoint and positive elements in the C^*-algebra $\mathrm{Mor}(E)$ as follows.

Proposition 3.5.12 *Let* $T \in \mathrm{Mor}(E)$.

(a) $T = T^* \Leftrightarrow \forall x \in E,\ (x, Tx) = (x, Tx)^*$

(b) $T \geq 0 \Leftrightarrow \forall x \in E,\ (x, Tx) \geq 0$

Proof (a) The implication $\Rightarrow$ is obvious. Conversely, set $Q_T(x) = (x, Tx)$. Using the polarization identity

$$(x, Ty) = \frac{1}{4}\left(Q_T(x + y) - i\,Q_T(x + iy) - Q_T(x - y) + i\,Q_T(x - iy)\right),$$

one easily gets $(x, Ty) = (Tx, y)$ for all $x, y \in E$; thus T is self-adjoint.

(b) If T is positive, there exists $S \in \mathrm{Mor}(E)$ such that $T = S^*S$. Then $(x, Tx) = (Sx, Sx)$ is positive for all x. Conversely, if $(x, Tx) \geq 0$ for all x, then T is self-adjoint by (a), and there exist positive elements T_+, T_- such that

$$T = T_+ - T_-, \qquad T_+T_- = T_-T_+ = 0.$$

It follows that

$$\forall x \in E,\ (x, T_+x) \geq (x, T_-x),$$
$$\forall z \in E,\ (T_-z, T_+T_-z) \geq (T_-z, T_-T_-z),$$
$$\forall z \in E,\ (z, (T_-)^3 z) \leq 0.$$

Because T_- is positive, T_-^3 is also positive and the last inequality implies $T_-^3 = 0$. It follows that $T_- = 0$ and then $T = T_+ \geq 0$. □

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3.5.2.2 Orthocompletion

Recall that for any subset S of E, $S^{\perp}$ is a Hilbert submodule of E. It is also worth noticing that any orthogonal submodules $F \perp G$ of E are direct summands.

The following properties are left to check as an exercise:

Proposition 3.5.13 *Let F, G be A-submodules of E.*

- $E^{\perp} = \{0\}$ *and* $\{0\}^{\perp} = E$.
- $F \subset G \Rightarrow G^{\perp} \subset F^{\perp}$.
- $F \subset F^{\perp\perp}$.
- *If $F \perp G$ and $F \oplus G = E$, then $F^{\perp} = G$ and $G^{\perp} = F$. In particular, F and G are Hilbert submodules.*

Definition 3.5.14 A Hilbert A-submodule F of E is said to be *orthocomplemented* if $F \oplus F^{\perp} = E$.

Remark 3.5.15 A Hilbert submodule is not necessarily orthocomplemented, even if it can be topologically complemented. For instance, consider $A = C([0, 1])$ and $J = C_0(]0, 1])$ as a Hilbert A-submodule of A. One easily checks that $J^{\perp} = \{0\}$; thus J is not orthocomplemented. On the other hand, $A = J \oplus \mathbb{C}$.

Lemma 3.5.16 *Let $T \in \mathrm{Mor}(E)$. Then*

- $\ker T^* = (\mathrm{Im}\, T)^{\perp}$,
- $\overline{\mathrm{Im}\, T} \subset (\ker T^*)^{\perp}$.

The proof is obvious. Note the difference in the second point from the case of bounded operators on Hilbert spaces (where equality always occurs). Thus, in general, $\ker T^* \oplus \overline{\mathrm{Im}\, T}$ is not the whole of E. Such a situation can occur when $\overline{\mathrm{Im}\, T}$ is not orthocomplemented.

Let us point out that we can have T^* injective without having $\mathrm{Im}\, T$ dense in E (for instance, $T : C[0, 1] \to C[0, 1]$, $f \mapsto tf$). Nevertheless, we have:

Theorem 3.5.17 *Let $T \in \mathrm{Mor}(E, F)$. The following assertions are equivalent:*

(i) $\mathrm{Im}\, T$ *is closed,*
(ii) $\mathrm{Im}\, T^*$ *is closed,*
(iii) 0 *is isolated in* $\mathrm{spec}(T^*T)$ *(or* $0 \notin \mathrm{spec}(T^*T)$*),*
(iv) 0 *is isolated in* $\mathrm{spec}(TT^*)$ *(or* $0 \notin \mathrm{spec}(TT^*)$*),*

and in that case $\mathrm{Im}\, T$*,* $\mathrm{Im}\, T^*$ *are orthocomplemented.*

Thus, under the assumption of the theorem $\ker T^* \oplus \mathrm{Im}\, T = F$, $\ker T \oplus \mathrm{Im}\, T^* = E$. Before proving the theorem, we gather some technical preliminaries into a lemma:

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Lemma 3.5.18 *Let* $T \in \mathrm{Mor}(E, F)$. *Then:*

(i) $T^*T \geq 0$. *We set* $|T| = \sqrt{T^*T}$.
(ii) $\overline{\mathrm{Im}\, T^*} = \overline{\mathrm{Im}\, |T|} = \overline{\mathrm{Im}\, T^*T}$.
(iii) Assume that $T(E_1) \subset F_1$ *for some Hilbert submodules* E_1, F_1. *Then* $T|_{E_1} \in \mathrm{Mor}(E_1, F_1)$.
(iv) If T *is onto, then* TT^* *is invertible (in* $\mathrm{Mor}(F)$*) and* $E = \ker T \oplus \mathrm{Im}\, T^*$.

Proof Proof of the lemma: (i) is obvious.

(ii) One has $T^*T(E) \subset T^*(F)$. Conversely,

$$T^* = \lim T^*(1/n + TT^*)^{-1}TT^*.$$

This is a convergence in norm, because

$$\|T^*(1/n + TT^*)^{-1}TT^* - T^*\| = \left\| \frac{1}{n}T^* \left(\frac{1}{n} + TT^* \right)^{-1} \right\| = O(1/\sqrt{n}).$$

It follows that $T^*(F) \subset \overline{T^*T(E)}$ and thus $\overline{\mathrm{Im}\, T^*} = \overline{\mathrm{Im}\, T^*T}$. Replacing T by $|T|$ yields the other equality.

(iii) Easy.

(iv) By the open-mapping theorem, there exists a positive real number $k > 0$ such that each $y \in F$ has a preimage x_y by T with $\|y\| \geq k\|x_y\|$. Using the Cauchy–Schwarz inequality for T^*y and x_y, we get

$$\|T^*y\| \geq k\|y\| \quad \forall y \in F. \tag{$*$}$$

Recall that in a C^*-algebra, the inequality $a^*a \leq b^*b$ is equivalent to $\|ax\| \leq \|bx\|$ for all $x \in A$. This can be adapted to Hilbert modules to show that $(*)$ implies $TT^* \geq k^2$ in $\mathrm{Mor}(F)$, so that TT^* is invertible. Then $p = T^*(TT^*)^{-1}T$ is an idempotent and $E = \ker p \oplus \mathrm{Im}\, p$. Moreover, $(TT^*)^{-1}T$ is onto, from which it follows that $\mathrm{Im}\, p = \mathrm{Im}\, T^*$. On the other hand, $T^*(TT^*)^{-1}$ is injective, so that $\ker p = \ker T$. □

Proof Proof of the theorem: Let us start with the implication (i) ⇒ (iv). By point (iii) of the lemma, $S := (T : E \to TE) \in \mathrm{Mor}(E, TE)$, and by point (iv) of the lemma SS^* is invertible. Because the spectra of SS^* and S^*S coincide outside 0 and because $S^*S = T^*T$, we get (iii).

The implication (iv) ⇒ (i): Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(0) = g(0) = 0$, $f(t) = 1$, $g(t) = 1/t$ for $t \neq 0$. Thus f and g are continuous on the spectrum of TT^*. Using the equalities $f(t)t = t$ and $tg(t) = f(t)$, we get $f(TT^*)TT^* = TT^*$ and $TT^*g(TT^*) = f(TT^*)$, from which we deduce $\mathrm{Im}\, f(TT^*) = \mathrm{Im}\, TT^*$. But $f(TT^*)$ is a projector (self-adjoint idempotent); hence $\mathrm{Im}\, TT^*$ is closed and orthocomplemented. Using point (ii) of the lemma and the inclusion $\mathrm{Im}\, TT^* \subset \mathrm{Im}\, T$ yields (i) (and also the orthocomplementability of $\mathrm{Im}\, T$).

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At this point we have the following equivalences: (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). Replacing T by T^*, we get (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). □

Another result which deserves to be stated is:

Proposition 3.5.19 *Let H be a Hilbert submodule of E, and $T : E \to F$ a A-linear map.*

- *H is orthocomplemented if and only if $i : H \hookrightarrow E \in \mathrm{Mor}(H, E)$.*
- *$T \in \mathrm{Mor}(E, F)$ if and only if the graph of T,*

$$\{(x, y) \in E \times F \mid y = Tx\},$$

is orthocomplemented.

3.5.2.3 Partial isometries

The following easy result is left as an exercise:

Proposition 3.5.20 *(and definition). Let $u \in \mathrm{Mor}(E, F)$. The following assertions are equivalent:*

(i) *u^*u is an idempotent,*
(ii) *uu^* is an idempotent,*
(iii) *$u^* = u^*uu^*$,*
(iv) *$u = uu^*u$.*

u is then called a partial isometry, with initial support $I = \mathrm{Im}\, u^*$ and final support $J = \mathrm{Im}\, u$.

Remark 3.5.21 If u is a partial isometry, then $\ker u = \ker u^*u$, $\ker u^* = \ker uu^*$, $\mathrm{Im}\, u = \mathrm{Im}\, uu^*$ and $\mathrm{Im}\, u^* = \mathrm{Im}\, u^*u$. In particular, u has closed range, and $E = \ker u \oplus \mathrm{Im}\, u^*$, $F = \ker u^* \oplus \mathrm{Im}\, u$, where the direct sums are orthogonal.

3.5.2.4 Polar decompositions

All homomorphisms do not admit a polar decomposition. For instance, consider $T \in \mathrm{Mor}(C[-1, 1])$ defined by $Tf = t \cdot f$ (here $C[-1, 1]$ is regarded as a Hilbert $C[-1, 1]$-module). T is self-adjoint, and $|T| : f \mapsto |t| \cdot f$. The equation $T = u|T|$, $u \in \mathrm{Mor}(C[-1, 1])$, leads to the constraint $u(1)(t) = \mathrm{sign}(t)$, so $u(1) \notin C[-1, 1]$ and u does not exist.

The next result clarifies the requirements for a polar decomposition to exist:

Theorem 3.5.22 *Let $T \in \mathrm{Mor}(E, F)$ such that $\overline{\mathrm{Im}\, T}$ and $\overline{\mathrm{Im}\, T^*}$ are orthocomplemented. Then there exists a unique $u \in \mathrm{Mor}(E, F)$, vanishing on $\ker T$, such that*

$$T = u|T|.$$

Moreover, u is a partial isometry with initial support $\overline{\mathrm{Im}\, T^*}$ and final support $\overline{\mathrm{Im}\, T}$.

Proof We first assume that T and T^* have dense range. Setting $u_n = T(1/n + T^*T)^{-1/2}$, we get a bounded sequence ($\|u_n\| \leq 1$) such that for all $y \in F$ we have $u_n(T^*y) = T(1/n + T^*T)^{-1/2}T^*y \to \sqrt{TT^*}(y)$. Thus, by density of $\operatorname{Im} T^*$, $u_n(x)$ converges for all $x \in E$. Let $v(x)$ denotes the limit. Replacing T by T^*, we also have that $u_n^*(y)$ converges for all $y \in F$, which yields $v \in \operatorname{Mor}(E, F)$. A careful computation shows that $u_n|T| - T$ goes to 0 in norm. Thus $v|T| = T$. The homomorphism v is unique by density of $\operatorname{Im}|T|$, and is unitary because $u_n^*u_n(x) \to x$ for all $x \in \operatorname{Im} T^*T$; this proves $v^*v = 1$, and similarly for vv^*.

Now consider the general case, and set $E_1 = \overline{\operatorname{Im} T^*}$, $F_1 = \overline{\operatorname{Im} T}$. One applies the first step to the restriction $T_1 \in \operatorname{Mor}(E_1, F_1)$ of T, and we denote by v_1 the unitary homomorphism constructed. We set $u(x) = v_1(x)$ if $x \in E_1$, and $u(x) = 0$ if $x \in E_1^\perp = \ker T$. This definition forces the uniqueness, and it is clear that u is a partial isometry with the claimed initial and final supports. □

Remark 3.5.23 u is the strong limit of $T(1/n + T^*T)^{-1/2}$.

3.5.2.5 Compact homomorphisms

Let $x \in E,\ y \in F$, and define $\theta_{y,x} \in \operatorname{Mor}(E, F)$ by

$$\theta_{y,x}(z) = y \cdot (x, z).$$

The adjoint is given by $\theta_{y,x}^* = \theta_{x,y}$. Then

Definition 3.5.24 We define $\mathcal{K}(E, F)$ to be the closure of the linear span of $\{\theta_{y,x};\ x \in E, y \in F\}$ in $\operatorname{Mor}(E, F)$.

One easily checks that

- $\|\theta_{y,x}\| \leq \|x\|\|y\|$ and $\|\theta_{x,x}\| = \|x\|^2$,
- $T\theta_{y,x} = \theta_{Ty,x}$ and $\theta_{y,x}S = \theta_{y,S^*x}$,
- $\mathcal{K}(E) := \mathcal{K}(E, E)$ is a closed two-sided ideal of $\operatorname{Mor}(E)$ (and hence a C^*-algebra).

We also prove:

Proposition 3.5.25

$$\mathcal{M}(\mathcal{K}(E)) \simeq \operatorname{Mor}(E),$$

where $\mathcal{M}(A)$ denotes the multiplier algebra of a C^*-algebra A.

