Hopf algebras,
from basics to applications to renormalization

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Introduction

These notes are an extended version of a series of lectures given at Bogota from 2nd to 6th december 2002. They aim to present a self-contained introduction to the Hopf-algebraic techniques which appear in the work of A. Connes and D. Kreimer on renormalization in Quantum Field Theory [CK1], [CK2], [BF]... Our point of view consists in revisiting a substantial part of their work in the abstract framework of connected graded Hopf algebras, i.e. Hopf algebras endowed with a compatible $\mathbb{Z}_+$ grading such that the degree zero component is one-dimensional.

Chapter I contains a few elements of Hopf algebra theory which can be found in any good introductory text on the subject ([Ab], [Sw], [Ka]...), as well as some basic tools from algebra which are necessary to understand the coradical filtration of a Hopf algebra.

Chapter II deals with connected graded and connected filtered Hopf algebras with emphasis on the convolution product. The main interest of these objects resides in the possibility to implement induction techniques with respect to the grading or the filtration : the starting point is the particular form of the coproduct on a connected filtered Hopf algebra $\mathcal{H}$:

$$\Delta x = x \otimes \mathbf{1} + \mathbf{1} \otimes x + \text{terms involving elements of strictly lower filtration degree.}$$

We derive the *Birkhoff decomposition* of any linear map on $\mathcal{H}$ with values into any commutative unital algebra $\mathcal{A}$ which sends the unit $\mathbf{1}$ to the unit $1_{\mathcal{A}}$, provided the algebra $\mathcal{A}$ is endowed with a *renormalization scheme*, i.e. a splitting :

$$\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$$

into two subalgebras, where $\mathcal{A}_+$ contains the unit $1_{\mathcal{A}}$. Those maps indeed form a group G under the convolution product, and the Birkhoff decomposition of any $\varphi \in G$ reads :

$$\varphi = \varphi_-^{*-1} * \varphi_+,$$

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where φ_- and φ_+ are elements of G, φ_- sends any element of $\mathcal{H}$ of positive degree to an element of $\mathcal{A}_-$, , and φ_+ is a map with values in $\mathcal{A}_+$. As an example, $\mathcal{A}$ is the algebra of germs of meromorphic functions at $z_0 \in \mathbb{C}$ endowed with the *minimal subtraction scheme*, i.e. $\mathcal{A}_+$ is the algebra of germs of holomorphic functions at z_0, and $\mathcal{A}_-$ is the subalgebra of polar parts $(z - z_0)^{-1}\mathbb{C}[(z - z_0)^{-1}]$.

The Birkhoff decomposition respects two particular subgroups of G, namely the group G_1 of algebra morphisms from $\mathcal{H}$ into germs of meromorphic functions (*characters*), and the group G_2 of elements φ of G such that $\varphi(xy) = \varphi(yx)$ (*cocycles*). We end up this chapter with some examples.

Chapter III is devoted to Feynman graphs, which are the original example coming from quantum field theory, where D. Kreimer discovered the underlying Hopf algebra structure.

In Chapter IV we define a bijective map $\widetilde{R} : G \to \mathfrak{g}$ where $\mathfrak{g}$ stands for the linear space of maps on $\mathcal{H}$ with values into the algebra of germs of meromorphic functions which sends the unit $\mathbf{1}$ to the constant function 0. It is uniquely defined by means of the equation :

$$\varphi \circ Y = \varphi * \widetilde{R}(\varphi),$$

where $Y : \mathcal{H} \to \mathcal{H}$ is the natural biderivation $x \mapsto |x|.x$ associated to the graduation. We give an explicit formula for this *renormalization map* and for its inverse. Finally, along the lines of [CK2] we explore more closely the interplay between the renormalization map $\widetilde{R}$ and Birkhoff decomposition. When $\mathcal{A}$ is the algebra of germs of meromorphic functions at $z_0 = 0$ endowed with the minimal subtraction scheme, the renormalization map $\widetilde{R}$ is closely related to the β-function of [CK2] as follows : for any $\varphi \in G_1$ with trivial positive part in the Birkhoff decomposition (i.e. $\varphi_+ = u_{\mathcal{A}} \circ \varepsilon$, where ε is the co-unit of $\mathcal{H}$ and $u_{\mathcal{A}}$ is the unit map of $\mathcal{A}$), we have :

$$\beta(\varphi) = z\widetilde{R}(\varphi).$$

(see theorem IV.4.4).

These notes find their origin in a workshop on renormalization in Quantum Field Theory held in Nancy every monday during year 2000-2001. I would like to thank all participants to this workshop, among them specially Philippe Bonneau, Malte Henkel, Mohsen Masmoudi, André Roux, Jérémie Unterberger and Tilmann Wurzbacher. Many thanks as well to the Mathematics Departments of Universidad de Los Andes and Universidad La Nacional at Bogota for the friendly atmosphere of this conference. I would like to thank Luis Fernandez and Sylvie Paycha for valuable remarks on previous versions of this text, as well as Kurusch Ebrahimi-Fard, Dirk Kreimer and Li Guo for illuminating discussions.

Enfin, un grand merci à toute l'équipe des Rencontres Mathématiques de Glanon pour avoir bien voulu publier ces notes dans ce volume des comptes-rendus.

Contents :

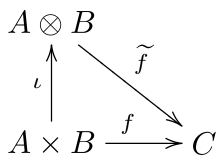
I. A few elements of Hopf algebra theory

I.1. Tensor algebra

Let k be any field, and let A, B be vector spaces over k. The *tensor product* $A \otimes B$ is a vector space over k which satisfies the following *universal property* : there exists a bilinear map :

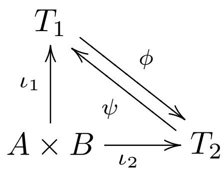
$$\iota \;: A \times B \longrightarrow A \otimes B$$
$$(a, b) \longmapsto a \otimes b$$

such that for any k-vector space C and for any bilinear map f from $A \times B$ into C there is a unique linear map $\widetilde{f} \;: A \otimes B \to C$ such that $f = \widetilde{f} \circ \iota$:



Proposition I.1.1.
Tensor product $A \otimes B$ exists and is unique up to isomorphism.

Proof. We show uniqueness first : if (T_1, ι_1) and (T_2, ι_2) are both candidates for playing the role of the tensor product, then the universal property applied to both tells us that there exist $\phi \;: T_1 \to T_2$ and $\psi \;: T_2 \to T_1$ such that $\iota_2 = \phi \circ \iota_1$ and $\iota_1 = \psi \circ \iota_2$.



Applying the universal property twice again shows that $\psi \circ \phi = I_{T_1}$ and $\phi \circ \psi = I_{T_2}$, whence uniqueness of the tensor product up to linear isomorphism.

For the existence we shall need axiom of choice in the infinite-dimensional case : let $(e_i)_{i \in E}$ and $(f_j)_{j \in F}$ be bases of A and B respectively. Then we take as $A \otimes B$ the vector space freely generated by $(c_{ij})_{i \in E, j \in F}$, and we set $\iota(e_i, f_j) = e_i \otimes f_j = c_{ij}$. For any bilinear map ϕ from $A \times B$ to another vector space C it is clear that the linear map $\widetilde{\phi}$ from $A \otimes B$ to C defined by :

$$\widetilde{\phi}(c_{ij}) = \phi(e_i, f_j)$$

is the only one such that $\widetilde{\phi} \circ \iota = \phi$. The vector space $A \otimes B$ constructed above fulfills then the universal property.

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Elements $a \otimes b$ in $A \otimes B$ are called *indecomposable*. They clearly generate $A \otimes B$. Tensor products $k \otimes A$ and $A \otimes k$ are canonically identified with A via $1 \otimes a \simeq a \simeq a \otimes 1$. When three vector spaces A, B, C are involved there is an isomorphism :

$$\begin{aligned} \alpha \ : (A \otimes B) \otimes C &\longrightarrow A \otimes (B \otimes C) \\ (a \otimes b) \otimes c &\longmapsto a \otimes (b \otimes c). \end{aligned}$$

Note that this isomorphism is *not* canonical, because tensor product $A \otimes B$ itself is only defined up to isomorphism. We shall denote by $A \otimes B \otimes C$ any of these two versions of the iterated tensor product.

Let A and B be two vector spaces. The *flip* $\tau : A \otimes B \to B \otimes A$ defined by $\tau(a \otimes b) = b \otimes a$ is an isomorphism. This generalizes to any finite collection of vector spaces as follows : any permutation $\sigma \in S_n$ defines a map :

$$\begin{aligned} \tau_\sigma : A_1 \otimes \cdots \otimes A_n &\longrightarrow A_{\sigma^{-1}1} \otimes \cdots \otimes A_{\sigma^{-1}n} \\ a_1 \otimes \cdots \otimes a_n &\longmapsto a_{\sigma^{-1}1} \otimes \cdots \otimes a_{\sigma^{-1}n}. \end{aligned}$$

In particular the group S_n acts on the tensor power $A^{\otimes n}$ by automorphisms. It makes then sense to consider the tensor power $A^{\otimes F}$ for any finite set, by choosing any arbitrary total order on F.

Now let us consider four k-vector spaces A_1, A_2, B_1, B_2.

Proposition I.1.2.
There is a canonical injection :

$$\mathcal{L}(A_1, A_2) \otimes \mathcal{L}(B_1, B_2) \underset{\sim}{\overset{j}{\hookrightarrow}} \mathcal{L}(A_1 \otimes B_1,\ A_2 \otimes B_2)$$

given by :

$$\big(j(f \otimes g)\big)(a \otimes b) = f(a) \otimes g(b).$$

In the case when all the spaces are finite-dimensional, this injection is an isomorphism.

Proof. The space $\mathcal{L}(A_1 \otimes B_1,\ A_2 \otimes B_2)$ together with the bilinear map :

$$\begin{aligned} \iota \ : \mathcal{L}(A_1, A_2) \times \mathcal{L}(B_1, B_2) &\longrightarrow \mathcal{L}(A_1 \otimes B_1,\ A_2 \otimes B_2) \\ (f, g) &\longmapsto \big(a \otimes b \mapsto f(a) \otimes g(b)\big) \end{aligned}$$

yields the map j, which is easily seen to be injective. In the finite-dimensional case we can either compute the dimensions or verify that the space $\mathcal{L}(A_1 \otimes B_1,\ A_2 \otimes B_2)$ together with the bilinear map ι fulfills the universal property. The details are left to the reader.

