

K-duality for pseudomanifolds with an isolated singularity

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Abstract. We associate to a pseudomanifold X with an isolated singularity a differentiable groupoid G which plays the role of the tangent space of X . We construct a Dirac element D and a Dual Dirac element λ which induce a Poincaré duality in K -theory between the C^* -algebras $C(X)$ and $C^*(G)$. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

K-dualité pour les pseudo-variétés à singularité isolée.

Résumé. *Etant donnée une pseudo-variété X ayant une singularité isolée, nous lui associons un groupoïde différentiable G qui joue le rôle d'espace tangent à X . Nous construisons un élément Dirac D ainsi qu'un élément dual-Dirac λ qui induisent une dualité de Poincaré en K -théorie entre les C^* -algèbres $C(X)$ et $C^*(G)$. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS*

1. The tangent bundle of a singular manifold

Let X be a pseudomanifold with an isolated singularity c , that is $X = cL \cup X_1$, where X_1 is a smooth compact manifold with boundary L glued along its boundary with the cone over L : $cL = L \times [0, 1] / L \times \{0\}$. The singularity c is then the image of $L \times \{0\}$ in cL . We denote by $M = L \times]-1, 1[\cup X_1$ the manifold obtained by gluing X_1 with $L \times]-1, 1[$ along the boundary.

If y is a point of M or of $X \setminus \{c\}$ which is in $L \times]-1, 1[$ we note $y_L \in L$ its tangential component and $k_y \in]-1, 1[$ its radial coordinate; the function k_y is smoothly extended to X_1 in such a way that $k_y \geq 1$ on X_1 . We set $M^+ = \{y \in M \mid k_y > 0\}$, $M^- = \{y \in M \mid k_y < 0\}$ and $\overline{M^+} = \{y \in M \mid k_y \geq 0\}$.

We suppose that the manifold M is equipped with a Riemannian metric whose injectivity radius is bigger than 1 and which is a product metric on $L \times]-1, 1[$.

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We define the groupoid G , with source map s , range map r and space of units $G^{(0)} = M$:

$$G = M^- \times M^- \cup T\overline{M^+} \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{r} \end{array} M .$$

It is the union of the bundle $T\overline{M^+} = \{(z, v) \in TM \mid z \in \overline{M^+}\}$ equipped with its standard groupoid structure over $\overline{M^+}$ and the pair groupoid $M^- \times M^- \rightrightarrows M^-$.

In order to equip G with a smooth structure, we take the usual structure of manifold on $M^- \times M^-$ and on $T\overline{M^+}$. We take a smooth positive function $\tau :]-1, +\infty[\rightarrow \mathbb{R}$ which satisfies $\tau^{-1}(\{0\}) = [0, +\infty[$. A local chart around boundary points of $T\overline{M^+}$ is provided by the following map :

$$E_G : \mathcal{V}(TM) \longrightarrow \mathcal{V}(G) , \begin{cases} (y, V) \mapsto (y, V) \text{ if } y \in \overline{M^+} \\ (y, V) \mapsto (y, \exp_y(-\tau(k_y)V)) \text{ if } y \in M^- \end{cases}$$

where \exp is the exponential map of the Riemannian manifold M , $\mathcal{V}(TM) = \{(y, V) \in TM ; -\tau(k_y)V \in \text{dom}(\exp_y)\}$ and $\mathcal{V}(G)$ is a neighborhood of $G^{(0)}$ in G . The groupoid G is called the *tangent bundle* of X .

Following the construction of A. Connes for smooth manifolds [3], we define the *tangent groupoid* of X in the following way :

$$\mathcal{G} = M \times M \times]0, 1] \cup G \times \{0\} \rightrightarrows M \times [0, 1] .$$

The groupoid \mathcal{G} is the union of the groupoid $G \times \{0\} \rightrightarrows M \times \{0\}$ and the pair groupoid over M parametrized by $]0, 1]$. We equip \mathcal{G} with a structure of smooth groupoid similarly as we did for G .