Proof One can show that for all $x \in E$ there is a unique $y \in E$ such that $x = y \cdot \langle y, y\rangle$ (a technical exercise: show that the limit $y = \lim x \cdot f_n(\sqrt{(x, x)})$ with $f_n(t) = t^{1/3}(1/n + t)^{-1}$ exists and satisfies the desired assertion). Consequently, E is a nondegenerate $\mathcal{K}(E)$-module (i.e., $\mathcal{K}(E) \cdot E = E$); indeed, $x = y \cdot \langle y, y\rangle = \theta_{y,y}(y)$. Using an approximate unit $(u_\lambda)_\Lambda$ for $\mathcal{K}(E)$, we can extend the $\mathcal{K}(E)$-module structure of E into an $\mathcal{M}(\mathcal{K}(E))$-module structure:

$$\forall T \in \mathcal{M}(\mathcal{K}(E)), x \in E, \quad T \cdot x = \lim_{\Lambda} T(u_\lambda) \cdot x.$$

The existence of the limit is a consequence of $x = \theta_{y,y}(y)$ and $T(u_\lambda) \cdot \theta_{y,y} = T(u_\lambda \theta_{y,y}) \to T(\theta_{y,y})$. The limit is $T(\theta_{y,y}) \cdot y$. By the uniqueness of y, this module structure, extending that of $\mathcal{K}(E)$, is unique.

Hence each $m \in \mathcal{M}(\mathcal{K}(E))$ gives rise to a map $M : E \to E$. For any x, z in E,

$$(z, M \cdot x) = (z, (m\theta_{y,y}) \cdot y) = ((m\theta_{y,y})^*(z), y);$$

thus M has an adjoint: $M \in \mathrm{Mor}(E)$, and M^* corresponds to m^*. The map $\rho : m \to M$ provides a $*$-homomorphism from $\mathcal{M}(\mathcal{K}(E))$ to $\mathrm{Mor}(E)$, which is the identity on $\mathcal{K}(E)$. On the other hand, let $\pi : \mathrm{Mor}(E) \to \mathcal{M}(\mathcal{K}(E)$ be the unique $*$-homomorphism, equal to the identity on $\mathcal{K}(E)$, associated to the inclusion $\mathcal{K}(E) \subset \mathrm{Mor}(E)$ as a closed ideal. We have $\pi \circ \rho = \mathrm{Id}$, and by uniqueness of the $\mathcal{M}(\mathcal{K}(E))$-module structure of E, $\rho \circ \pi = \mathrm{Id}$. □

Let us give some generic examples:

(i) Consider A as a Hilbert A-module. We know that for any $a \in A$, there exists $c \in A$ such that $a = cc^*c$. It follows that the map $\gamma_a : A \to A, b \mapsto ab$ is equal to θ_{c,c^*c} and thus is compact. We get a $*$-homomorphism $\gamma : A \to \mathcal{K}(A), a \mapsto \gamma_a$, which has dense image (the linear span of the θ's is dense in $\mathcal{K}(A)$) and is clearly injective, because $yb = 0$ for all $b \in A$ implies $y = 0$. Thus γ is an isomorphism:

$$\mathcal{K}(A) \simeq A.$$

In particular, $\mathrm{Mor}(A) \simeq \mathcal{M}(A)$, and if $1 \in A$, then $A \simeq \mathrm{Mor}(A) = \mathcal{K}(A)$.

(ii) For any n, one has in a similar way $\mathcal{K}(A^n) \simeq M_n(A)$ and $\mathrm{Mor}(A^n) \simeq M_n(\mathcal{M}(A))$. If moreover $1 \in A$, then

$$\mathrm{Mor}(A^n) = \mathcal{K}(A^n) \simeq M_n(A). \qquad \text{(i)}$$

For any Hilbert A-module E, we also have $\mathcal{K}(E^n) \simeq M_n(\mathcal{K}(E))$.

The relations (i) can be extended to arbitrary finitely generated Hilbert A-modules:

Proposition 3.5.26 *Let A be a unital C^*-algebra, and E a A-Hilbert module. Then the following are equivalent:*

(i) E is finitely generated.
(ii) $\mathcal{K}(E) = \mathrm{Mor}(E)$.
(iii) Id_E *is compact.*

In that case, E is also projective (i.e., it is a direct summand of A^n for some n).

For the proof we refer to [54].

3.5.3 Generalized Fredholm operators

Atkinson's theorem claims that for any bounded linear operator on a Hilbert space H, the assertion

$$\ker F \text{ and } \ker F^* \text{ are finite dimensional}$$

is equivalent to the following: There exists a linear bounded operator G such that $FG - \mathrm{Id}$, $GF - \mathrm{Id}$ are compact. This situation is a little more subtle on Hilbert A-modules, in that firstly all the kernel of homomorphisms are A-modules which are not necessarily free, and secondly, replacing the condition "finite dimensional" by "finitely generated" is not enough to recover the previous equivalence. This is why one uses the second assertion as a definition of Fredholm operator in the context of Hilbert modules, and we will see how to adapt Atkinson's classical result to this new setup.

Definition 3.5.27 The homomorphism $T \in \mathrm{Mor}(E, F)$ is a *generalized Fredholm operator* if there exists $G \in \mathrm{Mor}(F, E)$ such that

$$GF - \mathrm{Id} \in \mathcal{K}(E) \quad \text{and} \quad FG - \mathrm{Id} \in \mathcal{K}(F).$$

The following theorem is important to understand the next chapter on KK-theory.

Theorem 3.5.28 *Let A be a unital C^*-algebra, $\mathcal{E}$ a countably generated Hilbert A-module and F a generalized Fredholm operator on $\mathcal{E}$.*

(i) If $\mathrm{Im}\, F$ *is closed, then* $\ker F$ *and* $\ker F^*$ *are finitely generated Hilbert modules.*
(ii) There exists a compact perturbation G of F such that $\mathrm{Im}\, G$ *is closed.*

Proof (1) Because $\mathrm{Im}\, F$ is closed, so is $\mathrm{Im}\, F^*$, and both are orthocomplemented by, respectively, $\ker F^*$ and $\ker F$. Let $P \in \mathrm{Mor}(\mathcal{E})$ be the orthogonal projection on $\ker F$. Because F is a generalized Fredholm operator, there exists $G \in \mathrm{Mor}(\mathcal{E})$ such that $Q = 1 - GF$ is compact. In particular, Q is equal to Id on $\ker F$, and

$$QP : \mathcal{E} = \ker F \oplus \mathrm{Im}\, F^* \to \mathcal{E}, \qquad x \oplus y \mapsto x \oplus 0.$$

Because QP is compact, its restriction $QP|_{\ker F} : \ker F \to \ker F$ is also compact, but $QP|_{\ker F} = \mathrm{id}_{\ker F}$; hence Proposition 3.5.26 implies that $\ker F$ is finitely generated. The same argument works for $\ker F^*$.

(2) Let us denote by π the projection homomorphism

$$\pi : \mathrm{Mor}(\mathcal{E}) \to C(\mathcal{E}) := \mathrm{Mor}(\mathcal{E})/\mathcal{K}(\mathcal{E}).$$

Because $\pi(F)$ is invertible in $C(\mathcal{E})$, it has a polar decomposition: $\pi(F) = \omega \cdot |\pi(F)|$. Any unitary of $C(\mathcal{E})$ can be lifted to a partial isometry of Mor($\mathcal{E}$) [54]. Let U be such a lift of the unitary ω. Using $|\pi(F)| = \pi(|F|)$, it follows that

$$F = U|F| \mod \mathcal{K}(\mathcal{E}).$$

Because $\pi(|F|)$ is also invertible, and positive, we can form $\log(\pi(|F|))$ and choose a self-adjoint $H \in \mathrm{Mor}(\mathcal{E})$ with $\pi(H) = \log(\pi(|F|))$. Then

$$\pi(Ue^H) = \omega\pi(|F|) = \pi(F),$$

that is, Ue^H is a compact perturbation of F (and thus is a generalized Fredholm operator). The operator U is a partial isometry and hence has a closed image; and e^H is invertible in Mor($\mathcal{E}$), whence Ue^H has a closed image, and the theorem is proved. □

3.5.4 Tensor products

3.5.4.1 Inner tensor products

Let E be a Hilbert A-module, F a Hilbert B-module, and $\pi : A \to \mathrm{Mor}(F)$ a $*$-homomorphism. We define a sesquilinear form on $E \otimes_A F$ by setting

$$\forall x, x' \in E,\, y, y' \in F, \quad (x \otimes y, x' \otimes y')_{E\otimes F} := (y, (x, x')_E \cdot y')_F,$$

where we have set $a \cdot y = \pi(a)(y)$ to lighten the formula. This sesquilinear form is a B-valued scalar product: only the positivity axiom needs some explanation. Set

$$b = \left(\sum_i x_i \otimes y_i, \sum_i x_i \otimes y_i \right) = \sum_{i,j} (y_i, (x_i, x_j) \cdot y_j),$$

where π has been omitted. Let us set $P = ((x_i, x_j))_{i,j} \in M_n(A)$. The matrix P provides a (self-adjoint) compact homomorphism of A^n, which is positive because

$$\forall a \in A^n, \qquad (a, Pa)_{A^n} = \sum_{i,j} a_i^*(x_i, x_j)a_j = \left(\sum_i x_i a_i, \sum_j x_j a_j \right) \geq 0.$$

This means that $P = Q^*Q$ for some $Q \in M_n(A)$. On the other hand, one can consider P as a homomorphism on F^n, and setting $y = (y_1, \ldots, y_n) \in F^n$, we have

$$b = (y, Py) = (Qy, Qy) \geq 0.$$

Thus $E \otimes_A F$ is a pre-Hilbert module in the generalized sense (i.e., we do not require the inner product to be definite), and the Hausdorff completion of $E \otimes_A F$ is a Hilbert B-module denoted in the same way.

Proposition 3.5.29 *Let $T \in \mathrm{Mor}(E)$ and $S \in \mathrm{Mor}(F)$.*

- *$T \otimes 1 : x \otimes y \mapsto Tx \otimes y$ defines a homomorphism of $E \otimes_A F$.*
- *If S commutes with π, then $1 \otimes S : x \otimes y \mapsto x \otimes Sy$ is a homomorphism which commutes with any $T \otimes 1$.*

Remark 3.5.30

1. Even if T is compact, $T \otimes 1$ is not compact in general. The same is true for $1 \otimes S$ when defined.
2. In general $1 \otimes S$ is not even defined.

3.5.4.2 Outer tensor products

Now forget the homomorphism π, and consider the tensor product over $\mathbb{C}$ of E and F. We set

$$\forall x, x' \in E, y, y' \in F, \quad (x \otimes y, x' \otimes y')_{E \otimes F} := (x, x')_E \otimes (y, y')_F \in A \otimes B.$$

This defines a pre-Hilbert $A \otimes B$-module in the generalized sense (the proof of positivity uses similar arguments), where $A \otimes B$ denotes the spatial tensor product (as it will in the following, when not otherwise specified). The Hausdorff completion will be denoted $E \otimes_{\mathbb{C}} F$.

Example 3.5.31 Let H be a separable Hilbert space. Then

$$H \otimes_{\mathbb{C}} A \simeq H_A$$

3.5.4.3 Connections

We turn back to inner tensor products. We keep the notation of the Section 3.5.4.1. Connes and Skandalis [14] introduced the notion of connection to bypass the general nonexistence of $1 \otimes S$.

Definition 3.5.32 Consider two C^*-algebras A and B. Let E be a Hilbert A-module and F be a Hilbert B-module. Assume there is a $*$-morphism

$$\pi : A \to \mathcal{L}(F),$$

and take the inner tensor product $E \otimes_A F$. Given $x \in E$, we define a homomorphism

$$T_x : E \to E \otimes_A F,$$
$$y \mapsto x \otimes y,$$

whose adjoint is given by

$$T_x^* : E \otimes_A F \to F,$$
$$z \otimes y \mapsto \pi((x, z))y.$$

If $S \in \mathcal{L}(F)$, an *S-connection* on $E \otimes_A F$ is given by an element

$$G \in \mathcal{L}(E \otimes_A F)$$

such that for all $x \in E$,

$$T_x S - G T_x \in \mathcal{K}(F, E \otimes_A F),$$
$$S T_x^* - T_x^* G \in \mathcal{K}(E \otimes_A F, F).$$

Proposition 3.5.33

(1) If $[\pi, S] \subset \mathcal{K}(F)$, then there are S-connections.
(2) If G_i, $i = 1, 2$, are S_i-connections, then $G_1 + G_2$ is an $S_1 + S_2$-connection and $G_1 G_2$ is an $S_1 S_2$-connection.
(3) For any S-connection G, $[G, \mathcal{K}(E) \otimes 1] \subset \mathcal{K}(E \otimes_A F)$.
(4) The space of 0-connections is exactly

$$\{G \in \mathrm{Mor}(F, E \otimes_A F) \mid (\mathcal{K}(E) \otimes 1)G \text{ and } G(\mathcal{K}(E) \otimes 1) \text{ are subsets of } \mathcal{K}(E \otimes_A F)\}.$$

All these assertions are important for the construction of the Kasparov product. For the proof, see [14].

3.6 *KK*-Theory

3.6.1 Kasparov modules and homotopies

Given two C^*-algebras A and B, a *Kasparov A–B-module* (abbreviated "Kasparov module") is given by a triple

$$x = (\mathcal{E}, \pi, F)$$

where $\mathcal{E} = \mathcal{E}^0 \oplus \mathcal{E}^1$ is a $(\mathbb{Z}/2\mathbb{Z})$-graded countably generated Hilbert B-module, $\pi : A \to \mathcal{L}(\mathcal{E})$ is a $*$-morphism of degree 0 with respect to the grading, and $F \in \mathcal{L}(\mathcal{E})$ is of degree 1. These data are required to satisfy the following properties:

$$\pi(a)(F^2 - 1) \in \mathcal{K}(\mathcal{E}) \qquad \text{for all } a \in A,$$
$$[\pi(a), F] \in \mathcal{K}(\mathcal{E}) \qquad \text{for all } a \in A.$$

We denote the set of Kasparov A–B-modules by $E(A, B)$.

Let us immediately define the equivalence relation leading to KK-groups. We denote $B([0, 1]) := C([0, 1], B)$.