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Remark : There are "super" versions of the tensor product for super-vector spaces (i.e. $\mathbb{Z}_2$-graded vector spaces). Elements of vector spaces and linear morphisms are decomposed into degree zero (even) and degree one (odd) components. Then proposition I.1.2 must be modified by a minus sign if both g and a are odd : this is the *Koszul rule of signs* :

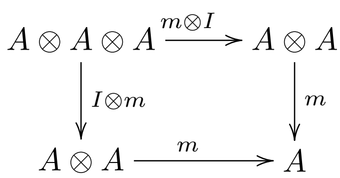
$$(f \otimes g)(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b),$$

according to which any transposition in such a formula brings a sign which is minus if and only if the two elements involved in the transposition are odd. We shall not use this $\mathbb{Z}_2$-graded framework in the sequel.

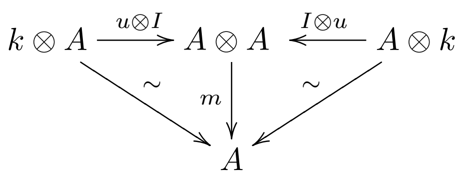
I.2. Algebras and modules

I.2.1. Basic definitions

A k-algebra is by definition a k-vector space A together with a bilinear map $m : A \otimes A \to A$ which is *associative*. The associativity is expressed by the commutativity of the following diagram :



The algebra A is *unital* if moreover there is a unit $\mathbf{1}$ in it. This is expressed by the commutativity of the following diagram :



where u is the map from k to A defined by $u(\lambda) = \lambda\mathbf{1}$. The algebra A is *commutative* if $m \circ \tau = m$, where $\tau : A \otimes A \to A \otimes A$ is the *flip*, defined by $\tau(a \otimes b) = b \otimes a$.

A subspace $J \subset A$ is called a *subalgebra* (resp. a *left ideal*, *right ideal*, *two-sided ideal*) of A if $m(J \otimes J)$ (resp. $m(J \otimes A)$, $m(A \otimes J)$, $m(J \otimes A + A \otimes J)$ is included in J.

I.2.2. Algebras and tensor product

To any vector space V we can associate its *tensor algebra* $T(V)$. As a vector space it is defined by :

$$T(V) = \bigoplus_{k \geq 0} V^{\otimes k},$$

with $V^{\otimes 0} = k$ and $V^{\otimes k+1} := V \otimes V^{\otimes k}$. The product is given by the *concatenation* :

$$m(v_1 \otimes \cdots \otimes v_p,\, v_{p+1} \otimes \cdots \otimes v_{p+q}) = v_1 \otimes \cdots \otimes v_{p+q}.$$

The embedding of $k = V^{\otimes 0}$ into $T(V)$ gives the unit map u. Tensor algebra $T(V)$ is also called the *free (unital) algebra generated by V*. This algebra is characterized by the following universal property : for any linear map φ from V to an algebra A there is a unique algebra morphism $\widetilde{\varphi}$ from $T(V)$ to A extending φ. The fact that this property characterizes $T(V)$ up to isomorphism is an easy exercise left to the reader.

Let A and B be unital k-algebras. We put a unital algebra structure on $A \otimes B$ in the following way :

$$(a_1 \otimes b_1).(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2.$$

The unit element $\mathbf{1}_{A\otimes B}$ is given by $\mathbf{1}_A \otimes \mathbf{1}_B$, and the associativity is clear. This multiplication is given by :

$$m_{A\otimes B} = (m_A \otimes m_B) \circ \tau_{23},$$

where $\tau_{23} : A \otimes B \otimes A \otimes B \to A \otimes A \otimes B \otimes B$ is defined by the flip of the two middle factors :

$$\tau_{23}(a_1 \otimes b_1 \otimes a_2 \otimes b_2) = a_1 \otimes a_2 \otimes b_1 \otimes b_2.$$

I.2.3. Modules

Let A be any unital algebra. A *left A-module* is a k-vector space M together with a map $\alpha : A \otimes M \to M$ such that the following diagrams commute :

The map α is called the action of the algebra A on M. For any $a \in A$ and $m \in M$ we usually denote by $a.m$ the action $\alpha(a \otimes m)$ of a on m. The two diagrams above express the identities :

$$(a.b).m = a.(b.m), \qquad \mathbf{1}.m = m$$

for any $a, b \in A$ and $m \in M$. The *right A-module* are defined similarly, replacing $A \otimes M$ with $M \otimes A$ (details are left to the reader). A linear subspace N of a left A-module M is called a *submodule* if $\alpha(A \otimes N) \subset N$. The intersection of all left submodules of M containing a subset P is called the *left submodule generated by P*.

A left module M is *simple* if it does not contain any submodule different from $\{0\}$ or M itself. If a left module M can be written as a direct sum of simple modules, M is said to be *semi-simple*.

Proposition I.2.1.
For any left maximal ideal J of an algebra A the quotient A/J is a simple left A-module, and conversely any simple left A-module is isomorphic to a simple left A-module of this form.

Proof. The first assertion is immediate. Conversely, let M a simple left module and let $m \in M - \{0\}$. Let J_m be the annihilator of m. It is a left ideal of A, and by simplicity of M the map :

$$\begin{aligned} \phi_m \ : A &\longrightarrow M \\ a &\longmapsto a.m \end{aligned}$$

gives rise to an morphism of left A-modules from A/J_m onto M. It is injective by definition of J_m, and surjectivity comes from the simplicity of the module M. So ϕ_m is an isomorphism. The left ideal J_m is maximal, which proves the proposition.

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Now let M be an A-module. We denote by A'_M the algebra of endomorphisms of M as an A-module, and we denote by A''_M the algebra of endomorphisms of M as an A'_M-module. Clearly any $a \in A$ gives rise to an element of A''_M. Following N. Jacobson [J Chap. 4.3] we shall give a proof of an important *density theorem* :

Theorem I.2.2.
Let M be a semi-simple A-module, and let $x_1, \ldots, x_n$ a finite collection of elements of M. Then for any $a'' \in A''_M$ there exists an element $a \in A$ such that $ax_i = a''x_i$ for any $i = 1, \ldots, n$.

Proof. First notice that any A-submodule N of M is an A''_M-submodule. To see this write (thanks to semi-simplicity) $M = N \oplus T$ where T is another A-submodule of M. The projection e on N with respect to this decomposition is an element of A'_M. For any $a'' \in A''_M$ we have then :

$$a''(N) = a'' \circ e(M) = e \circ a''(M) \subset N.$$

Consider then for any fixed positive integer n the semi-simple module M^n, direct sum of n copies of M. The algebra A'_{M^n} coincides with the algebra of $n \times n$ matrices over A'_M, and the diagonal matrices over A''_M form a subalgebra of A''_{M^n}, and thus realize an embedding of A''_M into A''_{M^n}.

Consider $x = (x_1, \ldots, x_n) \in M^n$. Then $N = A.x$ is an A-submodule of M^n. So it is an A''_{M^n}-submodule, hence an A''_M-submodule via the diagonal embedding above. Then for any $a'' \in A''_M$ there exists $a \in A$ such that $a''x = ax$, which proves the theorem.

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Corollary I.2.3.
On a semi-simple finite-dimensional module M the natural map from A into A''_M is surjective.

I.2.4. The Jacobson radical

Let A be a k-algebra. The *radical* $\operatorname{rad} M$ of a left module is by definition the intersection of all maximal submodules of M. When the module M is the algebra A itself, the radical $\operatorname{rad} A$ is the intersection of all maximal left ideals, and is called the *Jacobson radical* of the algebra A.

We shall give an alternative definition of the Jacobson radical : a *primitive ideal* is the annihilator of a simple module. In view of proposition I.2.1, any primitive ideal is the annihilator of A/J where J is a maximal left ideal. Of course a primitive ideal is two-sided.

Lemma I.2.4.
Any primitive ideal is an intersection of maximal left ideals.

Proof. Any primitive ideal J is by definition the annihilator of a simple module M. The annihilator J_m of any $m \in M - \{0\}$ is then a maximal left ideal containing J, and it is clear that we have :

$$J = \bigcap_{m \in M - \{0\}} J_m.$$

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Proposition I.2.5.
The Jacobson radical of A is the intersection of its primitive ideals.

Proof. Let us call P the intersection of all primitive ideals of A. By lemma I.2.4 and proposition I.2.1, P is indeed the intersection of all maximal left ideals.

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Lemma I.2.6 (Nakayama's lemma).
Let M a finitely generated A-module, and let N, L two submodules of M such that $M = L + N$ and $N \subset \operatorname{rad} M$. Then $L = M$.

Proof. Suppose that L is strictly contained in M. As M is finitely generated there exists a maximal nontrivial submodule $\widetilde{L}$ containing L. It contains N as well by definition of $\operatorname{rad} M$. Then $\widetilde{L}$ contains $L + N$, so $L + N$ cannot be equal to M.

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Corollary I.2.7.
The Jacobson radical of a finite-dimensional algebra is nilpotent.

Proof. Let A a finite-dimensional algebra with radical R. Observe first that for any A-module M we have the inclusion :

$$R.M \subset \operatorname{rad} M.$$

Indeed any maximal submodule N of M contains $J.M$ where J is a primitive ideal, namely the annihilator of M/N. Hence any maximal submodule of M contains $R.M$. Suppose now that A is finite dimensional and that J is an ideal of A such that $R.J = J$. A fortiori $\operatorname{rad} J = J$. Applying Nakayama's lemma I.2.6 to $M = J$ and $L = \{0\}$ we get $J = \{0\}$. We immediately deduce from this fact that for any positive integer n, R^n either contains strictly R^{n+1} or is equal to $\{0\}$. As A is finite-dimensional R^n is indeed equal to $\{0\}$ for some n.

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Remark : although the definition of the Jacobson ideal is not symmetric (because we used left ideals and left modules), the Jacobson ideal itself is a symmetric notion : in other words the Jacobson radicals of algebras A and A^{opp} coincide. In order to see this one can show that $\operatorname{Rad} A$ is the biggest two-sided ideal J such that $1-x$ is invertible for any $x \in J$ [B §6.3]. This definition is indeed symmetric.

I.2.5. Maximal two-sided ideals

It is easily seen that any maximal two-sided ideal in an algebra A is primitive. The converse is false in general : for example, in the enveloping algebra of the non-trivial two-dimensional Lie algebra, the ideal $\{0\}$ is primitive but not maximal [Di]. However we have the following result :

Proposition I.2.8.
Any finite-codimensional primitive ideal is maximal.