The groupoid G is amenable [1] so that its reduced C^* -algebra coincides with the maximal one and it is nuclear. The same occurs for \mathcal{G} . We denote respectively $C^*(G)$ and $C^*(\mathcal{G})$ these C^* -algebras. Moreover, up to isomorphisms, these C^* -algebras do not depend on the map τ used to define the smooth structure of G .

2. The Dirac element

The partition $M \times \{0\} \cup M \times]0, 1]$ of $\mathcal{G}^{(0)}$ into a saturated open subset and a saturated closed subset induces the following exact sequence of C^* -algebras [5] :

$$0 \longrightarrow C^*(\mathcal{G}|_{M \times]0, 1]}) \longrightarrow C^*(\mathcal{G}) \xrightarrow{e_0} C^*(\mathcal{G}|_{M \times \{0\}}) = C^*(G) \longrightarrow 0$$

where the first map is the inclusion and e_0 is the evaluation map at 0. The C^* -algebra $C^*(\mathcal{G}|_{M \times]0, 1]})$ is isomorphic to $\mathcal{K} \otimes C_0(]0, 1])$ which is contractible. So, since $C^*(\mathcal{G})$ is nuclear, the element $[e_0]$ of $KK(C^*(\mathcal{G}), C^*(G))$ corresponding to e_0 , is invertible. We denote $[e_0]^{-1} \in KK(C^*(G), C^*(\mathcal{G}))$ its inverse. Let $e_1 : C^*(\mathcal{G}) \rightarrow C^*(\mathcal{G}|_{M \times \{1\}}) = \mathcal{K}$ be the evaluation map at 1. We let b be the generator of $KK(\mathcal{K}, \mathbb{C})$. We set :

$$\partial = [e_0]^{-1} \otimes_{C^*(\mathcal{G})} [e_1] \otimes_{\mathcal{K}} b \in KK(C^*(G), \mathbb{C}) .$$

The algebra $C(X)$ maps to the center of the multiplier algebra of $C^*(G)$. Let $\Psi : C^*(G) \otimes C(X) \rightarrow C^*(G)$ be the morphism induced by multiplication and $[\Psi]$ be the corresponding element in $KK(C^*(G) \otimes C(X), C^*(G))$.

The *Dirac element* is then defined as

$$D = [\Psi] \otimes_{C^*(G)} \partial \in KK(C^*(G) \otimes C(X), \mathbb{C}) .$$

3. The dual Dirac element

We are looking for an element λ in $KK(\mathbb{C}, C^*(G) \otimes C(X))$, that is a continuous family $(\lambda_y)_{y \in X}$ of elements of $KK(\mathbb{C}, C^*(G))$. In order to be dual to the Dirac element D , λ is constructed so that :

- (i) λ is in the image of $(i_{\mathcal{O}})_* : KK(\mathbb{C}, C^*(G \times X|_{\mathcal{O}})) \rightarrow KK(\mathbb{C}, C^*(G \times X))$ where \mathcal{O} is an open subset of $\mathcal{U} = \{(x, y) \in M \times X \mid k_x < 1, k_y < 1\} \cup \{(x, y) \in M \times X \mid d(x, y) < 1\}$.
- (ii) The equality $\lambda \otimes_{C^*(G)} \partial = 1 \in K^0(X)$ holds.

We first assign to each point y of X an open subset O_y of M which is a ball centered on y when $k_y \geq 1$, which is contained in M^- when $k_y \leq 1/2$ and equal to M^- when $k_y \leq \varepsilon$ (where $0 < \varepsilon < 1/2$). Furthermore the set $\mathcal{O} = \cup_{y \in X} O_y \times \{y\}$ is an open subset of $M \times X$ contained in \mathcal{U} .

Notice that $K(C^*(G|_{O_y})) \simeq \mathbb{Z}$ for each $y \in X$.

We pull back the vector bundle of differential forms on M to G and \mathcal{G} using their range maps and then to $G \times X$ and $\mathcal{G} \times X$ using the first projection. We denote all these bundles by the same letter Λ . The following step is the construction of a continuous family $(\beta_y)_{y \in X}$, where $\beta_y = (F_y, C^*(G|_{O_y}, \Lambda))$ is an element of $E(\mathbb{C}, C^*(G|_{O_y}))$ which class is a generator of $K(C^*(G|_{O_y})) \simeq \mathbb{Z}$.