Definition 3.6.1 A *homotopy* between two Kasparov A–B-modules $x = (\mathcal{E}, \pi, F)$ and $x' = (\mathcal{E}', \pi', F')$ is a Kasparov A–$B([0, 1])$-module $\tilde{x}$ such that

$$(ev_{t=0})_*(\tilde{x}) = x,$$
$$(ev_{t=1})_*(\tilde{x}) = x'. \tag{3.12}$$

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Here $ev_{t=\cdot}$ is the evaluation map at $t = \cdot$. Homotopy between Kasparov A–B-modules is an equivalence relation. If there exists a homotopy between x and x', we write $x \sim_h x'$.

The set of homotopy classes of Kasparov A–B-modules is denoted $KK(A, B)$.

There is a natural *sum* on $E(A, B)$: if $x = (\mathcal{E}, \pi, F)$ and $x' = (\mathcal{E}', \pi', F')$ belong to $E(A, B)$, their sum $x + x' \in E(A, B)$ is defined by

$$x + x' = (\mathcal{E} \oplus \mathcal{E}', \pi \oplus \pi', F \oplus F').$$

A Kasparov A–B-module $x = (\mathcal{E}, \pi, F)$ is called *degenerate* if for all $a \in A$ one has $\pi(a)(F^2 - 1) = 0$ and $[\pi(a), F] = 0$. Then:

Proposition 3.6.2 *Degenerate elements of* $E(A, B)$ *are homotopic to* $(0, 0, 0)$.

The sum of Kasparov A–B*-modules provides* $KK(A, B)$ *with an abelian group structure.*

Proof Let $x = (\mathcal{E}, \pi, F) \in E(A, B)$ be a degenerate element. Set $\tilde{x} = (\tilde{\mathcal{E}}, \tilde{\pi}, \tilde{F}) \in E(A, B([0, 1]))$ with

$$\tilde{\mathcal{E}} = C_0([0, 1[\,, \mathcal{E}),$$
$$\tilde{\pi}(a)\xi(t) = \pi(a)\xi(t),$$
$$\tilde{F}\xi(t) = F\xi(t).$$

Then $\tilde{x}$ is a homotopy between x and $(0, 0, 0)$.

One can easily show that the sum of Kasparov modules makes sense at the level of their homotopy classes. Thus $KK(A, B)$ admits a commutative semigroup structure with $(0, 0, 0)$ as a neutral element. Finally, the opposite in $KK(A, B)$ of $x = (\mathcal{E}, \pi, F) \in E(A, B)$ may be represented by

$$(\mathcal{E}^{op}, \pi, -F).$$

where $\mathcal{E}^{op}$ is $\mathcal{E}$ with the opposite graduation: $(\mathcal{E}^{op})^i = \mathcal{E}^{1-i}$. Indeed, the module $(\mathcal{E}, \pi, F) \oplus (\mathcal{E}^{op}, \pi, -F)$ is homotopically equivalent to the degenerate module

$$\left(\mathcal{E} \oplus \mathcal{E}^{op}, \pi \oplus \pi, \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}\right).$$

This can be realized with the homotopy

$$G_t = \cos\left(\frac{\pi t}{2}\right)\begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix} + \sin\left(\frac{\pi t}{2}\right)\begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}. \qquad \square$$

3.6.2 Operations on Kasparov modules

Let us explain the functoriality of KK-groups with respect to its variables. The following two operations on Kasparov modules make sense on KK-groups:

- *Pushforward along $*$-morphisms: covariance in the second variable.* Let $x = (\mathcal{E}, \pi, F) \in E(A, B)$, and let $g : B \to C$ be a $*$-morphism. We define an element $g_*(x) \in E(A, C)$ by
$$g_*(x) = (\mathcal{E} \otimes_g C, \pi \otimes 1, F \otimes \mathrm{Id}),$$
where $\mathcal{E} \otimes_g C$ is the inner tensor product of the Hilbert B-module $\mathcal{E}$ with the Hilbert C-module C endowed with the left action of B given by g.
- *Pullback along $*$-morphisms: contravariance in the first variable.* Let $x = (\mathcal{E}, \pi, F) \in E(A, B)$, and let $f : C \to A$ be a $*$-morphism. We define an element $f^*(x) \in E(C, B)$ by
$$f^*(x) = (\mathcal{E}, \pi \circ f, F).$$

Provided with these operations, KK-theory is a bifunctor from the category (of pairs) of C^*-algebras to the category of abelian groups.

We recall another useful operation in KK-theory:

- *Suspension*: Let $x = (\mathcal{E}, \pi, F) \in E(A, B)$, and let D be a C^*-algebra. We define an element $\tau_D(x) \in E(A \otimes D, B \otimes D)$ by
$$\tau_D(x) = (\mathcal{E} \otimes_{\mathbb{C}} D, \pi \otimes 1, F \otimes \mathrm{Id}).$$
Here we take the external tensor product $\mathcal{E} \otimes_{\mathbb{C}} D$, which is a $B \otimes D$-Hilbert module.

3.6.3 Examples of Kasparov modules and of homotopies between them

3.6.3.1 Kasparov modules coming from homomorphisms between C^*-algebras

Let A, B be two C^*-algebras, and $f : A \to B$ a $*$-homomorphism. Because $\mathcal{K}(B) \simeq B$, the expression
$$[f] := (B, f, 0)$$
defines a Kasparov A–B-module. If A and B are $\mathbb{Z}_2$-graded, f has to be a homomorphism of degree 0 (i.e., respecting the grading).

3.6.3.2 *Atiyah's* Ell

Let X be a compact Hausdorff topological space. Take $A = C(X)$ to be the algebra of continuous functions on X and let $B = \mathbb{C}$. Then
$$E(A, B) = \mathrm{Ell}(X),$$

the ring of generalized elliptic operators on X as defined by Atiyah. We give two concrete examples of such Kasparov modules:

- Assume X is a compact smooth manifold, let $A = C(X)$ as before, and let $B = \mathbb{C}$. Let E and E' be two smooth vector bundles over X, and denote by π the action of $A = C(X)$ by multiplication on $L^2(X, E) \oplus L^2(X, E')$. Given a zero-order elliptic pseudodifferential operator

$$P : C^\infty(E) \to C^\infty(E')$$

with parametrix $Q : C^\infty(E') \to C^\infty(E)$, the triple

$$x_P = \left(L^2(X, E) \oplus L^2(X, E'),\ \pi,\ \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix} \right)$$

defines an element in $E(A, B) = E(C(X), \mathbb{C})$.

- Let X be a compact spin^c manifold of dimension $2n$, let $A = C(X)$ be as before, and let $B = \mathbb{C}$. Denote by $S = S^+ \oplus S^-$ the complex spin bundle over X, and let

$$\not{D} : L^2(X, S) \to L^2(X, S)$$

be the corresponding Dirac operator. Let π be the action of $A = C(X)$ by multiplication on $L^2(X, S)$. Then the triple

$$x_{\not{D}} = \left(L^2(X, S),\ \pi,\ \frac{\not{D}}{\sqrt{1 + \not{D}^2}} \right)$$

defines an element in $E(A, B) = E(C(X), \mathbb{C})$.

3.6.3.3 Compact perturbations

Let $x = (\mathcal{E}, \pi, F) \in E(A, B)$. Let $P \in \mathrm{Mor}(\mathcal{E})$ satisfy

$$\forall a \in A, \qquad \pi(a) \cdot P \in \mathcal{K}(\mathcal{E}) \text{ and } P \cdot \pi(a) \in \mathcal{K}(\mathcal{E}). \tag{3.13}$$

Then

$$x \sim_h (\mathcal{E}, \pi, F + P).$$

The homotopy is the obvious one: $(\mathcal{E} \otimes C([0, 1]), \pi \otimes \mathrm{Id}, F + tP)$. In particular, when B is unital, we can always choose a representative $(\mathcal{E}, \pi, G)$ with $\mathrm{Im}\, G$ closed (cf. Theorem 3.5.28).

3.6.3.4 (Quasi) Self-adjoint representatives

There exists a representative $(\mathcal{E}, \pi, G)$ of $x = (\mathcal{E}, \pi, F) \in E(A, B)$ satisfying

$$\pi(a)(G - G^*) \in \mathcal{K}(\mathcal{E}). \tag{3.14}$$

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Just take $(\mathcal{E} \otimes C([0,1]), \pi \otimes \mathrm{Id}, F_t)$ as a homotopy, where

$$F_t = (tF^*F + 1)^{1/2} F (tF^*F + 1)^{-1/2}.$$

Then $G = F_1$ satisfies (3.14). Now, $H = (G + G^*)/2$ is self-adjoint, and $P = (G - G^*)/2$ satisfies (3.13); thus $(\mathcal{E}, \pi, H)$ is another representative of x.

Note that (3.14) is often useful in practice and is added as an axiom in many definitions of KK-theory, like the original one of Kasparov. It was observed in [49] that it could be omitted.

3.6.3.5 Stabilization and unitarily equivalent modules

Any Kasparov module $(E, \pi, F) \in E(A, B)$ is homotopic to a Kasparov module $(\widehat{\mathcal{H}}_B, \rho, G)$, where $\widehat{\mathcal{H}}_B = \mathcal{H}_B \oplus \mathcal{H}_B$ is the standard graded Hilbert B-module. Indeed, add to (E, π, F) the degenerate module $(\widehat{\mathcal{H}}_B, 0, 0)$, and consider a grading-preserving isometry $u : E \oplus \widehat{\mathcal{H}}_B \to \widehat{\mathcal{H}}_B$ provided by Kasparov's stabilization theorem. Then, set $\widetilde{E} = E \oplus \widehat{\mathcal{H}}_B$, $\widetilde{F} = F \oplus 0$, $\widetilde{\pi} = \pi \oplus 0$, $\rho = u\widetilde{\pi}u^*$, $G = u\widetilde{F}u^*$, and consider the homotopy

$$\left(\widetilde{E} \oplus \widehat{\mathcal{H}}_B, \widetilde{\pi} \oplus \rho, \begin{pmatrix} \cos(\frac{t\pi}{2}) & -u^* \sin(\frac{t\pi}{2}) \\ u \sin(\frac{t\pi}{2}) & \cos(\frac{t\pi}{2}) \end{pmatrix} \begin{pmatrix} \widetilde{F} & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} \cos(\frac{t\pi}{2}) & u^* \sin(\frac{t\pi}{2}) \\ -u \sin(\frac{t\pi}{2}) & \cos(\frac{t\pi}{2}) \end{pmatrix}\right) \tag{3.15}$$

between $(E, \pi, F) \oplus (\widehat{\mathcal{H}}_B, 0, 0) = (\widetilde{E}, \widetilde{\pi}, \widetilde{F})$ and $(\widehat{\mathcal{H}}_B, \rho, G)$. Above, J denotes the operator

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

defined on $\widehat{\mathcal{H}}_B$.

One says that two Kasparov modules $(E_i, \pi_i, F_i) \in E(A, B)$, $i = 1, 2$, are unitarily equivalent when there exists a grading-preserving isometry $v : E_1 \to E_2$ such that

$$vF_1v^* = F_2 \quad \text{and} \quad \forall a \in A,\ v\pi_1(a)v^* - \pi_2(a) \in \mathcal{K}(\mathcal{E}_2).$$

Unitarily equivalent Kasparov modules are homotopic. Indeed, one can replace (E_i, π_i, F_i), $i = 1, 2$, by homotopically, equivalent modules $(\widehat{\mathcal{H}}_B, \rho_i, G_i)$, $i = 1, 2$. It follows from the preceding construction that the new modules $(\widehat{\mathcal{H}}_B, \rho_i, G_i)$ remain unitarily equivalent, and one immediately adapts (3.15) to a homotopy between then.

3.6.3.6 Relationship with ordinary K-theory

Let B be a unital C^*-algebra. A finitely generated ($\mathbb{Z}/2\mathbb{Z}$-graded) projective B-module $\mathcal{E}$ is a submodule of some $B^N \oplus B^N$ and can then be endowed with a Hilbert

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B-module structure. On the other hand, $\mathrm{Id}_{\mathcal{E}}$ is a compact morphism (Proposition 3.5.26); thus

$$(\mathcal{E}, \iota, 0) \in E(\mathbb{C}, B),$$

where ι is just multiplication by complex numbers. This provides a group homomorphism $K_0(B) \to KK(\mathbb{C}, B)$.

Conversely, let $(\mathcal{E}, 1, F) \in E(\mathbb{C}, B)$ be any Kasparov module where we have chosen F with closed range (see Theorem 3.5.28): $\ker F$ is then a finitely generated $\mathbb{Z}/2\mathbb{Z}$-graded projective B-module. Consider $\widetilde{\mathcal{E}} = \{\xi \in C([0,1], \mathcal{E}) \mid \xi(1) \in \ker F\}$ and $\widetilde{F}(\xi) : t \mapsto F(\xi(t))$. The triple $(\widetilde{\mathcal{E}}, 1, \widetilde{F})$ provides a homotopy between $(\mathcal{E}, 1, F)$ and $(\ker F, 1, 0)$. This also gives an inverse of the previous group homomorphism.

3.6.3.7 A nontrivial generator of $KK(\mathbb{C}, \mathbb{C})$

In the special case $B = \mathbb{C}$, we get $KK(\mathbb{C}, \mathbb{C}) \simeq K_0(\mathbb{C}) \simeq \mathbb{Z}$, and under this isomorphism, the triple

$$\left(L^2(\mathbb{R})^2, 1, \frac{1}{\sqrt{1+H}} \begin{pmatrix} 0 & -\partial_x + x \\ \partial_x + x & 0 \end{pmatrix}\right), \qquad \text{where} \quad H = -\partial_x^2 + x^2, \tag{3.16}$$

corresponds to $+1$. The reader can check as an exercise that $\partial_x + x$ and H are essentially self-adjoint as unbounded operators on $L^2(\mathbb{R})$, that H has a compact resolvent and that $\partial_x + x$ has a Fredholm index equal to $+1$. It follows that the Kasparov module in (3.16) is well defined and satisfies the required claim.