Before giving a proof of this result, we need the following definitions : an algebra A is *simple* if it does not contain any proper two-sided ideal. A *division algebra* is an algebra such that any nonzero element is inversible.

Lemma I.2.9.
Let D be a division algebra. Then the algebra $M_n(D)$ of square $n \times n$ matrices over D is simple.

Proof. Let us consider for any $i, j \in \{1, \ldots, n\}$ the elementary matrix e_{ij} with vanishing entries except the one on the i$^{\text{th}}$ row and j$^{\text{th}}$ column which is equal to the unit of D. Denote by I_i the left ideal of $M_n(D)$ consisting of matrices such that all columns vanish except the i$^{\text{th}}$ column. These left ideals are all simple and isomorphic as $M_n(D)$-modules. Now let I a nonzero two-sided ideal of $M_n(D)$. Let X a nonzero element of I, and x_{kl} a nonzero entry of the matrix X. Then for any $i \in \{0, \ldots, n\}$ the product $e_{ik}X$ belongs to $I_i \cap I$ and is different from 0.

Then $I_i \cap I \neq \{0\}$. As I_i is simple as a left module for each i that means that I contains all the left ideals I_i, and hence $I = M_n(D)$.

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There is a converse to this result, namely any simple *artinian* algebra (a fortiori any simple finite-dimensional algebra) is isomorphic to $M_n(D)$ where D is a division algebra. This is a particular case of the *Wedderburn-Artin theorem*, which gives a complete description of *semi-simple algebras* [DF Chap. 1], [DK Chap. 2].

Proof of Proposition I.2.8 : a finite-dimensional primitive ideal I is the annihilator of a simple finite-dimensional module M. By simplicity of M the algebra $D = A'_M$ is a division algebra (Schur's lemma). The action of A on M yields (thanks to corollary I.2.3) a surjective algebra morphism from A onto A''_M, and hence an algebra isomorphism from A/J onto A''_M. But A''_M is a matrix algebra over D, and then is simple (according to lemma I.2.7). So A/J is a simple algebra, which amounts to say that J is maximal as a two-sided ideal.

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Corollary I.2.10.
In a finite-dimensional algebra, primitive ideals and maximal two-sided ideals coincide. In particular the Jacobson radical is the intersection of all maximal two-sided ideals in this case.

I.3. Coalgebras and comodules

This paragraph is mostly borrowed from M.E. Sweedler's book [Sw], particularly chapters 1, 2, 8 and 9.

I.3.1. Coalgebras

Coalgebras are objects wich are somehow dual to algebras : axioms for coalgebras are derived from axioms for algebras by reversing the arrows of the corresponding diagrams :

A k-coalgebra is by definition a k-vector space C together with a bilinear map $\Delta : C \to C \otimes C$ which is *co-associative*. The co-associativity is expressed by the commutativity of the following diagram :

$$\begin{array}{ccc} C \otimes C \otimes C & \xleftarrow{\Delta \otimes I} & C \otimes C \\ \uparrow{\scriptstyle I \otimes \Delta} & & \uparrow{\scriptstyle \Delta} \\ C \otimes C & \xleftarrow{\Delta} & C \end{array}$$

Coalgebra C is *co-unital* if moreover there is a co-unit ε such that the following diagram commutes :

$$\begin{array}{ccccc} k \otimes C & \xleftarrow{\varepsilon \otimes I} & C \otimes C & \xrightarrow{I \otimes \varepsilon} & C \otimes k \\ & \nwarrow{\scriptstyle \sim} & \uparrow{\scriptstyle \Delta} & \nearrow{\scriptstyle \sim} & \\ & & C & & \end{array}$$

A subspace $J \subset C$ is called a *subcoalgebra* (resp. a *left coideal, right coideal, two-sided coideal*) of C if $\Delta(J)$ is contained in $J \otimes J$ (resp. $J \otimes C$, $C \otimes J$, $J \otimes C + C \otimes J$) is included in J. The duality alluded to above can be made more precise :

Proposition I.3.1.
1)The linear dual C^* of a co-unital coalgebra C is a unital algebra, with product (resp. unit map) the transpose of the coproduct (resp. of the co-unity).

2)Let J a linear subspace of C. Denote by $J^\perp$ the orthogonal of J in C^*. Then :

J is a two-sided coideal if and only if $J^\perp$ is a subalgebra of C^*.
J is a left coideal if and only if $J^\perp$ is a left ideal of C^*.
J is a right coideal if and only if $J^\perp$ is a rightt ideal of C^*.
J is a subcoalgebra if and only if $J^\perp$ is a two-sided ideal of C^*.

Proof. For any subspace K of C^* we shall denote by $K^\perp$ the subspace of those elements of C on which any element of K vanishes. It coincides with the intersection of the orthogonal of K with C, via the canonical embedding $C \hookrightarrow C^{**}$. So we have for any linear subspaces $J \subset C$ and $K \subset C^*$:

$$J^{\perp\perp} = J, \qquad\qquad K^{\perp\perp} \supset K.$$

Suppose that J is a two-sided coideal. Take any ξ, η in $J^\perp$. For any $x \in J$ we have :

$$< \xi\eta, x > = < \xi \otimes \eta, \Delta x > = 0,$$

as $\Delta x \subset J \otimes C + C \otimes J$. So $J^\perp$ is a subalgebra of C^*. Conversely if $J^\perp$ is a subalgebra then :

$$\Delta J \subset (J^\perp \otimes J^\perp)^\perp = J \otimes C + C \otimes J,$$

which proves the first assertion. We leave it to the reader as an exercice to prove the three other assertions along the same lines. Dually we have the following :

Proposition I.3.2.
Let K a linear subspace of C^*. Then :

$K^\perp$ is a two-sided coideal if and only if K is a subalgebra of C^*.
$K^\perp$ is a left coideal if and only if K is a left ideal of C^*.
$K^\perp$ is a right coideal if and only if K is a rightt ideal of C^*.
$K^\perp$ is a subcoalgebra if and only if K is a two-sided ideal of C^*.

The linear dual $(C \otimes C)^*$ naturally contains the tensor product $C^* \otimes C^*$. Take as a multiplication the restriction of ${}^t\Delta$ to $C^* \otimes C^*$:

$$m = {}^t\Delta \ : C^* \otimes C^* \longrightarrow C^*,$$

and put $u = {}^t\varepsilon \ : k \to C^*$. It is easily seen, by just reverting the arrows of the corresponding diagrams, that coassociativity of Δ implies associativity of m, and that the co-unit property for ε implies that u is a unit.

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The coalgebra C is *cocommutative* if $\tau \circ \Delta = \Delta$, where $\tau : C \otimes C \to C \otimes C$ is the flip. It will be convenient to use *Sweedler's notation* :

$$\Delta x = \sum_{(x)} x_1 \otimes x_2.$$

Cocommutativity expresses then as :

$$\sum_{(x)} x_1 \otimes x_2 = \sum_{(x)} x_2 \otimes x_1.$$

Coassociativity reads in Sweedler's notation :

$$(\Delta \otimes I) \circ \Delta(x) = \sum_{(x)} x_{1:1} \otimes x_{1:2} \otimes x_2 = \sum_{(x)} x_1 \otimes x_{2:1} \otimes x_{2:2} = (I \otimes \Delta) \circ \Delta(x),$$

We shall sometimes write the iterated coproduct as :

$$\sum_{(x)} x_1 \otimes x_2 \otimes x_3.$$

Sometimes we shall even mix the two ways of using Sweedler's notation for the iterated coproduct, in the case we want to keep partially track of how we have constructed it [DNR]. For example,

$$\begin{aligned}\Delta_3(x) &= (I \otimes \Delta \otimes I) \circ (\Delta \otimes I) \circ \Delta(x)\\ &= (I \otimes \Delta \otimes I)(\sum_{(x)} x_1 \otimes x_2 \otimes x_3)\\ &= \sum_{(x)} x_1 \otimes x_{2:1} \otimes x_{2:2} \otimes x_3.\end{aligned}$$

To any vector space V we can associate its *tensor coalgebra* $T^c(V)$. It is isomorphic to $T(V)$ as a vector space. The coproduct is given by the *deconcatenation* :

$$\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_{p=0}^{n} (v_1 \otimes \cdots \otimes v_p) \bigotimes (v_{p+1} \otimes \cdots \otimes v_n).$$

The co-unit is given by the natural projection of $T^c(V)$ onto k.

Let C and D be unital k-coalgebras. We put a co-unital coalgebra structure on $C \otimes D$ in the following way : the comultiplication is given by :

$$\Delta_{C \otimes D} = \tau_{23} \circ (\Delta_C \otimes \Delta_D),$$

where τ_{23} is again the flip of the two middle factors, and the co-unity is given by $\varepsilon_{C \otimes D} = \varepsilon_C \otimes \varepsilon_D$.

I.3.2. Comodules

Let C be any co-unital coalgebra. A *left C-comodule* is a k-vector space M together with a map $\Phi : M \to C \otimes M$ such that the following diagrams commute :

$$\begin{array}{ccc} C \otimes C \otimes M & \xleftarrow{\Delta \otimes I} & C \otimes M \\ \uparrow{\scriptstyle I \otimes \Phi} & & \uparrow{\scriptstyle \Phi} \\ C \otimes M & \xleftarrow{\Phi} & M \end{array} \qquad\qquad \begin{array}{ccc} k \otimes M & \xleftarrow{\varepsilon \otimes I} & C \otimes M \\ & \nwarrow{\scriptstyle \sim} & \uparrow{\scriptstyle \Phi} \\ & & C \end{array}$$

The notion of *right C-comodule* is defined similarly. A linear subspace N of a left C-comodule M is called a *subcomodule* if $\Phi(C) \subset C \otimes N$. The intersection of all left subcomodules of M containing a subset P is again a subcomodule, called the *left subcomodule generated by P*.