When O_y is a subset of M^+ it is natural to state $F_y = a_y$ where $a_y \in C_b^\infty(T^*O_y, \text{End}\Lambda)$ is the symbol of a pseudo-differential operator G_y which satisfies :

- G_y belongs to $\Psi_c^0(O_y, \Lambda) + C_b^\infty(O_y, \text{End}\Lambda)$ and $G_y^2 - 1$ belongs to $\Psi_c^{-1}(O_y, \Lambda)$,
- G_y is of the form $G_y = \begin{pmatrix} 0 & G_y^- \\ G_y^+ & 0 \end{pmatrix}$ with respect to the usual decomposition of $\Lambda = \Lambda^{ev} \oplus \Lambda^{odd}$,
- G_y^+ is surjective and $\text{Ker}(G_y^+) = \mathbb{C} \cdot \epsilon_y$ where ϵ_y belongs to $C_b^\infty(O_y, \Lambda)$,
- the family $(G_y)_{\substack{y \in X \\ k_y > \varepsilon}}$ defines a continuous section of $B(H)$ where H is the bundle $\bigcup_{\substack{y \in X \\ k_y > \varepsilon}} L^2(O_y, \Lambda)$.

The existence of such an operator is a consequence of theorem (19.2.12) of [6].

When y comes closer to the singularity c , we gradually pass from a situation where $G|_{O_y} = TO_y$ to a situation where $G|_{O_y} = O_y \times O_y$. The pseudo-differential calculus on groupoids, first introduced by A. Connes in [2] (see also [4]), enables us to construct a continuous family $(F_y)_{y \in X, k_y > \varepsilon}$ such that up to a compact operator $F_y = a_y$ when $G|_{O_y} = TO_y$ and $F_y = G_y$ when $G|_{O_y} = O_y \times O_y$. Thus we “replace” the symbol a_y by the operator G_y .

The last case is when O_y becomes equal to M^- . We first choose an appropriate extension of F_y which belongs to $B(L^2(O_y, \Lambda))$ into an element of $B(L^2(M^-, \Lambda))$. Afterwards, thanks to the properties of the operator G_y listed above, we can replace F_y by a constant operator F_c .

We construct in this way a continuous family β_y of elements of $E(\mathbb{C}, C^*(G|_{O_y}))$. This family induces an element β of $KK(\mathbb{C}, C^*(G \times X|_{\mathcal{O}}))$. We set $\lambda = (i_{\mathcal{O}})_*(\beta)$. The dual-Dirac element λ satisfies the properties (i) and (ii) mentioned above.

4. The Poincaré duality

THEOREM 4.1. – *The Dirac element D and the dual-Dirac element λ induce a Poincaré duality between the C^* -algebras $C^*(G)$ and $C(X)$, that is :*

$$\lambda \otimes_{C^*(G)} D = 1_{C(X)} \in KK(C(X), C(X)) \text{ and } \lambda \otimes_{C(X)} D = 1_{C^*(G)} \in KK(C^*(G), C^*(G)).$$

Idea of the proof : Let us consider the morphisms $\Psi', \Delta' : C^*(G \times X) \otimes C(X) \rightarrow C^*(G) \otimes C(X)$ given by $\Psi'(f \otimes g \otimes h) = \Psi(f \otimes h) \otimes g$ and $\Delta'(f \otimes g \otimes h) = f \otimes \Delta(h \otimes g)$ where $\Delta : C(X) \otimes C(X) \rightarrow C(X)$ is the multiplication map. Their restrictions $C^*(G \times X|_{\mathcal{O}}) \otimes C(X) \rightarrow C^*(G) \otimes C(X)$ are homotopic, hence, since $\lambda = (i_{\mathcal{O}})_*(\beta)$, the following equality holds :

$$\lambda \otimes_{C^*(G)} [\Psi] = \lambda \otimes_{C(X)} [\Delta].$$

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This ensures that

$$\lambda \otimes_{C^*(G)} D = (\lambda \otimes_{C^*(G)} \partial) \otimes_{C(X)} [\Delta] = 1_{C(X)} .$$