3.6.4 Ungraded Kasparov modules and KK_1

Triples $(\mathcal{E}, \pi, F)$ satisfying the properties (3.12) can arise with no natural grading for $\mathcal{E}$, and consequently with no diagonal–antidiagonal decompositions for π, F. We refer to those as ungraded Kasparov A–B-modules, and the corresponding set is denoted by $E^1(A, B)$. The direct sum is defined in the same way, as well as the homotopy, which this time is an element of $E^1(A, B[0,1])$. The homotopy defines an equivalence relation on $E^1(A, B)$, and the quotient inherits an abelian group structure as before.

Let C_1 be the complex Clifford algebra of the vector space $\mathbb{C}$ provided with the obvious quadratic form [33]. It is the C^*-algebra $\mathbb{C} \oplus \varepsilon\mathbb{C}$ generated by ε satisfying $\varepsilon^* = \varepsilon$ and $\varepsilon^2 = 1$. Assigning to ε the degree 1 yields a $\mathbb{Z}/2\mathbb{Z}$-grading on C_1. We have:

Proposition 3.6.3 *The map*

$$\begin{aligned} E^1(A, B) &\longrightarrow E(A, B \otimes C_1), \\ (\mathcal{E}, \pi, F) &\longmapsto (\mathcal{E} \otimes C_1, \pi \otimes \mathrm{Id}, F \otimes \varepsilon) \end{aligned} \tag{3.17}$$

induces an isomorphism between the quotient of $E^1(A, B)$ under homotopy and $KK_1(A, B) = KK(A, B \otimes C_1)$.

Proof The grading of C_1 gives the one of $\mathcal{E} \otimes C_1$, and the map (3.17) easily gives a homomorphism c from $KK_1(A, B)$ to $KK(A, B \otimes C_1)$.

Now let $y = (\mathcal{E}, \pi, F) \in E(A, B \otimes C_1)$. Multiplication by ε on the right of $\mathcal{E}$ makes sense, even if B is not unital, and one has $\mathcal{E}_1 = \mathcal{E}_0\varepsilon$. It follows that $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1 \simeq \mathcal{E}_0 \oplus \mathcal{E}_0$ and that any $T \in \mathrm{Mor}(\mathcal{E})$, thanks to the $B \otimes C_1$-linearity, has the following expression:

$$T = \begin{pmatrix} Q & P \\ P & Q \end{pmatrix}, \qquad P, Q \in \mathrm{Mor}_B(\mathcal{E}_0).$$

Thus $F = \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix}$, $\pi = \begin{pmatrix} \pi_0 & 0 \\ 0 & \pi_0 \end{pmatrix}$ and $c^{-1}[y] = [\mathcal{E}_0, \pi_0, P]$. □

Remark 3.6.4 The opposite of $(\mathcal{E}, \pi, F)$ in $KK_1(A, B)$ is represented by $(\mathcal{E}, \pi, -F)$. One may wonder why we have to decide if a Kasparov module is graded or not. Actually, if we forget the $\mathbb{Z}/2\mathbb{Z}$ grading of a graded Kasparov A–B-module $x = (\mathcal{E}, \pi, F)$ and consider it as an ungraded module, then we get the trivial class in $KK_1(A, B)$. Let us prove this claim.

The grading of x implies that $\mathcal{E}$ has a decomposition $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$ for which F has degree 1, that is,

$$F = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix}.$$

Now

$$G_t = \cos(t\pi/2)\, F + \sin(t\pi/2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.18}$$

provides an homotopy in KK_1 between x and

$$\left(\mathcal{E}, \pi, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right).$$

Because the latter is degenerate, the claim is proved.

Example 3.6.5 Take again the example of the Dirac operator $D\!\!\!/$ introduced in Section 3.6.3.2 on a spin^c manifold X whose dimension is odd. There is no natural

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$\mathbb{Z}/2\mathbb{Z}$ grading for the spinor bundle. The previous triple $x_{\not{D}}$ provides this time an interesting class in $E^1(C(X), \mathbb{C})$.

3.6.5 The Kasparov product

In this subsection we construct the product

$$KK(A, B) \otimes KK(B, C) \to KK(A, C).$$

It satisfies the properties given in Section 3.4. Actually:

Theorem 3.6.6 *Let $x = (\mathcal{E}, \pi, F)$ in $E(A, B)$ and $x = (\mathcal{E}', \pi', F')$ in $E(B, C)$ be two Kasparov modules. Set*

$$\mathcal{E}'' = \mathcal{E} \otimes_B \mathcal{E}'$$

and

$$\pi'' = \pi \otimes 1.$$

Then there exists a unique – up to homotopy – F'-connection on $\mathcal{E}''$ denoted by F'' such that

- $(\mathcal{E}'', \pi'', F'') \in E(A, C)$,
- $\pi''(a)\left[F'', F \otimes 1\right]\pi''(a)$ *is nonnegative modulo $\mathcal{K}(\mathcal{E}'')$ for all $a \in A$.*

$(\mathcal{E}'', \pi'', F'')$ is the Kasparov product of x and x'. It enjoys all the properties described in Section 3.4.

Proof Idea of the proof: We only explain the construction of the operator F''. For a complete proof, see for instance [14, 30]. A naive idea for F'' could be $F \otimes 1 + 1 \otimes F'$, but the trouble is that the operator $1 \otimes F'$ is in general not well defined. We can overcome this first difficulty by replacing the not well defined $1 \otimes F'$ by any F'-connection G on $\mathcal{E}''$, and try $F \otimes 1 + G$. We then stumble on a second problem, namely that the properties of Kasparov module are not satisfied in general with this candidate for F'': for instance, $(F^2 - 1) \otimes 1 \in \mathcal{K}(\mathcal{E}) \otimes 1 \not\subset \mathcal{K}(\mathcal{E}'')$ as soon as $\mathcal{E}''$ is not finitely generated.

The case of tensor products of elliptic self-adjoint differential operators on a closed manifold M gives us a hint towards the right way. If D_1 and D_2 are two such operators and H_1, H_2 the natural L^2 spaces on which they act, then the bounded operator on $H_1 \otimes H_2$ given by

$$\frac{D_1}{\sqrt{1 + D_1^2}} \otimes 1 + 1 \otimes \frac{D_2}{\sqrt{1 + D_2^2}} \tag{3.19}$$

inherits the same problem as $F \otimes 1 + G$, but

$$D'' := \frac{1}{\sqrt{2 + D_1^2 \otimes 1 + 1 \otimes D_2^2}}(D_1 \otimes 1 + 1 \otimes D_2)$$

has better properties: $D''^2 - 1$ and $[C(M), D'']$ belong to $\mathcal{K}(H_1 \otimes H_2)$. Note that

$$D'' = \sqrt{M} \cdot \frac{D_1}{\sqrt{1 + D_1^2}} \otimes 1 + \sqrt{N} \cdot 1 \otimes \frac{D_2}{\sqrt{1 + D_2^2}}$$

with

$$M = \frac{1 + D_1^2 \otimes 1}{2 + D_1^2 \otimes 1 + 1 \otimes D_2^2} \quad \text{and} \quad N = \frac{1 + 1 \otimes D_2^2}{2 + D_1^2 \otimes 1 + 1 \otimes D_2^2}.$$

The operators M, N are bounded on $H_1 \otimes H_2$, are positive, and satisfy $M + N = 1$. We thus see that in that case, the naive idea (3.19) can be corrected by combining the operators involved with some adequate "partition of unity."

Turning back to our problem, this calculation leads us to look for an adequate operator F'' in the following form:

$$F'' = \sqrt{M} \cdot F \otimes 1 + \sqrt{N} G.$$

We need to have that F'' is a F'-connection, and satisfies $a \cdot (F''^2 - 1) \in \mathcal{K}(E'')$ and $[a, F''] \in \mathcal{K}(E'')$ for all $a \in A$ (by a we mean $\pi''(a)$). Using the previous form for F'', a small computation shows that these assertions become true if all the following conditions hold:

($\imath$) M is a 0-connection (equivalently, N is a 1-connection),
($\imath\imath$) $[M, F \otimes 1]$, $N \cdot [F \otimes 1, G]$, $[G, M]$, $N(G^2 - 1)$ belong to $\mathcal{K}(E'')$,
($\imath\imath\imath$) $[a, M]$, $N \cdot [G, a]$ belong to $\mathcal{K}(E'')$.

At this point there is a miracle:

Theorem 3.6.7 (Kasparov's technical theorem) *Let J be a C^*-algebra, and denote by $\mathcal{M}(J)$ its multiplier algebra. Assume there are two subalge-bras A_1, A_2 of $\mathcal{M}(J)$ and a linear subspace $\triangle \subset \mathcal{M}(J)$ such that*

$$A_1 A_2 \subset J,$$

$$[\triangle, A_1] \subset J.$$

Then there exist two nonnegative elements $M, N \in \mathcal{M}(J)$ with $M + N = 1$ such that

$$M A_1 \subset J,$$
$$N A_2 \subset J,$$
$$[M, \triangle] \subset J.$$

For a proof, see [25].

Now, to get $(\imath)$, $(\imath\imath)$, $(\imath\imath\imath)$, we apply this theorem with

$$A_1 = C^*\langle \mathcal{K}(\mathcal{E}) \otimes 1,\ \mathcal{K}(\mathcal{E}'')\rangle,$$
$$A_2 = C^*\langle G^2 - 1,\ [G, F \otimes 1],\ \big[G, \pi''\big]\rangle,$$
$$\triangle = Vect\langle \pi''(A),\ G,\ F \otimes 1\rangle.$$

This gives us the correct F''. □

3.6.6 Equivalence and duality in KK-theory

With the Kasparov product come the following notions:

Definition 3.6.8 Let A, B be two C^*-algebras.

- One says that A and B are KK*-equivalent* if there exist $\alpha \in KK(A, B)$ and $\beta \in KK(B, A)$ such that

  $$\alpha \otimes \beta = 1_A \in KK(A, A) \quad \text{and} \quad \beta \otimes \alpha = 1_B \in KK(B, B).$$

  In that case, the pair (α, β) is called a *KK-equivalence*, and it gives rise to isomorphisms

  $$KK(A \otimes C, D) \simeq KK(B \otimes C, D) \quad \text{and} \quad KK(C, A \otimes D) \simeq KK(C, B \otimes D)$$

  given by Kasparov products for all C^*-algebras C, D.
- One says that A and B are KK*-dual (or Poincaré dual)* if there exist $\delta \in KK(A \otimes B, \mathbb{C})$ and $\lambda \in KK(\mathbb{C}, A \otimes B)$ such that

  $$\lambda \underset{B}{\otimes} \delta = 1 \in KK(A, A) \quad \text{and} \quad \lambda \underset{A}{\otimes} \delta = 1 \in KK(B, B).$$

  In that case, the pair (λ, δ) is called a KK-duality, and it gives rise to isomorphisms

  $$KK(A \otimes C, D) \simeq KK(C, B \otimes D) \quad \text{and} \quad KK(C, A \otimes D) \simeq KK(B \otimes C, B \otimes D)$$

  given by Kasparov products for all C^*-algebras C, D.

We continue this subsection with classical computations illustrating these notions.

3.6.6.1 Bott periodicity

Let $\beta \in KK(\mathbb{C}, C_0(\mathbb{R}^2))$ be represented by the Kasparov module:

$$(\mathcal{E}, \pi, C) = \left(C_0(\mathbb{R}^2) \oplus C_0(\mathbb{R}^2),\ 1,\ \frac{1}{\sqrt{1+c^2}} \begin{pmatrix} 0 & c_- \\ c_+ & 0 \end{pmatrix} \right),$$

where c_+, c_- are the operators given by pointwise multiplication by $x - \imath y$ and $x + \imath y$, respectively, and

$$c = \begin{pmatrix} 0 & c_- \\ c_+ & 0 \end{pmatrix}.$$

Let $\alpha \in KK(C_0(\mathbb{R}^2), \mathbb{C})$ be represented by the Kasparov module:

$$(\mathcal{H}, \pi, F) = \left(L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2),\ \pi,\ \frac{1}{\sqrt{1+D^2}} \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \right),$$

where $\pi : C_0(\mathbb{R}^2) \to \mathcal{L}(L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2))$ is the action given by multiplication of functions, the operators D_+ and D_- are given by

$$D_+ = \partial_x + \imath \partial y,$$
$$D_- = -\partial_x + \imath \partial y,$$

and $D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$.

Theorem 3.6.9 *α and β provide a KK-equivalence between $C_0(\mathbb{R}^2)$ and $\mathbb{C}$.*

This is the Bott periodicity theorem in the bivariant K-theory framework.