It will be convenient to use again Sweedler's notation :

$$\Phi(m) = \sum_{(m)} m_1 \otimes m_0,$$

with $m_0 \in M$ and $m_1 \in C$. Comodule property reads in Sweedler's notation :

$$(\Phi \otimes I) \circ \Phi(m) = \sum_{(x)} m_{1:1} \otimes m_{1:2} \otimes m_0 = \sum_{(m)} m_1 \otimes m_{0:1} \otimes m_{0:0} = (I \otimes \Delta) \circ \Phi(m),$$

We shall sometimes write the iterated coproduct as :

$$\sum_{(m)} m_1 \otimes m_2 \otimes m_0.$$

For a right comodule we have a similar behaviour with m_0 on the left. The notion of comodule is dual to the notion of module in the sense that any left (resp. right) C-comodule M admits a right (resp. left) C^*-module structure. To be precise suppose for the moment that Φ is any linear map from M to $M \otimes C$, and define $\alpha_\Phi : C^* \otimes M \to M$ as the following composition :

$$C^* \otimes M \xrightarrow{I \otimes \Phi} C^* \otimes M \otimes C \xrightarrow{\tau \otimes I} M \otimes C^* \otimes C \xrightarrow{I \otimes <-,->} M \otimes k \xrightarrow{\sim} M.$$

Then :

Proposition I.3.3.
(M, Φ) is a right (resp. left) C-comodule if and only is (M, α_Φ) is a left (resp. right) C^*-module.

Proof. See Sweedler [Sw] section 2.1.

•

Note that the duality property is not perfect : if the linear dual of a coalgebra is always an algebra, the linear dual of an algebra is not in general a coalgebra. However the *restricted dual* A° of an algebra A is a coalgebra. It is defined as the space of linear forms on A vanishing on some finite-codimensional ideal. Along the same lines, for a coalgebra C the only left C^*-modules related to a right C-comodule structure via proposition I.3.3 are the *rational* left C^*-modules, i.e. those left modules such that the linear map :

$$\begin{aligned} \rho : M &\longrightarrow \mathcal{L}(C^*, M) \\ m &\longmapsto (x \mapsto x.m) \end{aligned}$$

has image included in $M \otimes C$ via the embedding :

$$\begin{aligned} j : M \otimes C &\longrightarrow \mathcal{L}(C^*, M) \\ m \otimes c &\longmapsto (x \mapsto < x, c > m). \end{aligned}$$

See [Sw] for details. We come now to the fundamental theorem of comodule structure theory :

Theorem I.3.4.
Let M be a left comodule over a coalgebra C. For any element $m \in M$ the subcomodule generated by m is finite-dimensional.

Proof. There is a finite collection $(c_i)_{i=1\cdots s}$ of linearly independant elements of C and a collection $(m_i)_{i=1\cdots s}$ of elements of M such that

$$\Phi(m) = \sum_{i=1}^{s} m_i \otimes c_i.$$

Let N be the linear subspace of M generated by $m_1, \ldots, m_s$. Let us show that N is a left subcomodule of M. First note that thanks to the co-unit axiom we have :

$$m = (I \otimes \varepsilon) \circ \Phi(m) = \sum_{j=1}^{s} \varepsilon(c_j) m_j,$$

hence $m \in N$. On the other hand, considering linear forms $(f_i)_{i=1\cdots s}$ on C such that $f_i(c_j) = \delta_i^j$ we have :

$$\begin{aligned} \Phi(m_i) &= (I \otimes I \otimes f_i)\big(\Phi(m_i) \otimes c_i\big) \\ &= (I \otimes I \otimes f_i)\Big(\sum_{j=1}^{s} \Phi(m_j) \otimes c_j\Big) \\ &= (I \otimes I \otimes f_i) \circ (\Phi \otimes I) \circ \Phi(m) \\ &= (I \otimes I \otimes f_i) \circ (I \otimes \Delta) \circ \Phi(m) \\ &= \sum_{j=1}^{s} m_j \otimes (I \otimes f_i)(\Delta c_j), \end{aligned}$$

whence $\Phi(m_i) \in N \otimes C$, which proves the theorem.

•

Corollary I.3.5.
Let M be a left comodule over a coalgebra C. Then any left subcomodule of M generated by a finite set is finite-dimensional.

Proof. remark that if $P = \{m_1, \ldots, m_n\}$, the left subcomodule generated by P is the sum of the left comodules generated by the m_j's, and then apply theorem I.3.4.

•

I.3.3. Structure of coalgebras

Let C be a coalgebra. Any intersection of subcoalgebras is a subcoalgebra. To see this consider any family $(D_\alpha)_{\alpha\in\Lambda}$ of subcoalgebras of C. Then $I := \sum_\alpha D_\alpha^\perp$ is a two-sided ideal of C^*, as a sum of two-sided ideals. Hence $I^\perp$ is a subcoalgebra according to proposition I.3.2. But $I^\perp$ is indeed the intersection of the subcoalgebras D_α.

In particular, the intersection of all subcoalgebras containing a given subset P of C will be called the subcoalgebra generated by P. We can now state the fundamental theorem of coalgebra theory :

Theorem I.3.6.
Let C be a coalgebra. Then the subcoalgebra generated by one single element x is finite-dimensional.

Proof. The coalgebra C is a left comodule over itself. Let N be the left subcomodule generated by x. According to theorem I.3.4, N is finite-dimensional. Then $N^\perp$ has finite codimension, equal to $\dim N$. It is a left ideal thanks to proposition I.3.1. The quotient space $E = C^*/N^\perp$ is a finite-dimensional left module over $C^\perp$. Let K be the annihilator of this left module. As kernel of the associated representation $\rho : C^* \to \operatorname{End} E$ it has clearly finite codimension, and it is a two-sided ideal.

Now $K^\perp$ is a subcoalgebra according to proposition I.3.2. Moreover it is finite-dimensional, as $\dim K^\perp = \operatorname{codim} K^{\perp\perp} \leq \operatorname{codim} K$. Finally $K \subset N^\perp$ implies that $N^{\perp\perp} \subset K^\perp$. A fortiori $N \subset K^\perp$, so x belongs to $K^\perp$. The subcoalgebra generated by x is then included in the finite-dimensional subcoalgebra $K^\perp$, which proves the theorem.

•

A coalgebra C is said to be *irreducible* if two nonzero subcoalgebras of C have always nonzero intersection. A *simple* coalgebra is a coalgebra which does not contain any proper subcoalgebra. A coalgebra C will be called *pointed* if any simple subcoalgebra of C is one-dimensional.

Lemma I.3.7.
Any coalgebra C contains a simple subcoalgebra.

Proof. According to theorem I.3.6 we may suppose that C is finite-dimensional, and the lemma is immediate in this case.

•

Proposition I.3.8.
A coalgebra C is irreducible if and only if it contains a unique simple subcoalgebra.

Proof. Suppose C irreducible, and suppose that D_1 and D_2 are two simple subcoalgebras. The intersection $D_1 \cap D_2$ is nonzero, and hence, by simplicity, $D_1 = D_2$. Conversely suppose that E is the only simple subcoalgebra of C, and let D any subcoalgebra. According to lemma I.3.7 we have $E \subset D$, hence E is included in any intersection of subcoalgebras, which proves that C is irreducible.

•

Lemma I.3.9.
Let $(C_\alpha)_{\alpha\in\Lambda}$ a family of subcoalgebras of a coalgebra C such that C is the direct sum of the C_α's. Then for any subcoalgebra D we have :

$$D = \bigoplus_{\alpha\in\Lambda} D \cap C_\alpha.$$

Proof. The sum is indeed direct and included in D. To prove the reverse inclusion, pick any y in D and decompose it inside C :

$$y = \sum_{\alpha\in\Lambda} y_\alpha$$

with $y_\alpha \in C_\alpha$ (finite sum). Consider for any $\gamma \in \Lambda$ the linear form f_γ defined by :

$$\begin{aligned} f_\gamma|_{C_\alpha} &= \varepsilon|_{C_\alpha} \text{ si } \alpha = \gamma \\ &= 0 \text{ si } \alpha \neq \gamma. \end{aligned}$$

Then $f_\gamma(y) = \varepsilon(y_\gamma)$. Now we have :

$$\begin{aligned} (I \otimes f_\gamma) \circ \Delta(y) &= \sum_\alpha (I \otimes f_\gamma) \circ \Delta(y_\alpha) \\ &= \sum_\alpha \sum_{(y_\alpha)} (y_\alpha)_1 f_\gamma\big((y_\alpha)_2\big) \\ &= \sum_{(y_\gamma)} (y_\gamma)_1 \varepsilon\big((y_\gamma)_2\big) \\ &= (I \otimes \varepsilon) \circ \Delta(y_\gamma) \\ &= y_\gamma. \end{aligned}$$

This shows that y_γ in in D, which proves the lemma.

•

Let us define the *coradical* of a coalgebra C as the sum R of its simple subcoalgebras. As indicated by the terminology this notion is dual to the notion of Jacobson radical of an algebra : cf. proposition I.3.11 below.

Proposition I.3.10.
Let R be the coradical of a coalgebra C. Then for any subcoalgebra D the coradical R_D of D is equal to $R \cap D$.

Proof. Any simple subcoalgebra of D is a simple subcoalgebra of C, so $R_D \subset R \cap D$. Conversely by lemma I.3.9 $R \cap D$ is a direct sum of simple subcoalgebras of D, hence $R \cap D \subset R_D$.

•

Proposition I.3.11.
If C is a finite-dimensional coalgebra with coradical R, then $R^\perp$ is the Jacobson radical of the algebra C^*.

Proof. If S is a simple subcoalgebra of C it is clear from dimension considerations that $S^\perp$ is a maximal two-sided ideal of C^*. Conversely any maximal two-sided ideal of C^* is the orthogonal of a simple subcoalgebra, so $R^\perp$ is indeed the intersection of all maximal two-sided ideals of C^*. Finally (lemma I.2.8), maximal two-sided and primitive ideals of a finite-dimensional algebra coincide.

•

I.3.4. The wedge

Let C be a coalgebra and X, Y two linear subspaces of C. We define :

$$X \wedge Y = \{x \in C,\ \Delta x \in X \otimes C + C \otimes Y\}.$$

We define as well inductively :

$$\wedge^0 X = \{0\}, \qquad \wedge^n X = (\wedge^{n-1} X) \wedge X.$$

The alternative definition in terms of the algebra structure on C^* is often more manageable, and its verification is straightforward :

$$X \wedge Y = (X^\perp Y^\perp)^\perp.$$

Here is a first application of this definition :

Proposition I.3.12.
1) The wedge is associative : $(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z)$.
2) If X is a left coideal then $\{0\} \wedge X = X$.
3) If X is a left coideal and Y is a right coideal, then $X \wedge Y$ is a subcoalgebra of C.
4) The wedge of two subcoalgebras is a subcoalgebra.
5) If $X \subset X'$ and $Y \subset Y'$, then $X \wedge Y \subset X' \wedge Y'$.