In order to show the second equality, we study the invariance of the element $\lambda \otimes_{C(X)} [\Psi]$ under the *flip* automorphism f of $C^*(G \times G)$, that is the automorphism induced by $(\gamma, \eta) \in G \times G \mapsto (\eta, \gamma)$. The motivation comes from the equality

$$\begin{aligned} ((\lambda \otimes_{C(X)} [\Psi]) \otimes_{C^*(G \times G)} [f]) \otimes_{C^*(G)} \partial &= (\lambda \otimes_{C^*(G)} \partial) \otimes_{C(X)} [\Psi] = 1_{C^*(G)} , \text{ which implies} \\ \lambda \otimes_{C(X)} D - 1_{C^*(G)} &= ((\lambda \otimes_{C(X)} [\Psi]) \otimes_{C^*(G \times G)} [\text{id} - f]) \otimes_{C^*(G)} \partial . \end{aligned}$$

If B is a symmetric geodesically convex neighborhood of the diagonal of $M^+ \times M^+$ contained in the range of \exp , then the flip automorphism of $C^*(TB)$ is homotopic to identity. Let $C = L \times]-1, 1[\subset M$ and $F = M \times M \setminus C \times C$. We denote by $r_* : KK(C^*(G), C^*(G \times G)) \rightarrow KK(C^*(G), C^*(G \times G|_F))$ the morphism induced by the restriction r .

Using again that λ is in the image of $(i_{\mathcal{O}})_*$ and that $\mathcal{O} \cap M^+ \times M^+$ is a small enough neighborhood of the diagonal, we show that $r_*(\lambda \otimes_{C(X)} [\Psi])$ is invariant under the flip automorphism of $C^*(G \times G|_F)$. Then, the long exact sequence in KK -theory associated to :

$$0 \longrightarrow C^*(G \times G|_{C \times C}) \xrightarrow{i_{C \times C}} C^*(G \times G) \xrightarrow{r} C^*(G \times G|_F) \longrightarrow 0$$

ensures that $(\lambda \otimes_{C(X)} [\Psi]) \otimes_{C^*(G \times G)} [\text{id} - f]$ belongs to the image of $(i_{C \times C})_* : KK(C^*(G), C^*(G \times G|_{C \times C})) \rightarrow KK(C^*(G), C^*(G \times G))$. This enables us to show that $\lambda \otimes_{C(X)} D - 1_{C^*(G)}$ is in the image of $(i_C)_* : KK(C^*(G), C^*(G|_C)) \rightarrow KK(C^*(G), C^*(G))$ where i_C is the inclusion morphism of $C^*(G|_C)$ into $C^*(G)$.

On the other hand the equality $\lambda \otimes_{C^*(G)} D = 1_{C(X)}$ ensures that $\lambda \otimes_{C(X)} D - 1_{C^*(G)}$ is in the kernel of the map $(\cdot \otimes_{C^*(G)} D) : KK(C^*(G), C^*(G)) \rightarrow KK(C^*(G) \otimes C(X), \mathbb{C})$. To finish the proof we show that this map is injective in restriction to the image of $(i_C)_*$. This last point comes from the fact that the inclusion $i^{\mathcal{K}}$ of $C^*(G|_{M^-}) \simeq \mathcal{K}$ into $C^*(G|_C)$ induces a KK -equivalence between \mathcal{K} and $C^*(G|_C)$; and $(\cdot \otimes_{C^*(G)} D) \circ (i_C)_* \circ (i^{\mathcal{K}})_* = e_c^*$ where $e_c^* : KK(C^*(G), \mathbb{C}) \rightarrow KK(C^*(G) \otimes C(X), \mathbb{C})$ comes from the evaluation map at c , $e_c : C(X) \rightarrow \mathbb{C}$. \square

A consequence of the preceding theorem is a Poincaré duality between $C_0(T^*\overline{M^+})$ and $C_0(M^+)$. Thus we get a second Poincaré duality for manifolds with boundary, the first one has been stated by G. Kasparov in [7] for the algebras $C_0(T^*M^+)$ and $C(\overline{M^+})$.

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