Proof Let us begin with the computation of $\beta \otimes \alpha \in KK(\mathbb{C}, \mathbb{C})$. We have an identification

$$\mathcal{E} \underset{C_0(\mathbb{R}^2)}{\otimes} \mathcal{H} \simeq \mathcal{H} \oplus \mathcal{H}, \tag{3.20}$$

where on the right, the first copy of $\mathcal{H}$ stands for

$$\mathcal{E}_0 \underset{C_0(\mathbb{R}^2)}{\otimes} \mathcal{H}_0 \oplus \mathcal{E}_1 \underset{C_0(\mathbb{R}^2)}{\otimes} \mathcal{H}_1$$

and the second for

$$\mathcal{E}_0 \underset{C_0(\mathbb{R}^2)}{\otimes} \mathcal{H}_1 \oplus \mathcal{E}_1 \underset{C_0(\mathbb{R}^2)}{\otimes} \mathcal{H}_0.$$

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One checks directly that under this identification the operator

$$G = \frac{1}{\sqrt{1+D^2}} \begin{pmatrix} 0 & 0 & D_- & 0 \\ 0 & 0 & 0 & -D_+ \\ D_+ & 0 & 0 & 0 \\ 0 & -D_- & 0 & 0 \end{pmatrix} \tag{3.21}$$

is an F-connection. On the other hand, under the identification (3.20), the operator $C \otimes 1$ gives

$$\frac{1}{\sqrt{1+c^2}} \begin{pmatrix} 0 & 0 & 0 & c_- \\ 0 & 0 & c_+ & 0 \\ 0 & c_- & 0 & 0 \\ c_+ & 0 & 0 & 0 \end{pmatrix}. \tag{3.22}$$

It immediately follows that $\beta \otimes \alpha$ is represented by

$$\delta = \left(\mathcal{H} \oplus \mathcal{H}, 1, \frac{1}{\sqrt{1+c^2+D^2}} \mathbf{D} \right), \tag{3.23}$$

where

$$\mathbf{D} = \begin{pmatrix} 0 & \mathbf{D}_- \\ \mathbf{D}_+ & 0 \end{pmatrix}$$

with

$$\mathbf{D}_+ = \begin{pmatrix} D_+ & c_- \\ c_+ & -D_- \end{pmatrix} \quad \text{and} \quad \mathbf{D}_- = \mathbf{D}_+^*.$$

Observe that, denoting by ρ the rotation in $\mathbb{R}^2$ of angle $\pi/4$, we have

$$\begin{pmatrix} \rho^{-1} & 0 \\ 0 & \rho \end{pmatrix} \begin{pmatrix} 0 & \mathbf{D}_- \\ \mathbf{D}_+ & 0 \end{pmatrix} \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \rho^{-1}\mathbf{D}_-\rho^{-1} \\ \rho\mathbf{D}_+\rho & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \imath(\partial_y - y) & -\partial_x + x \\ 0 & 0 & \partial_x + x & -\imath(\partial_y + y) \\ \imath(\partial_y + y) & -\partial_x + x & 0 & 0 \\ \partial_x + x & \imath(-\partial_y + y) & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & x - \partial_x \\ x + \partial_x & 0 \end{pmatrix} \otimes 1 + 1 \otimes \begin{pmatrix} 0 & \imath(\partial_y - y) \\ \imath(\partial_y + y) & 0 \end{pmatrix}.$$

Of course,

$$\delta \sim_h \left(\mathcal{H} \oplus \mathcal{H}, 1, \frac{1}{\sqrt{1+c^2+D^2}} \begin{pmatrix} 0 & \rho^{-1}\mathbf{D}_-\rho^{-1} \\ \rho\mathbf{D}_+\rho & 0 \end{pmatrix} \right),$$

and the preceding computation shows that δ coincides with the Kasparov product $u \otimes u$ with $u \in KK(\mathbb{C}, \mathbb{C})$ given by

$$u = \left(L^2(\mathbb{R})^2, 1, \frac{1}{\sqrt{1 + x^2 + \partial_x^2}} \begin{pmatrix} 0 & x - \partial_x \\ x + \partial_x & 0 \end{pmatrix} \right).$$

A simple exercise shows that $\partial_x + x : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is essentially self-adjoint with one-dimensional kernel and zero-dimensional cokernel; thus $1 = u = u \otimes u \in KK(\mathbb{C}, \mathbb{C})$.

Let us turn to the computation of $\alpha \otimes \beta \in KK(C_0(\mathbb{R}^2), C_0(\mathbb{R}^2))$. It is a Kasparov product over $\mathbb{C}$, and thus it commutes:

$$\alpha \otimes \beta = \tau_{C_0(\mathbb{R}^2)}(\beta) \otimes \tau_{C_0(\mathbb{R}^2)}(\alpha), \tag{3.24}$$

but we must observe that the two copies of $C_0(\mathbb{R}^2)$ in $\tau_{C_0(\mathbb{R}^2)}(\beta)$ and $\tau_{C_0(\mathbb{R}^2)}(\alpha)$ play different roles: on should think of the first copy as functions of the variable $u \in \mathbb{R}^2$, and of the second as functions of the variable $v \in \mathbb{R}^2$. It follows that one cannot directly factorize $\tau_{C_0(\mathbb{R}^2)}$ on the right-hand side of (3.24) in order to use the value of $\beta \otimes \alpha$. This is where a classical argument, known as the rotation trick of Atiyah, is necessary:

Lemma 3.6.10 *Let $\phi : C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}^2) \to C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}^2)$ be the flip automorphism: $\phi(f)(u, v) = f(v, u)$. Then*

$$[\phi] = 1 \in KK(C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}^2), C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}^2)).$$

Proof Proof of the lemma: Let us denote by I_2 the identity matrix of $M_2(\mathbb{R})$. Use a continuous path of isometries of $\mathbb{R}^4$ connecting

$$\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}.$$

This gives a homotopy $(C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}^2), \phi, 0) \sim_h (C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}^2), \mathrm{Id}, 0)$. □

Now

$$\begin{aligned} \alpha \otimes \beta &= \tau_{C_0(\mathbb{R}^2)}(\beta) \otimes \tau_{C_0(\mathbb{R}^2)}(\alpha) = \tau_{C_0(\mathbb{R}^2)}(\beta) \otimes [\phi] \otimes \tau_{C_0(\mathbb{R}^2)}(\alpha) \\ &= \tau_{C_0(\mathbb{R}^2)}(\beta \otimes \alpha) = \tau_{C_0(\mathbb{R}^2)}(1) = 1 \in KK(C_0(\mathbb{R}^2), C_0(\mathbb{R}^2)). \end{aligned} \tag{3.25}$$

□

3.6.6.2 Self-duality of $C_0(\mathbb{R})$

With the same notation as before, we get:

Corollary 3.6.11 *The algebra $C_0(\mathbb{R})$ is Poincaré dual to itself.*

Other examples of Poincaré-dual algebras will be given later.

Proof The automorphism ψ of $C_0(\mathbb{R})^{\otimes^3}$ given by $\psi(f)(x, y, z) = f(z, x, y)$ is homotopic to the identity; thus

$$\begin{aligned} \beta \underset{C_0(\mathbb{R})}{\otimes} \alpha &= \tau_{C_0(\mathbb{R})}(\beta) \otimes \tau_{C_0(\mathbb{R})}(\alpha) = \tau_{C_0(\mathbb{R})}(\beta) \otimes [\psi] \otimes \tau_{C_0(\mathbb{R})}(\alpha) \\ &= \tau_{C_0(\mathbb{R})}(\beta \otimes \alpha) = \tau_{C_0(\mathbb{R})}(1) = 1 \in KK(C_0(\mathbb{R}), C_0(\mathbb{R})). \end{aligned} \quad (3.26)$$

□

Exercise 3.6.12 With $C_1 = \mathbb{C} \oplus \varepsilon\mathbb{C}$ the Clifford algebra of $\mathbb{C}$, consider

$$\beta_c = \left(C_0(\mathbb{R}) \otimes C_1, 1, \frac{x}{\sqrt{x^2+1}} \otimes \varepsilon \right) \in KK(\mathbb{C}, C_0(\mathbb{R}) \otimes C_1),$$

$$\alpha_c = \left(L^2(\mathbb{R}, \Lambda^*\mathbb{R}), \pi, \frac{1}{\sqrt{1+\Delta}}(d+\delta) \right) \in KK(C_0(\mathbb{R}) \otimes C_1, \mathbb{C}),$$

where $(d+\delta)(a+bdx) = -b' + a'dx$, $\Delta = (d+\delta)^2$, and $\pi(f \otimes \varepsilon)$ sends $a + bdx$ to $f(b + adx)$. Show that β_c, α_c provide a KK-equivalence between $\mathbb{C}$ and $C_0(\mathbb{R}) \otimes C_1$. (Hints: compute directly $\beta_c \otimes \alpha_c$; then use the commutativity of the Kasparov product over $\mathbb{C}$ and check that the flip of $(C_0(\mathbb{R}) \otimes C_1)^{\otimes^2}$ is 1 to conclude about the computation of $\alpha_c \otimes \beta_c$.)

3.6.6.3 A simple Morita equivalence

Let $\imath_n = (M_{1,n}(\mathbb{C}), 1, 0) \in E(\mathbb{C}, M_n(\mathbb{C}))$, where the $M_n(\mathbb{C})$-module structure is given by multiplication by matrices on the right. Note that $[\imath_n]$ is also the class of the homomorphism $\mathbb{C} \to M_n(\mathbb{C})$ given by the upper left corner inclusion. Let also $\jmath_n = (M_{n,1}(\mathbb{C}), m, 0) \in E(M_n(\mathbb{C}), \mathbb{C})$, where m is multiplication by matrices on the left. It follows in a straightforward way that

$$\imath_n \otimes \jmath_n \sim_h (\mathbb{C}, 1, 0) \quad \text{and} \quad \jmath_n \otimes \imath_n \sim_h (M_n(\mathbb{C}), 1, 0);$$

thus $\mathbb{C}$ and $M_n(\mathbb{C})$ are KK-equivalent, and this is an example of a Morita equivalence. The map in K-theory associated with $\jmath : \cdot \otimes \jmath_n : K_0(M_n(\mathbb{C})) \to \mathbb{Z}$ is just the trace homomorphism. Similarly, let us consider the Kasparov elements $\imath \in E(\mathbb{C}, \mathcal{K}(\mathcal{H}))$ associated to the homomorphism $\imath : \mathbb{C} \to \mathcal{K}(\mathcal{H})$ given by the choice of a rank one projection, and $\jmath = (\mathcal{H}, m, 0) \in E(\mathcal{K}(\mathcal{H}), \mathbb{C})$, where m is just the action of compact operators on $\mathcal{H}$: they provide a KK-equivalence between $\mathcal{K}$ and $\mathbb{C}$.

3.6.6.4 $C_0(\mathbb{R})$ and C_1.

We leave the proof of the following result as an exercise:

Proposition 3.6.13 *The algebras $C_0(\mathbb{R})$ and C_1 are KK-equivalent.*

Proof Hint: Consider

$$\widetilde{\alpha} = \left(L^2(\mathbb{R}, \Lambda^*\mathbb{R}), m, \frac{1}{\sqrt{1+\Delta}}(d+\delta)\right) \in KK(C_0(\mathbb{R}), C_1),$$

where d, δ, Δ are defined in the previous exercise, $m(f)(\xi) = f\xi$, and the C_1-right module structure of $L^2(\mathbb{R}, \Lambda^*\mathbb{R})$ is given by $(a + bdx) \cdot \varepsilon = -ib + iadx$. Consider also

$$\widetilde{\beta} = \left(C_0(\mathbb{R})^2, \varphi, \frac{x}{\sqrt{1+x^2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \in KK(C_1, C_0(\mathbb{R})),$$

where $\varphi(\varepsilon)(f, g) = (-ig, if)$. Prove that they provide the desired *KK*-equivalence. □

Exercise 3.6.14

(i) Check that $\tau_{C_1} : KK(A, B) \to KK(A \otimes C_1, B \otimes C_1)$ is an isomorphism.
(ii) Check that under τ_{C_1} and the Morita equivalence $M_2(\mathbb{C}) \sim \mathbb{C}$, the elements α_c, β_c of the previous exercise coincide with $\widetilde{\alpha}, \widetilde{\beta}$ and recover the *KK*-equivalence between C_1 and $C_0(\mathbb{R})$.

Remark 3.6.15 At this point, one sees that $KK_1(A, B) = KK(A, B(\mathbb{R}))$ $(B(\mathbb{R}) := C_0(\mathbb{R}) \otimes B)$ can also be presented in the following different ways:

$$E_1(A, B)/\sim_h \simeq KK(A, B \otimes C_1) \simeq KK(A \otimes C_1, B) \simeq KK(A(\mathbb{R}), B).$$

3.6.7 Computing the Kasparov product without its definition

Computing the product of two Kasparov modules is in general quite hard, but we are often in one of the following situations.

3.6.7.1 Use of the functorial properties

Thanks to the functorial properties listed in Section 3.4, many products can be deduced from known, already computed ones. For instance, in the proof of Bott periodicity (the *KK*-equivalence between $\mathbb{C}$ and $C_0(\mathbb{R}^2)$) one had to compute two products: the first one was directly computed; the second one was deduced from the first using the properties of the Kasparov product and a simple geometric fact. There are numerous examples of this kind.

3.6.7.2 Maps between K-theory groups

Let A, B be two unital (if not, add a unit) C^*-algebras, let $x \in KK(A, B)$ be given by a Kasparov module $(\mathcal{E}, \pi, F)$ where F has a closed range, and assume that we

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are interested in the map $\phi_x : K_0(A) \to K_0(B)$ associated with x in the following way:

$$y \in K_0(A) \simeq KK(\mathbb{C}, A), \qquad \phi_x(y) = y \otimes x.$$

This product takes a particularly simple form when y is represented by $(\mathcal{P}, 1, 0)$ with $\mathcal{P}$ a finitely generated projective A-module (see Section 3.6.3.6):

$$y \otimes x = \left(\mathcal{P} \underset{A}{\otimes} \mathcal{E}, 1 \otimes \pi, G\right) = (\ker(G), 1, 0),$$

where G is an arbitrary F-connection.

3.6.7.3 Kasparov elements constructed from homomorphisms

Sometimes, Kasparov classes $y \in KK(B, C)$ can be explicitly represented as Kasparov products of classes of homomorphisms with inverses of such classes. Assume for instance that $y = [e_0]^{-1} \otimes [e_1]$, where $e_0 : \mathcal{C} \to B$, $e_1 : \mathcal{C} \to C$ are homomorphisms of C^*-algebras and e_0 produces an invertible element in KK-theory (for instance, $\ker e_0$ is K-contractible and B is nuclear or $\mathcal{C}$, B K-nuclear; see [16,50]). Then computing a Kasparov product $x \otimes y$ where $x \in KK(A, B)$ amounts to lifting x to $KK(A, \mathcal{C})$ (that is, finding $x' \in KK(A, \mathcal{C})$ such that $(e_0)_*(x') = x$) and restricting this lift to $KK(A, C)$ (that is, evaluating $x'' = (e_1)_*(x')$). It follows from the properties of the product that $x'' = x \otimes y$.

Example 3.6.16 Consider the tangent groupoid $\mathcal{G}_{\mathbb{R}}$ of $\mathbb{R}$, and let $\delta = [e_0]^{-1} \otimes [e_1] \otimes \mu$ be the associated deformation element: $e_0 : C^*(\mathcal{G}_{\mathbb{R}}) \to C^*(T\mathbb{R}) \simeq C_0(\mathbb{R}^2)$ is evaluation at $t = 0$, $e_1 : C^*(\mathcal{G}_{\mathbb{R}}) \to C^*(\mathbb{R} \times \mathbb{R}) \simeq \mathcal{K}(L^2(\mathbb{R})) \simeq \mathcal{K}$ is evaluation at $t = 1$, and $\mu = (L^2(\mathbb{R}), m, 0) \in KK(\mathcal{K}, \mathbb{C})$ gives the Morita equivalence $\mathcal{K} \sim \mathbb{C}$.