Proof. According to the definition we have :

$$(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z) = (X^\perp Y^\perp Z^\perp)^\perp.$$

Let X (resp. Y) be a left (resp. right) coideal. According to proposition I.3.1, $X^\perp$ is a left ideal of C^* and $Y^\perp$ is a right ideal. The product $(X^\perp Y^\perp)$ is then a two-sided ideal. Second and third assertions follow by applying proposition I.3.1 again, and by noticing that :

$$\{0\} \wedge X = (C^*.X^\perp)^\perp = X^{\perp\perp} = X.$$

4) is an immediate consequence of 3), and 5) is clear.

•

The wedge admits the following comodule version : let M be a right C-comodule with coaction $\Phi : M \to M \otimes C$. Let N be a subspace of M and X be a subspace of C. We define :

$$N \wedge X = \{x \in M,\ \Phi x \in N \otimes C + C \otimes X\}.$$

One can check that if X is a right coideal the $N \wedge X$ is a subcomodule, and if N is a subcomodule then $N \subset N \wedge X$.

Proposition I.3.13.
Let R be the coradical of a coalgebra C, and let M be a right C-comodule. Then for $\{0\} \subset M$ we have :

$$\bigcup_n \{0\} \wedge (\wedge^n M) = M.$$

Proof. Let $x \in M$, and let N be a finite-dimensional subcomodule containing x (which exists thanks to theorem I.3.4). If Φ denotes the coaction, then $\Phi(N) \subset N \otimes X$ where X is a finite-dimensional subspace of C. Let D be a finite-dimensional subcoalgebra containing X (which exists thanks to theorem I.3.6). It is clear that N is a right D-comodule.

Applying Proposition I.3.10, the coradical of D is $R_0 = R \cap D$. Proposition I.3.11 says that $R_0^\perp$ is the Jacobson radical of D^*, which is nilpotent by corollary I.2.7 : there exists a positive integer n such that $(R_0^\perp)^n = \{0\}$. Dualizing we get that $\wedge_D^n R_0 = D$, where the subscribe D reminds with respect to which coalgebra the wedge operation is performed. Clearly we have :

$$\wedge_D^n R_0 \subset \wedge_C^n R_0 \subset \wedge_C^n R,$$

the second inclusion coming from assertion 5) of proposition I.3.12. We have then :

$$D \subset \wedge^n R$$

(we have dropped the subscribe "C" here), hence the inclusions :

$$N \subset \{0\} \wedge D \subset \{0\} \wedge (\wedge^n R).$$

The initial element x belongs then to $\{0\} \wedge (\wedge^n R)$ for some n, which proves the assertion.

•

I.3.5. The coradical filtration

Let C be a coalgebra with coradical R. We consider for any integer $i \geq 0$:

$$C^i = \wedge^{i+1} R.$$

The following proposition is an immediate consequence of propositions I.3.12 and I.3.13 :

Proposition I.3.14.
$(C^i)_{i\geq 0}$ is an increasing sequence of subcoalgebras of C, and we have :

$$C = \bigcup_{i\geq 0} C^i.$$

The coalgebra C is then endowed with an increasing filtration by subcoalgebras : its *coradical filtration.*

Proposition I.3.15.
The coproduct is compatible with the coradical filtration, in the sense that the following inclusion holds :

$$\Delta(C^n) \subset \sum_{i=0}^{n} C^i \otimes C^{n-i}.$$

Proof. For any $i \in \{0, \ldots, n+1\}$ we have :

$$\wedge^{n+1} R = (\wedge^i R) \wedge (\wedge^{n-i+1} R).$$

This is immediate for $i = 1, \ldots, n$, and comes from proposition I.3.12 assertion 2) for $i = 0$ and $i = n+1$. We have then (setting $C^{-1} = \{0\}$) :

$$\Delta C^n \subset \bigcap_{i=0}^{n+1} (C \otimes C^{n-i} + C^{i-1} \otimes C).$$

The right-hand side (RHS) is contained in $C^n \otimes C^n$. Choose any supplementary subspace D_i of C^{i-1} inside C^i : so $C^0 = D_0$, and $C^i = D_0 \oplus \cdots \oplus D_i$. We have :

$$\begin{aligned}(RHS) &= \bigcap_{i=0}^{n+1} \bigoplus_{r \leq i-1 \text{ or } s \leq n-i} D_r \otimes D_s \\ &= \bigoplus_{r+s\leq n} D_r \otimes D_s \\ &= \sum_{i=0}^{n} C^i \otimes C^{n-i},\end{aligned}$$

which proves the result.

•

I.4. Convolution product

Let A be an algebra and C be a coalgebra (over the same field k). Then there is an associative product on $\mathcal{L}(C, A)$ called the *convolution product*. It is given by :

$$\varphi * \psi = m_A \circ (\varphi \otimes \psi) \circ \Delta_C.$$

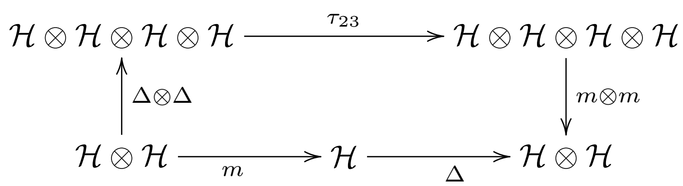
In Sweedler's notation it reads :

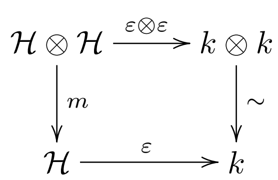
$$\varphi * \psi(x) = \sum_{(x)} \varphi(x_1)\psi(x_2).$$

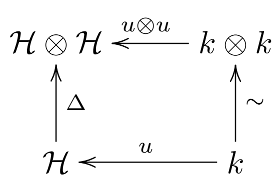
The associativity is a direct consequence of both associativity of A and coassociativity of C. We shall study this product more thoroughly in the next chapters.

I.5. Bialgebras and Hopf algebras

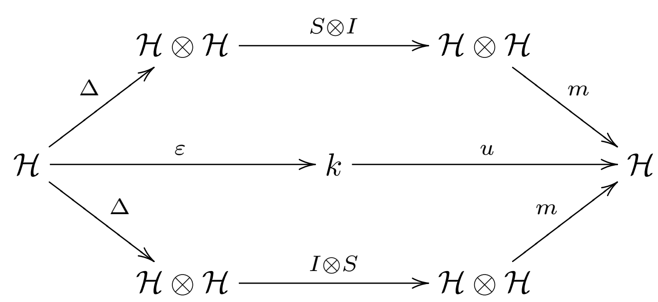
A (unital and co-unital) *bialgebra* is a vector space $\mathcal{H}$ endowed with a structure of unital algebra (m, u) and a structure of co-unital coalgebra (Δ, ε) which are compatible. The compatibility requirement is that Δ is an algebra morphism (or equivalently that m is a coalgebra morphism), ε is an algebra morphism and u is a coalgebra morphism. It is expressed by the commutativity of the three following diagrams :







A *Hopf algebra* is a bialgebra $\mathcal{H}$ together with a linear map $S : \mathcal{H} \to \mathcal{H}$ called the *antipode*, such that the following diagram commutes :



In Sweedler's notation it reads :

$$\sum_{(x)} S(x_1)x_2 = \sum_{(x)} x_1 S(x_2) = (u \circ \varepsilon)(x).$$

In other words the antipode is an inverse of the identity I for the convolution product on $\mathcal{L}(H, H)$. The unit for the convolution is the map $u \circ \varepsilon$.

Let $\mathcal{H}$ be a bialgebra. A *primitive element* in $\mathcal{H}$ is an element x such that $\Delta x = x \otimes 1 + 1 \otimes x$. A *grouplike element* is a nonzero element x such that $\Delta x = x \otimes x$. Note that grouplike elements make sense in any coalgebra.

I.6. Examples

I.6.1. The Hopf algebra of a group

Let G be a group, and let kG be the group algebra (over the field k). It is by definition the vector space freely generated by the elements of G : the product of G extends uniquely to a bilinear map from $kG \times kG$ into kG, hence a multiplication $m : kG \otimes kG \to kG$, which is associative. The neutral element of G gives the unit for m.

The space kG is also endowed with a co-unital coalgebra structure, given by :

$$\Delta(\sum \lambda_i g_i) = \sum \lambda_i . g_i \otimes g_i$$

and :

$$\varepsilon(\sum \lambda_i g_i) = \sum \lambda_i.$$

This defines the *coalgebra of the set G* : it does not take into account the extra group structure on G, as the algebra structure does.

Proposition I.6.1.
The vector space kG endowed with the algebra and coalgebra structures defined above is a Hopf algebra. The antipode is given by :

$$S(g) = g^{-1}, g \in G.$$

Proof. The compatibility of the product and the coproduct is an immediate consequence of the following computation : for any $g, h \in G$ we have :

$$\Delta(gh) = gh \otimes gh = (g \otimes g)(h \otimes h) = \Delta g . \Delta h.$$

Now $m(S \otimes I)\Delta(g) = g^{-1}g = e$ and similarly for $m(I \otimes S)\Delta(g)$. But $e = u \circ \varepsilon(g)$ for any $g \in G$, so map S is indeed the antipode.

•

Remark : if G were only a semigroup, the same construction would lead to a bialgebra structure on kG : the Hopf algebra structure (i.e. the existence of an antipode) reflects the group structure (the existence of the inverse). We have $S^2 = I$ in this case, but involutivity of the antipode is not true for general Hopf algebras.

I.6.2. Tensor algebras

There is a natural structure of cocommutative Hopf algebra on the tensor algebra $T(V)$ of any vector space V. Namely we define the coproduct Δ as the unique algebra morphism from $T(V)$ into $T(V) \otimes T(V)$ such that :

$$\Delta(1) = 1 \otimes 1, \qquad \Delta(x) = x \otimes 1 + 1 \otimes x, \ x \in V.$$

We define the co-unit as the algebra morphism such that $\varepsilon(1) = 1$ and $\varepsilon_{|_V} = 0$ This endows $T(V)$ with a cocommutative bialgebra structure. We claim that the principal anti-automorphism :

$$S(x_1 \otimes \cdots \otimes x_n) = (-1)^n x_n \otimes \cdots \otimes x_1$$

verifies the axioms of an antipode, so that $T(V)$ is indeed a Hopf algebra. For $x \in V$ we have $S(x) = -x$, hence $S * I(x) = I * S(x) = 0$. As V generates $T(V)$ as an algebra it is easy to conclude.