Let $\beta \in KK(\mathbb{C}, C_0(\mathbb{R}^2))$ be the element used in Section 3.6.6.1. Then $\beta \otimes \delta \in KK(\mathbb{C}, \mathbb{C})$ is easy to compute. The lift $\beta' \in KK(\mathbb{C}, C^*(\mathcal{G}_{\mathbb{R}}))$ is produced using the pseudodifferential calculus for groupoids (see Section 3.7) and can be presented as a family $\beta' = (\beta_t)$ with

$$\beta_0 = \beta, \qquad t > 0,$$
$$\beta_t = \left(C^*\left(\mathbb{R} \times \mathbb{R}, \frac{dx}{t}\right), 1, \frac{1}{\sqrt{1 + x^2 + t^2\partial_x^2}} \begin{pmatrix} 0 & x - t\partial_x \\ x + t\partial_x & 0 \end{pmatrix}\right).$$

After restricting at $t = 1$ and applying the Morita equivalence, only the index of the Fredholm operator appearing in β_1 remains – that is, $+1$ – and this proves $\beta \otimes \delta = 1$.

Observe that by uniqueness of the inverse, we can conclude that $\delta = \alpha$ in $KK(C_0(\mathbb{R}^2), \mathbb{C})$.

Example 3.6.17 (Boundary homomorphisms in long exact sequences.) Let

$$0 \to I \underset{i}{\to} A \underset{p}{\to} B \to 0$$

be a short exact sequence of C^*-algebras. We assume that either it admits a completely positive, norm-decreasing linear section or I, A, B are K-nuclear [50]. Let $C_p = \{(a,\varphi) \in A \oplus C_0([0,1[, B) \mid p(a) = \varphi(0)\}$ be the cone of the homomorphism $p : A \to B$, and denote by d the homomorphism $C_0(]0,1[, B) \hookrightarrow C_p$ given by $d(\varphi) = (0,\varphi)$, and by e the homomorphism $I \to C_p$ given by $e(a) = (a, 0)$. Thanks to the hypotheses, $[e]$ is invertible in KK-theory. One can set $\delta = [d] \otimes [e]^{-1} \in KK(C_0(\mathbb{R}) \otimes B, I)$ and use the Bott periodicity $C_0(\mathbb{R}^2) \underset{\text{KK}}{\sim} \mathbb{C}$ in order to identify

$$KK_2(C, D) = KK(C_0(\mathbb{R}^2) \otimes C, D) \simeq KK(C, D).$$

Then the connecting maps in the long exact sequences

$$\begin{aligned} &\cdots \to KK_1(I, D) \to KK(B, D) \\ &\overset{i^*}{\to} KK(A, D) \overset{p^*}{\to} KK(I, D) \to KK_1(B, D) \to \cdots, \\ &\cdots \to KK_1(C, B) \to KK(C, I) \\ &\overset{i_*}{\to} KK(C, A) \overset{p_*}{\to} KK(C, B) \to KK_1(C, I) \to \cdots \end{aligned}$$

are given by the appropriate Kasparov products with δ.

III. Index Theorems

3.7 Introduction to pseudodifferential operators on groupoids

The historical motivation for developing pseudodifferential calculus on groupoids comes from Connes, who implicitly introduced this notion for foliations. Later on, this calculus was axiomatized and studied on general groupoids by several authors [38, 39, 52].

The following example illustrates how pseudodifferential calculus on groupoids arises in our approach to index theory. If P is a partial differential operator on $\mathbb{R}^n$,

$$P(x, D) = \sum_{|\alpha| \leq d} c_\alpha(x) D_x^\alpha,$$

we may associate to it the asymptotic operator

$$P(x, tD) = \sum_{|\alpha| \leq d} c_\alpha(x) (t D_x)^\alpha$$

by introducing a parameter $t \in]0, 1]$ in front of each ∂_{x_j}. Here we use the usual convention: $D_x^\alpha = (-i\partial_{x_1})^{\alpha_1} \cdots (-i\partial_{x_n})^{\alpha_n}$. We would like to give an (interesting) meaning to the limit $t \to 0$. Of course we would not be happy with $tD \to 0$.

To investigate this question, let us look at $P(x, tD)$ as a left multiplier on $C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times]0, 1])$ rather as a linear operator on $C^\infty(\mathbb{R}^n)$:

$$\begin{aligned}
P(x, tD_x)u(x, y, t) &= \int e^{(x-z)\cdot\xi} P(x, t\xi)u(z, y, t)\,dzd\xi \\
&= \int e^{\frac{x-z}{t}\cdot\xi} P(x, \xi)u(z, y, t)\frac{dzd\xi}{t^n} \\
&= \int e^{(X-Z).\xi} P(x, \xi)u(x - t(X - Z), x - tX, t)\,dZd\xi.
\end{aligned}$$

In the last line we introduced the notation $X = \frac{x-y}{t}$ and performed the change of variables $Z = \frac{z-y}{t}$.

At this point, assume that u has the following behavior near $t = 0$:

$$u(x, y, t) = \widetilde{u}\left(y, \frac{x-y}{t}, t\right), \qquad \text{where } \widetilde{u} \in C^\infty(\mathbb{R}^{2n} \times [0, 1]).$$

It follows that

$$\begin{aligned}
P(x, tD_x)u(x, x - tX, t) &= \int e^{(X-Z)\cdot\xi} P(x, \xi)\widetilde{u}(x - tX, Z, t)\,dZd\xi \\
&\xrightarrow{t\to 0} \int e^{(X-Z)\cdot\xi} P(x, \xi)\widetilde{u}(x, Z, 0)\,dZd\xi \\
&= P(x, D_X)\widetilde{u}(x, X, 0).
\end{aligned}$$

Some observations:

- $P(x, D_X)$ is a partial differential operator in the variable X with constant coefficients, depending smoothly on a parameter x and with symbol coinciding with that of $P(x, D_x)$, in the sense that $\sigma(P(x, D_X)(x, X, \xi) = P(x, \xi)$. In particular, $P(x, D_X)$ is invariant under the translation $X \mapsto X + X_0$. Of course, $P(x, D_X)$ is nothing else, up to a Fourier transformation in X, than the symbol $P(x, \xi)$ of $P(x, D_x)$. In other words, denoting by $S_X(T\mathbb{R}^n)$ the space of smooth functions $f(x, X)$ rapidly decreasing in X and by $\mathcal{F}_X$ the Fourier transform with respect to the variable X, we have a commutative diagram

$$\begin{array}{ccc}
S_X(T\mathbb{R}^n) & \xrightarrow{P(x,D_X)} & S_X(T\mathbb{R}^n) \\
{\scriptstyle \mathcal{F}_X}\big\downarrow & & {\scriptstyle \mathcal{F}_X}\big\downarrow \\
S_\xi(T^*\mathbb{R}^n) & \xrightarrow{P(x,\xi)} & S_\xi(T^*\mathbb{R}^n)
\end{array}$$

where $P(x, D_X)$ acts as a left multiplier on the convolution algebra $S_X(T\mathbb{R}^n)$, and $P(x, \xi)$ acts as a left multiplier on the function algebra $S_\xi(T^*\mathbb{R}^n)$ (equipped with the pointwise multiplication of functions).

- u and $\widetilde{u}$ are related by the bijection

$$\begin{aligned}\phi : \mathbb{R}^{2n} \times [0, 1] &\longrightarrow \mathcal{G}_{\mathbb{R}^n},\\ (x, X, t) &\longmapsto (x - tX, x, t) \qquad \text{if } t > 0,\\ (x, X, 0) &\longmapsto (x, X, 0)\end{aligned}$$

($\phi^{-1}(x, y, t) = (y, (x - y)/t, t)$, $\phi^{-1}(x, X, 0) = (x, X, 0)$). In fact, the smooth structure of the tangent groupoid $\mathcal{G}_{\mathbb{R}^n}$ of the manifold $\mathbb{R}^n$ (see Section 3.2.7) is defined by requiring that ϕ be a diffeomorphism. Thus $\widetilde{u} \in C^\infty(\mathbb{R}^{2n} \times [0, 1])$ means $u \in C^\infty(\mathcal{G}_{\mathbb{R}^n})$.

Thus $P(x, D_X)$ is another way to look at, and even another way to define, the symbol of $P(x, D_x)$. What is important for us is that it arises as a limit of a family P_t constructed with P, and the pseudodifferential calculus on the tangent groupoid of $\mathbb{R}^n$ will enable us to give a rigorous meaning to this limit and perform interesting computations.

The following material is taken from [38,39,52]. Let G be a Lie groupoid, with unit space $G^{(0)} = V$ and with a smooth (right) Haar system $d\lambda$. We assume that V is a compact manifold and that the s-fibers G_x, $x \in V$, have no boundary. We denote by U_γ the map induced on functions by right multiplication by γ, that is,

$$U_\gamma : C^\infty(G_{s(\gamma)}) \longrightarrow C^\infty(G_{r(\gamma)});\ U_\gamma f(\gamma') = f(\gamma'\gamma).$$

Definition 3.7.1 A G-operator is a continuous linear map $P : C^\infty_c(G) \to C^\infty(G)$ such that

(i) P is given by a family $(P_x)_{x\in V}$ of linear operators $P_x : C^\infty_c(G_x) \to C^\infty(G_x)$, and

$$\forall f \in C^\infty_c(G), \quad P(f)(\gamma) = P_{s(\gamma)} f_{s(\gamma)}(\gamma),$$

where f_x stands for the restriction $f|_{G_x}$.

(ii) The following invariance property holds

$$U_\gamma P_{s(\gamma)} = P_{r(\gamma)} U_\gamma.$$

Let P be a G-operator, and denote by $k_x \in C^{-\infty}(G_x \times G_x)$ the Schwartz kernel of P_x, for each $x \in V$, as obtained from the Schwartz kernel theorem applied to the manifold G_x provided with the measure $d\lambda_x$.

Thus, using property (i) in the definition,

$$\forall \gamma \in G, f \in C^\infty(G), \quad Pf(\gamma) = \int_{G_x} k_x(\gamma, \gamma') f(\gamma') d\lambda_x(\gamma') \qquad (x = s(\gamma)).$$

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Next,

$$U_\gamma Pf(\gamma') = Pf(\gamma'\gamma) = \int_{G_x} k_x(\gamma'\gamma, \gamma'') f(\gamma'') d\lambda_x(\gamma'') \qquad (x = s(\gamma)),$$

and

$$\begin{aligned} P(U_\gamma f)(\gamma') &= \int_{G_y} k_y(\gamma', \gamma'') f(\gamma''\gamma) d\lambda_y(\gamma''), \qquad (y = r(\gamma)) \\ &\overset{\eta=\gamma''\gamma}{=} \int_{G_x} k_y(\gamma', \eta\gamma^{-1}) f(\eta) d\lambda_x(\eta), \qquad (x = s(\gamma)), \end{aligned}$$

where the last line uses the invariance property of Haar systems. Property (ii) is equivalent to the following equalities of distributions on $G_x \times G_x$, for all $x \in V$:

$$\forall \gamma \in G, \quad k_x(\gamma'\gamma, \gamma'') = k_y(\gamma', \gamma''\gamma^{-1}) \qquad (x = s(\gamma),\ y = r(\gamma)).$$

Setting $k_P(\gamma) := k_{s(\gamma)}(\gamma, s(\gamma))$, we get $k_x(\gamma, \gamma') = k_P(\gamma\gamma'^{-1})$, and the linear operator $P : C_c^\infty(G) \to C^\infty(G)$ is given by

$$P(f)(\gamma) = \int_{G_x} k_P(\gamma\gamma'^{-1}) d\lambda_x(\gamma') \qquad (x = s(\gamma)).$$

We may consider k_P as a single distribution on G acting on smooth functions on G by convolution. With a slight abuse of terminology, we will refer to k_P as the Schwartz (or convolution) kernel of P.

We say that P is *smoothing* if k_P lies in $C^\infty(G)$, and is *compactly supported* or *uniformly supported* if k_P is compactly supported (which implies that each P_x is properly supported).

Let us develop some examples of G-operators.

Examples 3.7.2

(i) If $G = G^{(0)} = V$ is just a set, then $G_x = \{x\}$ for all $x \in V$. Then in Definition 3.7.1, property (i) is empty, and property (ii) implies that a G-operator is given by pointwise multiplication by a smooth function $P \in C^\infty(V)$: $Pf(x) = P(x) \cdot f(x)$.

(ii) Let $G = V \times V$, the pair groupoid, and let the Haar system $d\lambda$ be given in the obvious way by a single measure dy on V:

$$d\lambda_x(y) = dy \qquad \text{under the identification } G_x = V \times \{x\} \simeq V.$$

It follows that for any G-operator P,

$$Pg(z, x) = \int_{V\times\{x\}} k_P(z, y) g(y, x) d\lambda_x(y, x) = \int_V k_P(z, y) g(y, x) dy,$$

which immediately proves that $P_x = P_y$ as linear operators on $C^\infty(V)$ under the obvious identifications $V \simeq V \times \{x\} \simeq V \times \{y\}$.

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(iii) Let $p : X \to Z$ be a submersion, and $G = X \underset{Z}{\times} X = \{(x, y) \in X \times X \mid p(x) = p(y)\}$ the associated subgroupoid of the pair groupoid $X \times X$. The manifold G_x can be identified with the fiber $p^{-1}(p(x))$. Property (ii) implies that for any G-operator P, we have $P_x = P_y$ as linear operators on $p^{-1}(p(x))$ as soon as $y \in p^{-1}(p(x))$. Thus, P is actually given by a family $\tilde{P}_z$, $z \in Z$, of operators on $p^{-1}(z)$, with the relation $P_x = \tilde{P}_{p(x)}$.

(iv) Let $G = E$ be the total space of a (Euclidean, Hermitian) vector bundle $p : E \to V$, with $r = s = p$. The Haar system $d_x w$, $x \in V$, is given by the metric structure on the fibers of E. We have here

$$Pf(v) = \int_{E_x} k_P(v - w) f(w) d_x w \qquad (x = p(v)).$$

Thus, for all $x \in V$, P_x is a convolution operator on the linear space E_x.

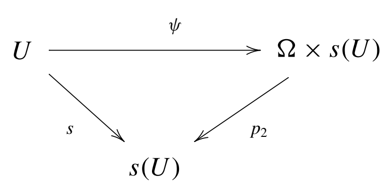
(v) Let $G = \mathcal{G}_V = TV \times \{0\} \sqcup V \times V \times \,]0, 1]$ be the tangent groupoid of V. It can be viewed as a family of groupoids G_t parametrized by $[0, 1]$, where $G_0 = TV$ and $G_t = V \times V$ for $t > 0$. A $\mathcal{G}_V$-operator is given by a family P_t of G_t-operators, and $(P_t)_{t>0}$ is a family of operators on $C_c^\infty(V)$ parametrized by t, whereas P_0 is a family of translation-invariant operators on $T_x V$ parametrized by $x \in V$. The $\mathcal{G}_V$-operators are thus a blend of examples (ii) and (iv).

We now turn to the definition of pseudodifferential operators on a Lie groupoid G.

Definition 3.7.3 A G-operator P is a G-pseudodifferential operator of order m if:

(i) The Schwartz kernel k_P is smooth outside $G^{(0)}$.

(ii) For every distinguished chart $\psi : U \subset G \to \Omega \times s(U) \subset \mathbb{R}^{n-p} \times \mathbb{R}^p$ of G,



the operator $(\psi^{-1})^* P \psi^* : C_c^\infty(\Omega \times s(U)) \to C_c^\infty(\Omega \times s(U))$ is a smooth family parametrized by $s(U)$ of pseudodifferential operators of order m on Ω.

We will use few properties of this calculus and only provide some examples and a list of properties. The reader can find a complete presentation in [37–39, 51, 52].

Examples 3.7.4 In the previous five examples (Examples 3.7.3), a G-pseudodifferential operator is:

(i) an operator given by pointwise multiplication by a smooth function on V;

(ii) a single pseudodifferential operator on V;

(iii) a smooth family parametrized by Z of pseudodifferential operators in the fibers (this coincides with the notion of [7]);
(iv) a family parametrized by $x \in V$ of convolution operators in E_x such that the underlying distribution k_P identifies with the Fourier transform of a symbol on E (that is, a smooth function on E satisfying the standard decay conditions with respect to its variable in the fibers);
(v) the data provided by an asymptotic pseudodifferential operator on V together with its complete symbol, the choice of it depending on the gluing in $\mathcal{G}_V$ (this is quite close to the notions studied in [8, 22, 23]).

It turns out that the space $\Psi_c^*(G)$ of compactly supported G-pseudodifferential operators is an involutive algebra.

The principal symbol of a G-pseudodifferential operator P of order m is defined as a function $\sigma_m(P)$ on $A^*(G) \setminus G^{(0)}$ by

$$\sigma_m(P)(x, \xi) = \sigma_{pr}(P_x)(x, \xi),$$

where $\sigma_{pr}(P_x)$ is the principal symbol of the pseudodifferential operator P_x on the manifold G_x. Conversely, suppose we are given a symbol f of order m on $A^*(G)$ together with the following data:

(i) a smooth embedding $\theta : \mathcal{U} \to AG$, where $\mathcal{U}$ is a open set in G containing $G^{(0)}$, such that $\theta(G^{(0)}) = G^{(0)}$, $(d\theta)|_{G^0} = \mathrm{Id}$, and $\theta(\gamma) \in A_{s(\gamma)}G$ for all $\gamma \in \mathcal{U}$;
(ii) a smooth, compactly supported map $\phi : G \to \mathbb{R}_+$ such that $\phi^{-1}(1) = G^{(0)}$.

Then we get a G-pseudodifferential operator $P_{f,\theta,\phi}$ with the formula ($u \in C_c^\infty(G)$)

$$P_{f,\theta,\phi}u(\gamma) = \int_{\substack{\gamma' \in G_{s(\gamma)},\\ \xi \in A^*_{r(\gamma)}(G)}} e^{-i\theta(\gamma'\gamma^{-1})\cdot\xi} f(r(\gamma), \xi)\phi(\gamma'\gamma^{-1})u(\gamma')d\lambda_{s(\gamma)}(\gamma').$$

The principal symbol of $P_{f,\theta,\phi}$ is just the leading part of f.

The principal symbol map respects pointwise product, whereas the product law for total symbols is much more involved. An operator is *elliptic* when its principal symbol never vanishes, and in that case, as in the classical situation, it has a parametrix inverting it modulo $\Psi_c^{-\infty}(G) = C_c^\infty(G)$.

Operators of negative order in $\Psi_c^*(G)$ are actually in $C^*(G)$, whereas zero-order operators are in the multiplier algebra $\mathcal{M}(C^*(G))$.

All these definitions and properties immediately extend to the case of operators acting between sections of bundles on $G^{(0)}$ pulled back to G with the range map r. The space of compactly supported pseudodifferential operators on G acting on sections of r^*E and taking values in sections of r^*F will be denoted by $\Psi_c^*(G, E, F)$. If $F = E$, we get an algebra denoted by $\Psi_c^*(G, E)$.

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Examples 3.7.5

(i) The family given by $P_t = P(x, tD_x)$ for $t > 0$ and $P_0 = P(x, D_X)$ described in the introduction of this section is a G-pseudodifferential operator with G the tangent groupoid of $\mathbb{R}^n$.

(ii) More generally, let V be a closed manifold endowed with a Riemannian metric. We denote by exp the exponential map associated with the metric. Let f be a symbol on V. We get a $\mathcal{G}_V$-pseudodifferential operator P by setting

$$(t > 0) \quad P_t u(x, y, t) = \int_{z \in V, \xi \in T_x^* V} e^{\frac{\exp_x^{-1}(z)}{t} \cdot \xi} f(x, \xi) u(z, y) \frac{dz d\xi}{t^n},$$

$$P_0 u(x, X, 0) = \int_{Z \in T_x V, \xi \in T_x^* V} e^{(X-Z)\cdot\xi} f(x, \xi) u(x, Z) dZ d\xi.$$

Moreover, P_1 is a pseudodifferential operator on the manifold V, which admits f as a complete symbol.

3.8 Index theorem for smooth manifolds

The purpose of this section is to present a proof of the Atiyah–Singer index theorem using deformation groupoids and show how it generalizes to conical pseudomanifolds. The results presented here come from recent works of the authors together with a joint work with V. Nistor [18–20]; we refer to [19, 20] for the proofs.

3.8.1 The KK-element associated to a deformation groupoid

Before going to the description of the index maps, let us describe a useful and classical construction [13, 27].

Let G be a smooth deformation groupoid (Definition 3.2.6):

$$G = G_1 \times \{0\} \cup G_2 \times]0, 1] \rightrightarrows G^{(0)} = M \times [0, 1].$$

One can consider the saturated open subset $M \times]0, 1]$ of $G^{(0)}$. Using the isomorphisms $C^*(G|_{M\times]0,1]}) \simeq C^*(G_2) \otimes C_0(]0, 1])$ and $C^*(G|_{M\times\{0\}}) \simeq C^*(G_1)$, we obtain the following exact sequence of C^*-algebras:

$$0 \longrightarrow C^*(G_2) \otimes C_0(]0, 1]) \xrightarrow{i_{M\times]0,1]}} C^*(G) \xrightarrow{ev_0} C^*(G_1) \longrightarrow 0,$$

where $i_{M\times]0,1]}$ is the inclusion map and ev_0 is the *evaluation map* at 0, that is, ev_0 is the map coming from the restriction of functions to $G|_{M\times\{0\}}$. We assume now that $C^*(G_1)$ is nuclear. Because the C^*-algebra $C^*(G_2) \otimes C_0(]0, 1])$ is contractible, the long exact sequence in KK-theory shows that the group homomorphism

$$(ev_0)_* = \cdot \otimes [ev_0] : KK(A, C^*(G)) \to KK(A, C^*(G_1))$$

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is an isomorphism for each C^*-algebra A. In particular, with $A = C^*(G)$ we get that $[ev_0]$ is invertible in KK-theory: there is an element $[ev_0]^{-1}$ in $KK(C^*(G_1), C^*(G))$ such that $[ev_0]\otimes[ev_0]^{-1} = 1_{C^*(G)}$ and $[ev_0]^{-1}\otimes[ev_0] = 1_{C^*(G_1)}$.

Let $ev_1 : C^*(G) \to C^*(G_2)$ be the evaluation map at 1, and $[ev_1]$ the corresponding element of $KK(C^*(G), C^*(G_2))$. The *KK-element associated to the deformation groupoid G* is defined by

$$\delta = [ev_0]^{-1}\otimes[ev_1] \in KK(C^*(G_1), C^*(G_2)).$$

We will meet several examples of this construction in the sequel.

3.8.2 The analytical index

Let M be a closed manifold, and consider its tangent groupoid:

$$\mathcal{G}_M^t := TM \times \{0\} \cup M \times M \times]0, 1] \rightrightarrows M \times [0, 1].$$

It is a deformation groupoid, and the preceding construction provides us a KK-element:

$$\partial_M = (e_1^M)_* \circ (e_0^M)_*^{-1} \in KK(C_0(T^*M), \mathcal{K}) \simeq KK(C_0(T^*M), \mathbb{C}),$$

where $e_i^M : C^*(\mathcal{G}_M^t) \to C^*(\mathcal{G}_M^t)|_{t=i}$ are evaluation homomorphisms.

The analytical index is then [13]

$$\begin{aligned}\mathrm{Ind}_a M := (e_1^M)_* \circ (e_0^M)_*^{-1} \ :\ KK(\mathbb{C}, C_0(T^*M)) &\to KK(\mathbb{C}, \mathcal{K}(L^2(M)) \\ &\simeq K_0(C_0(T^*M)) \simeq \mathbb{Z}\end{aligned}$$

or, in terms of the Kasparov product,

$$\mathrm{Ind}_a M = \cdot \otimes \partial_M.$$

Using the notion of pseudodifferential calculus for $\mathcal{G}_M^t$, it is easy to conclude that this map is the usual analytical index map. Indeed, let $f(x, \xi)$ be an elliptic zero-order symbol, and consider the $\mathcal{G}_M^t$-pseudodifferential operator, $P_f = (P_t)_{0\le t\le 1}$, defined as in Examples 3.7.5. Then f provides a K-theory class $[f] \in K_0(C^*(TM)) \simeq K_0(C_0(T^*M))$, whereas P provides a K-theory class $[P] \in K_0(C^*(\mathcal{G}_M^t))$, and

$$(e_0^M)_*([P]) = [f] \in K_0(C^*(TM)).$$

Thus

$$[f] \otimes [e_0^M]^{-1} \otimes [e_1^M] = [P_1] \in K_0(\mathcal{K}),$$

and $[P_1]$ coincides with $\mathrm{Ind}(P_1)$ under $K_0(\mathcal{K}) \simeq \mathbb{Z}$.

Because P_1 has principal symbol equal to the leading part of f, and because every class in $K_0(C_0(T^*M))$ can be obtained from a zero-order elliptic symbol, the claim is justified.

To be complete, let us explain that the analytical index map is the Poincaré dual of the homomorphism in K-homology associated with the obvious map: $M \to \{\cdot\}$. Indeed, thanks to the obvious homomorphism $\Psi : C^*(TM) \otimes C(M) \to C^*(TM)$ given by multiplication, ∂_M can be lifted into an element $D_M = \Psi_*(\partial_M) \in KK(C^*(TM) \otimes C(M), \mathbb{C}) = K^0(C^*(TM) \otimes C(M))$, called the *Dirac element*. This Dirac element yields the well-known Poincaré duality between $C_0(T^*M)$ and $C(M)$ [14, 19, 31], and in particular it gives an isomorphism

$$\cdot \underset{C^*(TM)}{\otimes} D_M : \ K_0(C^*(TM)) \xrightarrow{\simeq} K^0(C(M)),$$

whose inverse is induced by the principal symbol map.

One can then easily show the following proposition:

Proposition 3.8.1 *Let q: $M \to \cdot$ be the projection onto a point. The following diagram commutes:*

$$\begin{array}{ccc} K^0(T^*M) & \xrightarrow{\text{PD}} & K_0(M) \\ \text{Ind}_a \downarrow & & \downarrow q_* \\ \mathbb{Z} & \xrightarrow{=} & \mathbb{Z} \end{array}$$

3.8.3 The topological index

Take an embedding $M \to \mathbb{R}^n$, and let $p : N \to M$ be the normal bundle of this embedding. The vector bundle $TN \to TM$ admits a complex structure; thus we have a Thom isomorphism

$$T : K_0(C^*(TM)) \xrightarrow{\simeq} K_0(C^*(TN))$$

given by a KK-equivalence

$$T \in KK(C^*(TM), C^*(TN)).$$

T is called the *Thom element* [30].

The bundle N identifies with an open neighborhood of M into $\mathbb{R}^n$, so we have the excision map

$$j : C^*(TN) \to C^*(T\mathbb{R}^n).$$

Consider also $B : K_0(C^*(T\mathbb{R}^n)) \to \mathbb{Z}$ given by the isomorphism $C^*(T\mathbb{R}^n) \simeq C_0(\mathbb{R}^{2n})$ together with Bott periodicity.

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The *topological index map* Ind_t is the composition

$$K(C^*(TM)) \xrightarrow{T} K(C^*(TN)) \xrightarrow{j_*} K(C^*(T\mathbb{R}^n)) \underset{\simeq}{\xrightarrow{B}} \mathbb{Z}.$$

This classical construction can be reformulated with groupoids.