I.6.3. Enveloping algebras

Let $\mathfrak{g}$ a Lie algebra. The universal enveloping algebra is the quotient of the tensor algebra $T(\mathfrak{g})$ by the ideal J generated by $x \otimes y - y \otimes x - [x, y], \ x, y \in \mathfrak{g}$.

Lemma I.6.2.
J is a Hopf ideal, i.e. $\Delta(J) \subset \mathcal{H} \otimes J + J \otimes \mathcal{H}$ and $S(\mathcal{H}) \subset \mathcal{H}$.

Proof. The last assertion is immediate. The first comes easily from the fact that the ideal J is generated by primitive elements (according to proposition I.7.3 below) : indeed any ideal generated by primitive elements is a Hopf ideal (very easy and left to the reader). •

The quotient of a Hopf algebra by a Hopf ideal is a Hopf algebra. Hence the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a cocommutative Hopf algebra.

I.7. Some properties of Hopf algebras

We summarize in the proposition below the main properties of the antipode in a Hopf algebra :

Proposition I.7.1 (cf. [Sw] proposition 4.0.1).
Let $\mathcal{H}$ be a Hopf algebra with multiplication m, comultiplication Δ, unit $u : 1 \mapsto \mathbf{1}$, co-unit ε and antipode S. Then :

1) $S \circ u = u$ and $\varepsilon \circ S = \varepsilon$.

2) S is an algebra antimorphism and a coalgebra antimorphism, i.e. if τ denotes the flip we have :

$$m \circ (S \otimes S) \circ \tau = S \circ m, \qquad \tau \circ (S \otimes S) \circ \Delta = \Delta \circ S.$$

3) If $\mathcal{H}$ is commutative or cocommutative, then $S^2 = I$.

Proof. We follow closely the proof given by Chr. Kassel here [K]. Starting from $\varepsilon(\mathbf{1}) = 1$ and $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$, we get : $\mathbf{1} = m(S \otimes I)\Delta(\mathbf{1}) = S(\mathbf{1})\mathbf{1}$, so $S(\mathbf{1}) = \mathbf{1}$. We have for any $x \in \mathcal{H}$:

$$\varepsilon \circ u \circ \varepsilon(x) = \varepsilon(\varepsilon(x).\mathbf{1}) = \varepsilon(x).$$

Then,

$$\begin{aligned}\varepsilon(x) &= \varepsilon\big(m(S \otimes I)\Delta\big)(x) \\ &= \varepsilon\big(\sum_{(x)} S(x_1)x_2\big) \\ &= \sum_{(x)} (\varepsilon \circ S)(x_1)\varepsilon(x_2).\end{aligned}$$

On the other hand,

$$S(x) = S * (u \circ \varepsilon)(x) = \sum_{(x)} S(x_1)(u \circ \varepsilon)(x_2).$$

Hence,

$$(\varepsilon \circ S)(x) = \sum_{(x)} (\varepsilon \circ S)(x_1)\varepsilon(x_2).$$

This proves assertion 1). Now consider $m, N, P \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H}, \mathcal{H})$ defined by :

$$m(x \otimes y) = xy, \qquad N(x \otimes y) = S(y)S(x), \qquad P(x \otimes y) = S(xy).$$

Considering the convolution product $\widetilde{*}$ on $\mathcal{L}(\mathcal{H} \otimes \mathcal{H}, \mathcal{H})$ we shall prove :

Lemma I.7.2.

$$P \widetilde{*} m = u \circ \varepsilon_{\mathcal{H} \otimes \mathcal{H}} = m \widetilde{*} N.$$

Proof. We compute, with Sweedler's notation :

$$\begin{aligned}P \widetilde{*} m(x \otimes y) &= \sum_{(x \otimes y)} P\big((x \otimes y)_1\big)m\big((x \otimes y)_2\big) \\ &= \sum_{(x),(y)} P(x_1 \otimes y_1)m(x_2 \otimes y_2) \\ &= \sum_{(x),(y)} S(x_1y_1)x_2y_2 \\ &= (S * I)(xy) \\ &= (u \circ \varepsilon)(xy) \\ &= (u \circ \varepsilon_{\mathcal{H} \otimes \mathcal{H}})(x \otimes y),\end{aligned}$$

and :

$$
\begin{aligned}
m\widetilde{*}N(x\otimes y) &= \sum_{(x),(y)} m(x_1\otimes y_1)N(x_2\otimes y_2)\\
&= \sum_{(x),(y)} x_1y_1S(y_2)S(x_2)\\
&= \sum_{(x)} x_1(u\circ\varepsilon)(y)S(x_2)\\
&= (u\circ\varepsilon)(x)(u\circ\varepsilon)(y)\\
&= (u\circ\varepsilon_{\mathcal{H}\otimes\mathcal{H}})(x\otimes y),
\end{aligned}
$$

which proves the lemma.

•

End of proof of Proposition I.7.1 : as $u\circ\varepsilon_{\mathcal{H}\otimes\mathcal{H}}$ is the unit element for $\widetilde{*}$ lemma proves that both P and N are inverse of m for this convolution. Hence :

$$P = P\widetilde{*}(m\widetilde{*}N) = (P\widetilde{*}m)\widetilde{*}N = N,$$

which proves the first part of assertion 2). The second part is proved similarly using convolution in $\mathcal{L}(\mathcal{H},\mathcal{H}\otimes H)$ with m replaced with Δ, N replaced with $\tau\circ(S\otimes S)\circ\Delta$ and P replaced with $\Delta\circ S$. Indeed,

$$
\begin{aligned}
\big((\Delta\circ S)\widetilde{*}\Delta\big)(x) &= (m\otimes m)\circ\tau_{23}\circ\big((\Delta\circ S)\otimes\Delta\big)\circ\Delta(x)\\
&= (m\otimes m)\circ\tau_{23}\circ(\Delta\otimes\Delta)\circ(S\otimes I)\circ\Delta(x)\\
&= \Delta\circ m\circ(S\otimes I)\circ\Delta(x)\\
&= \Delta\big(u\circ\varepsilon(x)\big)\\
&= u\circ\varepsilon(x)\otimes u\circ\varepsilon(x),
\end{aligned}
$$

and :

$$
\begin{aligned}
\Delta\widetilde{*}\big(\tau\circ(S\otimes S)\circ\Delta\big)(x) &= (m\otimes m)\circ\tau_{23}\circ\Big(\Delta\otimes\big(\tau\circ(S\otimes S)\circ\Delta\big)\Big)\circ\Delta(x)\\
&= (m\otimes m)\circ\tau_{23}\circ\tau_{34}\circ(I\otimes I\otimes S\otimes S)\circ(\Delta\otimes\Delta)\circ\Delta(x)\\
&= (m\otimes m)\circ\tau_{23}\circ\tau_{34}\circ(I\otimes I\otimes S\otimes S)\circ(I\otimes\Delta\otimes I)\circ(\Delta\otimes I)\circ\Delta(x)\\
&= (m\otimes m)\circ\tau_{23}\circ\tau_{34}\circ(I\otimes I\otimes S\otimes S)\circ(I\otimes\Delta\otimes I)\Big(\sum_{(x)}x_1\otimes x_2\otimes x_3\Big)\\
&= (m\otimes m)\circ\tau_{23}\circ\tau_{34}\circ\Big(\sum_{(x)}x_1\otimes x_{2:1}\otimes Sx_{2:2}\otimes Sx_3\Big)\\
&= \sum_{(x)}x_1Sx_3\otimes x_{2:1}Sx_{2:2}\\
&= \sum_{(x)}x_1Sx_3\otimes(u\circ\varepsilon)(x_2)\\
&= \Big(\sum_{(x)}x_1Sx_2\Big)\otimes(u\circ\varepsilon)(x)\\
&= u\circ\varepsilon(x)\otimes u\circ\varepsilon(x).
\end{aligned}
$$

If $\mathcal{H}$ is commutative then :

$$u\circ\varepsilon(x)=\sum_{(x)}x_2S(x_1).$$

If $\mathcal{H}$ is cocommutative, this is again true, as :

$$u\circ\varepsilon(x)=(I*S)(x)=\sum_{(x)}x_1S(x_2).$$

Suppose that $\mathcal{H}$ is commutative or cocommutative. Then using assertion 2) we can compute :

$$
\begin{aligned}
(S\circ S)*S(x) &= \sum_{(x)}(S\circ S)(x_1)S(x_2)\\
&= S\big(\sum_{(x)}x_2S(x_1)\big)\\
&= S\big(u\circ\varepsilon(x)\big)\\
&= (u\circ\varepsilon)(x).
\end{aligned}
$$

Thus $S\circ S$ is a left inverse for S (for the convolution), and then $S\circ S=I$. This ends the proof of Proposition I.7.1.