First, let us give a description of T, or rather of its inverse, in terms of groupoids. Recall the construction of the Thom groupoid. We begin by pulling back TM over N in the groupoid sense. Let

$$^*p^*(TM) = N \underset{M}{\times} TM \underset{M}{\times} N \rightrightarrows N$$

and

$$\mathcal{T}_N = TN \times \{0\} \sqcup {}^*p^*(TM) \times]0, 1] \rightrightarrows N \times [0, 1].$$

This *Thom groupoid* and the Morita equivalence between $^*p^*(TM)$ and TM provide the KK-element

$$\tau_N \in KK(C^*(TN), C^*(TM)).$$

This element is defined exactly as ∂_M is. Precisely, the evaluation map at 0, namely $\tilde{e}_0 : C^*(\mathcal{T}_N) \to C^*(TN)$, defines an invertible KK-element. We let $\tilde{e}_1 : C^*(\mathcal{T}_N) \to C^*(^*p^*(TM))$ be the evaluation map at 1. The Morita equivalence between the groupoids TM and $^*p^*(TM)$ leads to a Morita equivalence between the corresponding C^*-algebra and thus to a KK-equivalence $\mathcal{M} \in KK(C^*(^*p^*(TM)), C^*(TM))$. Then

$$\tau_N := [\tilde{e}_0]^{-1} \otimes [\tilde{e}_1] \otimes \mathcal{M}.$$

We have the following:

Proposition 3.8.2 [20] *If T is the KK-equivalence giving the Thom isomorphism, then*

$$\tau_N = T^{-1}.$$

This proposition also applies to interpret the isomorphism $B : K_0(C^*(T\mathbb{R}^n)) \to \mathbb{Z}$. Indeed, consider the embedding $\cdot \hookrightarrow \mathbb{R}^n$. The normal bundle is just $\mathbb{R}^n \to \cdot$, and we get as before

$$\tau_{\mathbb{R}^n} \in KK(C^*(T\mathbb{R}^n), \mathbb{C}).$$

Using the previous proposition, we get $B = \cdot \otimes \tau_{\mathbb{R}^n}$.

Remark also that $\mathcal{T}_{\mathbb{R}^n} = \mathcal{G}_{\mathbb{R}^n}$, so that $\tau_{\mathbb{R}^n} = [e_0^{\mathbb{R}^n}]^{-1} \otimes [e_1^{\mathbb{R}^n}]$.

Finally, the topological index

$$\text{Ind}_t = \tau_{\mathbb{R}^n} \circ j_* \circ \tau_N^{-1}$$

is entirely described using (deformation) groupoids.

3.8.4 The equality of the indices

A last groupoid is necessary in order to prove the equality of index maps. Namely, this groupoid is obtained by recasting the construction of the Thom groupoid at the level of tangent groupoids:

$$\widetilde{\mathcal{T}}_N = \mathcal{G}_N \times \{0\} \sqcup {}^*(p \otimes \mathrm{Id}_{[0,1]})^*(\mathcal{G}_M) \times]0, 1]. \tag{3.27}$$

As before, this yields a class

$$\widetilde{\tau}_N \in KK(C^*(\mathcal{G}_N), C^*(\mathcal{G}_M)).$$

All maps in the diagram

$$\begin{array}{ccccc}
\mathbb{Z} & = & \mathbb{Z} & = & \mathbb{Z} \\
{\scriptstyle e_1^M}\big\uparrow & & {\scriptstyle e_1^N}\big\uparrow & & {\scriptstyle e_1^{\mathbb{R}^n}}\big\uparrow \\
K_0(C^*(\mathcal{G}_M)) & \xleftarrow{\otimes\widetilde{\tau}_N} & K_0(C^*(\mathcal{G}_N)) & \xrightarrow{\widetilde{j}_*} & K_0(C^*(\mathcal{G}_{\mathbb{R}^n})) \\
{\scriptstyle e_0^M}\big\downarrow\simeq & & {\scriptstyle e_0^N}\big\downarrow\simeq & & {\scriptstyle e_0^{\mathbb{R}^n}}\big\downarrow\simeq \\
K_0(C^*(TM)) & \underset{\simeq}{\xleftarrow{\otimes\tau_N}} & K_0(C^*(TN)) & \xrightarrow{j_*} & K_0(C^*(T\mathbb{R}^n))
\end{array} \tag{3.28}$$

are given by Kasparov products with

(i) classes of homomorphisms coming from restrictions or inclusions between groupoids,
(ii) inverses of such classes,
(iii) explicit Morita equivalences.

This easily yields the commutativity of the diagram (3.28). Having in mind the previous description of index maps using groupoids, this commutativity property just implies

$$\mathrm{Ind}_a = \mathrm{Ind}_t.$$

3.9 The case of pseudomanifolds with isolated singularities

As we explained earlier, the proof of the K-theoretical form of the Atiyah–Singer index theorem presented in this chapter easily extends to the case of pseudomanifolds with isolated singularities. This is achieved provided one uses the correct notion of *tangent space* of the pseudomanifold; for a pseudomanifold X with one conical point (the case of several isolated singularities is similar), this is the noncommutative tangent space defined in Section 3.2.5:

$$T^{\mathsf{S}}X = X^- \times X^- \cup T\overline{X^+} \rightrightarrows X^\circ.$$

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In the sequel, it will replace the ordinary tangent space of a smooth manifold. Moreover, it gives rise to another deformation groupoid, which will replace the ordinary tangent groupoid of a smooth manifold:

$$\mathcal{G}_X^t = T^{\mathsf{S}}X \times \{0\} \cup X^\circ \times X^\circ \times]0, 1] \rightrightarrows X^\circ \times [0, 1].$$

We call $\mathcal{G}_X^t$ the *tangent groupoid* of X. It can be provided with a smooth structure such that $T^{\mathsf{S}}X$ is a smooth subgroupoid. Moreover, both are amenable, so their reduced and maximal C^*-algebras coincide and are nuclear.

With these choices of $T^{\mathsf{S}}X$ as a tangent space for X and of $\mathcal{G}_X^t$ as a tangent groupoid, one can follow step by step all the constructions made in the previous section.

3.9.1 The analytical index

Using the partition $X^\circ \times [0, 1] = X^\circ \times \{0\} \cup X^\circ \times]0, 1]$ into saturated open and closed subsets of the units space of the tangent groupoid, we define the KK-element associated to the tangent groupoid of X:

$$\partial_X := [e_0]^{-1} \otimes [e_1] \in KK(C^*(T^{\mathsf{S}}X), \mathcal{K}) \simeq KK(C^*(T^{\mathsf{S}}X), \mathbb{C}),$$

where $e_0 : C^*(\mathcal{G}_X^t) \to C^*(\mathcal{G}_X^t|_{X^\circ \times \{0\}}) \simeq C^*(T^{\mathsf{S}}X)$ is the evaluation at 0, and $e_1 : C^*(\mathcal{G}_X^t) \to C^*(\mathcal{G}_X^t|_{X^\circ \times \{1\}}) \simeq \mathcal{K}(L^2(X))$ is the evaluation at 1.

Now we can define the analytical index exactly as we did for closed smooth manifolds. Precisely, the *analytical index* for X is set to be the map

$$\mathrm{Ind}_a^X = \cdot \otimes \partial_X : KK(\mathbb{C}, C^*(T^{\mathsf{S}}X)) \to KK(\mathbb{C}, \mathcal{K}(L^2(X^\circ))) \simeq \mathbb{Z}.$$

The interpretation of this map as the Fredholm index of an appropriate class of elliptic operators is possible and carried out in [34].

3.9.2 The Poincaré duality

Pursuing the analogy with smooth manifolds, we explain in this subsection that the analytical index map for X is Poincaré dual to the index map in K-homology associated to the obvious map : $X \to \{.\}$.

The algebras $C(X)$ and $C^\bullet(X) := \{f \in C(X) \mid f \text{ is constant on } cL\}$ are homotopic. If g belongs to $C^\bullet(X)$ and f to $C_c(T^{\mathsf{S}}X)$, let $g \cdot f$ be the element of $C_c(T^{\mathsf{S}}X)$ defined by $g \cdot f(\gamma) = g(r(\gamma))f(\gamma)$. This induces a $*$-morphism

$$\Psi : C(X) \otimes C^*(T^{\mathsf{S}}X) \to C^*(T^{\mathsf{S}}X).$$

The *Dirac element* is defined to be

$$D_X := [\Psi] \otimes \partial_X \in KK(C(X) \otimes C^*(T^{\mathsf{S}}X), \mathbb{C}).$$

We recall

Theorem 3.9.1 [19] *There exists a (dual-Dirac) element $\lambda_X \in KK(\mathbb{C}, C(X) \otimes C^*(T^{\mathsf{S}}X))$ such that*

$$\lambda_X \underset{C(X)}{\otimes} D_X = 1_{C^*(T^{\mathsf{S}}X)} \in KK(C^*(T^{\mathsf{S}}X), C^*(T^{\mathsf{S}}X)),$$
$$\lambda_X \underset{C^*(T^{\mathsf{S}}X)}{\otimes} D_X = 1_{C(X)} \in KK(C(X), C(X)).$$

This means that $C(X)$ and $C^*(T^{\mathsf{S}}X)$ are Poincaré dual.

Remark 3.9.2 The explicit construction of λ_X, which is heavy going and technical, can be avoided. In fact, the definitions of $T^{\mathsf{S}}X$, $\mathcal{G}^t_X$ and thus of D_X can be extended in a natural way to the case of an arbitrary pseudomanifold, and the proof of Poincaré duality can be done using a recursive argument on the depth of the stratification, starting with the case depth $= 0$, that is, with the case of smooth closed manifolds. This is the subject of [18].

The theorem implies that

$$KK(\mathbb{C}, C^*(T^{\mathsf{S}}X)) \simeq K_0(C^*(T^{\mathsf{S}}X)) \to K(C(X), \mathbb{C}) \simeq K^0(C(X)),$$
$$x \mapsto x \underset{C^*(T^{\mathsf{S}}X)}{\otimes} D_X$$

is an isomorphism. In [34], it is explained how to interpret its inverse as a principal symbol map, and one also get the analogue of Proposition 3.8.1:

Proposition 3.9.3 *Let $q : X \to \cdot$ be the projection onto a point. The following diagram commutes:*

$$\begin{array}{ccc} K_0(C^*(T^{\mathsf{S}}X)) & \xrightarrow{\text{PD}} & K_0(X) \\ {\scriptstyle \mathrm{Ind}^X_a}\downarrow & & \downarrow{\scriptstyle q_*} \\ \mathbb{Z} & \xrightarrow{=} & \mathbb{Z} \end{array}$$

3.9.3 The topological index

3.9.3.1 Thom isomorphism

Take an *embedding* $X \hookrightarrow c\mathbb{R}^n = \mathbb{R}^n \times [0, +\infty[/\mathbb{R}^n \times \{0\}$. This means that we have a map which restricts to an embedding $X^\circ \to \mathbb{R}^n \times]0, +\infty[$ in the usual sense and which sends c to the image of $\mathbb{R}^n \times \{0\}$ in $c\mathbb{R}^n$. Moreover, we require the embedding on $X^- = L \times]0, 1[$ to be of the form $j \times \mathrm{Id}$ where j is an embedding of L in $\mathbb{R}^n$.

Such an embedding provides a *conical normal bundle*. Precisely, let $p : N^\circ \to X^\circ$ be the normal bundle associated with $X^\circ \hookrightarrow \mathbb{R}^n \times]0, +\infty[$. We can identify

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$N^\circ|_{X^-} \simeq N^\circ|_L \times]0,1[$, and set

$$N = \bar{c}N^\circ|_L \cup N^\circ|_{X^+}.$$

Thus N is the pseudomanifold with an isolated singularity obtained by gluing the closed cone $\bar{c}N^\circ|_L := N^\circ|_L \times [0,1]/N^\circ|_L \times \{0\}$ with $N^\circ|_{X^+}$ along their common boundary $N^\circ|_L \times \{1\} = N^\circ|_{\partial X^+}$. Moreover, $p : N \to X$ is a conical vector bundle.

The *Thom groupoid* is then

$$\mathcal{T}_N = T^{\mathsf{S}}N \times \{0\} \sqcup {}^*p^*(T^{\mathsf{S}}X) \times]0,1].$$

It is a deformation groupoid. The corresponding KK-element gives the *inverse Thom element*:

$$\tau_N \in KK(C^*(T^{\mathsf{S}}N), C^*(T^{\mathsf{S}}X)).$$

Proposition 3.9.4 [20] *The following map is an isomorphism:*

$$K(C^*(T^{\mathsf{S}}N)) \xrightarrow{\cdot \otimes \tau_N} K(C^*(T^{\mathsf{S}}X)).$$

Roughly speaking, the inverse of $\cdot \otimes \tau_N$ is the *Thom isomorphism* for the "vector bundle" $T^{\mathsf{S}}N$ "over" $T^{\mathsf{S}}X$. One can show that it really restricts to usual Thom homomorphism on regular parts.

3.9.3.2 Excision

The groupoid $T^{\mathsf{S}}N$ is identified with an open subgroupoid of $T^{\mathsf{S}}c\mathbb{R}^n$, and we have an excision map

$$j : C^*(T^{\mathsf{S}}N) \to C^*(T^{\mathsf{S}}\mathbb{R}^n).$$

3.9.3.3 Bott element

Consider $c \hookrightarrow c\mathbb{R}^n$. The (conical) normal bundle is $c\mathbb{R}^n$ itself. Remark that $\mathcal{G}^t_{c\mathbb{R}^n} = \mathcal{T}_{c\mathbb{R}^n}$. Then

$$\tau_{c\mathbb{R}^n} \in KK(C^*(T^{\mathsf{S}}c\mathbb{R}^n), \mathbb{C})$$

gives an isomorphism

$$B = (\cdot \otimes \tau_{c\mathbb{R}^n}) : K_0(C^*(T^{\mathsf{S}}c\mathbb{R}^n)) \to \mathbb{Z}.$$

Definition 3.9.5 The *topological index* is the morphism

$$\mathrm{Ind}_t^X = B \circ j_* \circ \tau_N^{-1} : \ K_0(C^*(T^{\mathsf{S}}X)) \to \mathbb{Z}.$$

The following index theorem can be proved along the same lines as in the smooth case:

Theorem 3.9.6 *The following equality holds:*

$$\mathrm{Ind}_a^X = \mathrm{Ind}_t^X.$$

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