•

Proposition I.7.3.
1). If x is a primitive element then $S(x)=-x$.
2). The linear subspace Prim $\mathcal{H}$ *of primitive elements in $\mathcal{H}$ is a Lie algebra.*

Proof. If x is primitive, then $(\varepsilon\otimes\varepsilon)\circ\Delta(x) = 2\varepsilon(x)$. On the other hand, $(\varepsilon\otimes\varepsilon)\circ\Delta(x) = \varepsilon(x)$, so $\varepsilon(x) = 0$. Then :

$$0 = (u \circ \varepsilon)(x) = m(S \otimes I)\Delta(x) = S(x) - x.$$

Now let x and y be primitive elements of $\mathcal{H}$. Then we can easily compute :

$$\begin{aligned}\Delta(xy - yx) &= (x \otimes \mathbf{1} + \mathbf{1} \otimes x)(y \otimes \mathbf{1} + \mathbf{1} \otimes y) - (y \otimes \mathbf{1} + \mathbf{1} \otimes y)(x \otimes \mathbf{1} + \mathbf{1} \otimes x) \\ &= (xy - yx) \otimes \mathbf{1} + \mathbf{1} \otimes (xy + yx) + x \otimes y + y \otimes x - y \otimes x - x \otimes y \\ &= (xy - yx) \otimes \mathbf{1} + \mathbf{1} \otimes (xy - yx).\end{aligned}$$

•

II. Convolution product and regularization

II.1. Connected graded bialgebras

Let k be a field with vanishing characteristic. We shall denote by $k[[t]]$ the ring of formal series on k, and by $k[t^{-1}, t]]$ the field of Laurent series on k. A *graded Hopf algebra* on k is a graded k-vector space :

$$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$$

endowed with a product $m : \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}$, a coproduct $\Delta : \mathcal{H} \to \mathcal{H}\otimes\mathcal{H}$, a unity $u : k \to \mathcal{H}$, a co-unity $\varepsilon : \mathcal{H} \to k$ and an antipode $S : \mathcal{H} \to \mathcal{H}$ fulfilling the usual axioms of a Hopf algebra, and such that :

$$\begin{aligned}&\mathcal{H}_p.\mathcal{H}_q \subset \mathcal{H}_{p+q} \\ &\Delta(\mathcal{H}_n) \subset \bigoplus_{p+q=n} \mathcal{H}_p \otimes \mathcal{H}_q. \\ &S(\mathcal{H}_n) \subset \mathcal{H}_n\end{aligned}$$

If we do not ask for the existence of an antipode $\mathcal{H}$ we get the definition of a *graded bialgebra*. In a graded bialgebra $\mathcal{H}$ we shall consider the increasing filtration :

$$\mathcal{H}^n = \bigoplus_{p=0}^{n} \mathcal{H}_p.$$

Suppose moreover that $\mathcal{H}$ is *connected*, i.e. $\mathcal{H}_0$ is one-dimensional. Then we have :

$$\operatorname{Ker} \varepsilon = \bigoplus_{n\geq 1} \mathcal{H}_n.$$

Proposition II.1.1.
For any $x \in \mathcal{H}^n, n \geq 1$ *we can write :*

$$\Delta x = x \otimes \mathbf{1} + \mathbf{1} \otimes x + \widetilde{\Delta} x, \qquad \widetilde{\Delta} x \in \bigoplus_{p+q=n, p\neq 0, q\neq 0} \mathcal{H}_p \otimes \mathcal{H}_q.$$

The map $\widetilde{\Delta}$ *is coassociative on* $\operatorname{Ker}\varepsilon$ *and* $\widetilde{\Delta}_k = (I^{\otimes k-1} \otimes \widetilde{\Delta})(I^{\otimes k-2} \otimes \widetilde{\Delta}) ... \widetilde{\Delta}$ *sends* $\mathcal{H}^n$ *into* $(\mathcal{H}^{n-k})^{\otimes k+1}$.

Proof. Thanks to connectedness we clearly can write :

$$\Delta x = a(x \otimes 1) + b(1 \otimes x) + \widetilde{\Delta} x$$

with $a, b \in k$ and $\widetilde{\Delta} x \in \operatorname{Ker}\varepsilon \otimes \operatorname{Ker}\varepsilon$. The co-unity property then tells us that, with $k \otimes \mathcal{H}$ and $\mathcal{H} \otimes k$ canonically identified with $\mathcal{H}$:

$$\begin{aligned} x &= (\varepsilon \otimes I)(\Delta x) = bx \\ x &= (I \otimes \varepsilon)(\Delta x) = ax, \end{aligned}$$

hence $a = b = 1$. We shall use the following two variants of Sweedler's notation :

$$\begin{aligned} \Delta x &= \sum_{(x)} x_1 \otimes x_2, \\ \widetilde{\Delta} x &= \sum_{(x)} x' \otimes x'', \end{aligned}$$

the second being relevant only for $x \in \operatorname{Ker}\varepsilon$. if x is homogeneous of degree n we can suppose that the components x_1, x_2, x', x'' in the expressions above are homogeneous as well, and we have then $|x_1| + |x_2| = n$ and $|x'| + |x''| = n$ We easily compute :

$$\begin{aligned} (\Delta \otimes I)\Delta(x) = {} & x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x \\ & + \sum_{(x)} x' \otimes x'' \otimes 1 + x' \otimes 1 \otimes x'' + 1 \otimes x' \otimes x'' \\ & + (\widetilde{\Delta} \otimes I)\widetilde{\Delta}(x) \end{aligned}$$

and

$$\begin{aligned} (I \otimes \Delta)\Delta(x) = {} & x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x \\ & + \sum_{(x)} x' \otimes x'' \otimes 1 + x' \otimes 1 \otimes x'' + 1 \otimes x' \otimes x'' \\ & + (I \otimes \widetilde{\Delta})\widetilde{\Delta}(x), \end{aligned}$$

hence the co-associativity of $\widetilde{\Delta}$ comes from the one of Δ. Finally it is easily seen by induction on k that for any $x \in \mathcal{H}^n$ we can write :

$$\widetilde{\Delta}_k(x) = \sum_x x^{(1)} \otimes \cdots \otimes x^{(k+1)},$$

with $|x^{(j)}| \geq 1$. The grading imposes :

$$\sum_{j=1}^{k+1} |x^{(j)}| = n,$$

so the maximum possible for any degree $|x^{(j)}|$ is $n-k$.

•

II.2. Connected filtered bialgebras

A *filtered Hopf algebra* on k is a k-vector space together with an increasing $\mathbb{Z}_+$-indexed filtration :

$$\mathcal{H}^0 \subset \mathcal{H}^1 \subset \cdots \subset \mathcal{H}^n \subset \cdots, \bigcup_n \mathcal{H}^n = \mathcal{H}$$

endowed with a product $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$, a coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, a unit characteristic $u : k \to \mathcal{H}$, a co-unit $\varepsilon : \mathcal{H} \to k$ and an antipode $S : \mathcal{H} \to \mathcal{H}$ fulfilling the usual axioms of a Hopf algebra, and such that :

$$\begin{aligned}
&\mathcal{H}^p.\mathcal{H}^q \subset \mathcal{H}^{p+q} \\
&\Delta(\mathcal{H}^n) \subset \sum_{p+q=n} \mathcal{H}^p \otimes \mathcal{H}^q \\
&S(\mathcal{H}^n) \subset \mathcal{H}^n.
\end{aligned}$$

If we do not ask for the existence of an antipode $\mathcal{H}$ we get the definition of a *filtered bialgebra*. For any $x \in \mathcal{H}$ we set :

$$|x| = \min\{n \in \mathbb{N},\ x \in \mathcal{H}^n\}.$$

Any graded bialgebra or Hopf algebra is obviously filtered by the canonical filtration associated to the grading :

$$\mathcal{H}^n = \bigoplus_{i=0}^n \mathcal{H}_i,$$

and in that case, if x is an homogeneous element, x is of degree n if and only if $|x| = n$. We say that the filtered bialgebra $\mathcal{H}$ is connected if $\mathcal{H}^0$ is one-dimensional. There is an analogue of proposition II.1.1 in the connected filtered case :

Proposition II.2.1.
For any $x \in \mathcal{H}^n, n \geq 1$ we can write :

$$\Delta x = x \otimes \mathbf{1} + \mathbf{1} \otimes x + \widetilde{\Delta} x, \qquad \widetilde{\Delta} x \in \sum_{p+q=n, p\neq 0, q\neq 0} \mathcal{H}^p \otimes \mathcal{H}^q.$$

The map $\widetilde{\Delta}$ is coassociative on $\operatorname{Ker}\varepsilon$ *and $\widetilde{\Delta}_k = (I^{\otimes k-1} \otimes \widetilde{\Delta})(I^{\otimes k-2} \otimes \widetilde{\Delta})...\widetilde{\Delta}$ sends $\mathcal{H}^n$ into $(\mathcal{H}^{n-k})^{\otimes k+1}$.*

Proof. Straightforward adaptation of proof of proposition II.1.1.

•

The following theorem is due to S. Montgomery [Mo lemma 1.1].

Theorem II.2.2.
Let $\mathcal{H}$ be any pointed Hopf algebra. Then the coradical filtration endows $\mathcal{H}$ with a structure of filtered Hopf algebra.

Proof. It only remains to show that for any $n \in \mathbb{N}$ the inclusion $S(H^n) \subset H^n$ holds, and that for any $p, q \in \mathbb{N}$:

$$\mathcal{H}^p \mathcal{H}^q \subset \mathcal{H}^{p+q},$$

which, together with proposition I.3.15, will prove the result. Recall that a pointed coalgebra is a coalgebra in which any simple subcoalgebra is one-dimensional. In this case any simple subcoalgebra is linearly generated by a unique grouplike element. Any grouplike element g in a Hopf algebra admits an inverse Sg, where S is the antipode. It follows that the coradical $\mathcal{H}^0$ of a pointed Hopf algebra $\mathcal{H}$ is a Hopf subalgebra of $\mathcal{H}$, precisely the Hopf algebra of the group of the grouplike elements of $\mathcal{H}$ (cf. example I.6.1).

The proof proceeds by induction : inclusion $S\mathcal{H}_0 \subset \mathcal{H}_0$ obviously holds. Suppose that $S\mathcal{H}^k \subset \mathcal{H}^k$ for all $k \le n-1$. Using the definition of $\mathcal{H}^n$:

$$\mathcal{H}^n = \mathcal{H}^0 \wedge \mathcal{H}^{n-1} = \mathcal{H}^{n-1} \wedge H^0$$

and the formula :

$$Sx = \sum_{(x)} Sx_2 \otimes Sx_1,$$

(cf. proposition I.7.1) we deduce the inclusion $S\mathcal{H}^n \subset \mathcal{H}^n$. Now, suppose that the inclusion $\mathcal{H}^k \mathcal{H}^0 \subset \mathcal{H}^k$ holds for $k \le n-1$ (its is obviously the case for $k = 0$). Then we have :

$$\begin{aligned}\Delta(\mathcal{H}^n \mathcal{H}^0) &\subset (\mathcal{H}^0 \otimes \mathcal{H} + \mathcal{H} \otimes \mathcal{H}^{n-1})(\mathcal{H}^0 \otimes \mathcal{H}^0)\\ &\subset \mathcal{H}^0 \otimes \mathcal{H} + \mathcal{H} \otimes \mathcal{H}^{n-1}\mathcal{H}^0\\ &\subset \mathcal{H}^0 \otimes \mathcal{H} + \mathcal{H} \otimes \mathcal{H}^{n-1}.\end{aligned}$$

So $\mathcal{H}^n \mathcal{H}^0 \subset \Delta^{-1}(\mathcal{H}^0 \otimes \mathcal{H} + \mathcal{H} \otimes \mathcal{H}^{n-1}) = \mathcal{H}^n$. Similarly on the other side we have $\mathcal{H}^0 \mathcal{H}^n \subset \mathcal{H}^n$ for any n. Suppose now that the inclusion :

$$\mathcal{H}^p \mathcal{H}^q \subset \mathcal{H}^{p+q}$$

holds for any p, q such that $p + q \le n - 1$. Choose p, q with $p + q = n$ and compute :

$$\begin{aligned}\Delta(\mathcal{H}^p \mathcal{H}^q) &\subset (\mathcal{H}^0 \otimes \mathcal{H}^p + \mathcal{H}^p \otimes \mathcal{H}^{p-1})(\mathcal{H}^0 \otimes \mathcal{H}^q + \mathcal{H}^q \otimes \mathcal{H}^{q-1})\\ &\subset \mathcal{H}^0\mathcal{H}^0 \otimes \mathcal{H}^p\mathcal{H}^q + \mathcal{H}^p\mathcal{H}^0 \otimes \mathcal{H}^{p-1}\mathcal{H}^q + \mathcal{H}^0\mathcal{H}^q \otimes \mathcal{H}^p\mathcal{H}^{q-1} + \mathcal{H}^p\mathcal{H}^q \otimes \mathcal{H}^{p-1}\mathcal{H}^{q-1}\\ &\subset \mathcal{H}^0 \otimes \mathcal{H} + \mathcal{H} \otimes \mathcal{H}^{p+q-1}\end{aligned}$$

thanks to the induction hypothesis and the property already proved when one of the indices is equal to zero. Thus $\mathcal{H}^p \mathcal{H}^q \subset \mathcal{H}^{p+q}$, which finishes the proof of the theorem.

•

Remark 1 : The proof used only the property that the coradical is a Hopf subalgebra of $\mathcal{H}$. The pointedness of $\mathcal{H}$ implies this property but is not strictly necessary.

Remark 2 : the image of k under the unit map u is a one-dimensional simple subcoagebra of $\mathcal{H}$. If $\mathcal{H}$ is an irreducible coalgebra, by proposition I.3.8 it is the unique one, and then the coradical is $\mathcal{H}^0 = k.\mathbf{1}$. Any irreducible Hopf algebra is then pointed, and connected with respect to the coradical filtration.

II.3. The convolution product

An important result is that any connected filtered bialgebra is indeed a filtered Hopf algebra, in the sense that the antipode comes for free. We give a proof of this fact as well as a recursive formula for the antipode with the help of the *convolution product* :

Let $\mathcal{H}$ be a (connected filtered) bialgebra, and let $\mathcal{A}$ be any k-algebra : the convolution product on $\mathcal{L}(\mathcal{H}, \mathcal{A})$ is given by :

$$\begin{aligned} \varphi * \psi(x) &= m_{\mathcal{A}}(\varphi \otimes \psi)\Delta(x) \\ &= \sum_{(x)} \varphi(x_1)\psi(x_2). \end{aligned}$$

Proposition II.3.1.
The map $e = u_{\mathcal{A}} \circ \varepsilon$, given by $e(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}$ and $e(x) = 0$ for any $x \in \operatorname{Ker} \varepsilon$, is a unit for the convolution product. Moreover the set $G = \{\varphi \in \mathcal{L}(\mathcal{H}, \mathcal{A}),\ \varphi(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}\}$ endowed with the convolution product is a group.

Proof. The first statement is straightforward. To prove the second let us consider the formal series :

$$\begin{aligned} \varphi^{*-1}(x) &= \big(e - (e - \varphi)\big)^{*-1}(x) \\ &= \sum_{k \geq 0} (e - \varphi)^{*k}(x). \end{aligned}$$

Using $(e - \varphi)(\mathbf{1}) = 0$ we have immediately $(e - \varphi)^{*k}(\mathbf{1}) = 0$, and for any $x \in \operatorname{Ker} \varepsilon$:

$$(e - \varphi)^{*k}(x) = m_{\mathcal{A},k-1}(\varphi \otimes \cdots \otimes \varphi)\widetilde{\Delta}_{k-1}(x).$$

When $x \in \mathcal{H}^n$ this expression vanishes then for $k \geq n + 1$. The formal series ends up then with a finite number of terms for any x, which proves the result.

•

Corollary II.3.2.
Any connected filtered bialgebra $\mathcal{H}$ is a filtered Hopf algebra. The antipode is defined by :

$$S(x) = \sum_{k\geq 0} (u\varepsilon - I)^{*k}(x).$$

It is given by $S(\mathbf{1}) = \mathbf{1}$ and recursively by any of the two formulas for $x \in \operatorname{Ker}\varepsilon$:

$$S(x) = -x - \sum_{(x)} S(x')x''$$
$$S(x) = -x - \sum_{(x)} x'S(x'').$$

Proof. The antipode, when it exists, is the inverse of the identity for the convolution product on $\mathcal{L}(\mathcal{H}, \mathcal{H})$. One just needs then to apply proposition II.3.1 with $\mathcal{A} = \mathcal{H}$. The two recursive formulas come directly from the two equalities :

$$m(S \otimes I)\Delta(x) = m(I \otimes S)\Delta(x) = 0$$

fulfilled by any $x \in \operatorname{Ker}\varepsilon$.

•

Let $\mathfrak{g}$ be the subspace of $\mathcal{L}(\mathcal{H}, \mathcal{A})$ formed by the elements α such that $\alpha(\mathbf{1}) = 0$. It is clearly a subalgebra of $\mathcal{L}(\mathcal{H}, \mathcal{A})$ for the convolution product. We have :

$$G = e + \mathfrak{g}.$$

From now on we shall suppose that the ground field k is of characteristic zero. For any $x \in \mathcal{H}^n$ the exponential :

$$e^{*\alpha}(x) = \sum_{k\geq 0} \frac{\alpha^{*k}(x)}{k!}$$

is a finite sum (ending up at $k = n$). It is a bijection from $\mathfrak{g}$ onto G. Its inverse is given by :

$$\operatorname{Log}(1 + \alpha)(x) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \alpha^{*k}(x).$$

This sum again ends up at $k = n$ for any $x \in \mathcal{H}^n$. Let us introduce a decreasing filtration on $\mathcal{L} = \mathcal{L}(\mathcal{H}, \mathcal{A})$:

$$\mathcal{L}^n = \{\alpha \in \mathcal{L}, \alpha_{|\mathcal{H}^{n-1}} = 0\}.$$

Clearly $\mathcal{L}_0 = \mathcal{L}$ and $\mathcal{L}_1 = \mathfrak{g}$. We define the valuation $\operatorname{val}\varphi$ of an element φ of $\mathcal{L}$ as the biggest integer k such that φ is in $\mathcal{L}_k$. We shall consider in the sequel the ultrametric distance on $\mathcal{L}$ induced by the filtration :

$$d(\varphi, \psi) = 2^{-\operatorname{val}(\varphi-\psi)}.$$

For any $\alpha, \beta \in \mathfrak{g}$ let $[\alpha, \beta] = \alpha * \beta - \beta * \alpha$.

Proposition II.3.3.
1) We have the inclusion :

$$\mathcal{L}^p * \mathcal{L}^q \subset \mathcal{L}^{p+q}.$$

2) The metric space $\mathcal{L}$ endowed with then distance defined just above is complete.

Proof. Take any $x \in \mathcal{H}^{p+q-1}$, and any $\alpha \in \mathcal{L}_p$ and $\beta \in \mathcal{L}_q$. We have :

$$(\alpha * \beta)(x) = \sum_{(x)} \alpha(x_1)\beta(x_2).$$

Recall that we denote by $|x|$ the minimal n such that $x \in \mathcal{H}^n$. Since $|x_1| + |x_2| = |x| \leq p + q - 1$, either $|x_1| \leq p - 1$ or $|x_2| \leq q - 1$, so the expression vanishes. Now if (ψ_n) is a Cauchy sequence in $\mathcal{L}$ it is immediate to see that this sequence is *locally stationary*, i.e. for any $x \in \mathcal{H}$ there exists $N(x) \in \mathbb{N}$ such that $\psi_n(x) = \psi_{N(x)}(x)$ for any $n \geq N(x)$. Then the limit of (ψ_n) exists and is clearly defined by :

$$\psi(x) = \psi_{N(x)}(x).$$

•

As a corollary the Lie algebra $\mathcal{L}_1 = \mathfrak{g}$ is *pro-nilpotent*, in a sense that it is the projective limit of the Lie algebras $\mathfrak{g}/\mathcal{L}^n$, which are nilpotent.

II.4. Algebra morphisms and cocycles

Let $\mathcal{H}$ be a connected filtered Hopf algebra over k, and let $\mathcal{A}$ be a k-algebra. A *cocycle from $\mathcal{H}$ to $\mathcal{A}$* is a linear morphism $\tau : \mathcal{H} \to \mathcal{A}$ such that $\tau(xy) = \tau(yx)$ for any $x, y \in \mathcal{H}$. It is indeed a 1-cocycle in the cohomology of the Lie algebra $\mathcal{H}$ with values in $\mathcal{A}$ considered as a trivial $\mathcal{H}$-module. In the case where $\mathcal{A}$ is the ground field k cocycles are just traces.

We shall also consider algebra morphisms from $\mathcal{H}$ to $\mathcal{A}$. When the algebra $\mathcal{A}$ is commutative we shall call them slightly abusively *characters*. It is clear that any character in our sense is a cocycle. We recover of course the usual notion of character when the algebra $\mathcal{A}$ is the ground field k.

The notions of character and cocycle involve only the algebra structure of $\mathcal{H}$. Let us consider now the full Hopf algebra structure and see what happens with the convolution product :

Proposition II.4.1.
Let $\mathcal{H}$ be a connected filtered Hopf algebra over k, and let $\mathcal{A}$ be a k-algebra. Then,

1). The convolution of two cocycles in $\mathcal{L}(\mathcal{H}, \mathcal{A})$ is a cocycle.

2). If τ is a cocycle such that $\tau(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}$, then the inverse τ^{*-1} is a cocycle as well.

3). In the case of a commutative algebra $\mathcal{A}$ the characters from $\mathcal{H}$ to $\mathcal{A}$ form a group Γ under the convolution product, and for any $\chi \in \Gamma$ the inverse is given by :

$$\chi^{*-1} = \chi \circ S.$$

Proof. Using the fact that Δ is an algebra morphism we have for any $x, y \in \mathcal{H}$:

$$f * g(xy) = \sum_{(x)(y)} f(x_1 y_1) g(x_2 y_2).$$

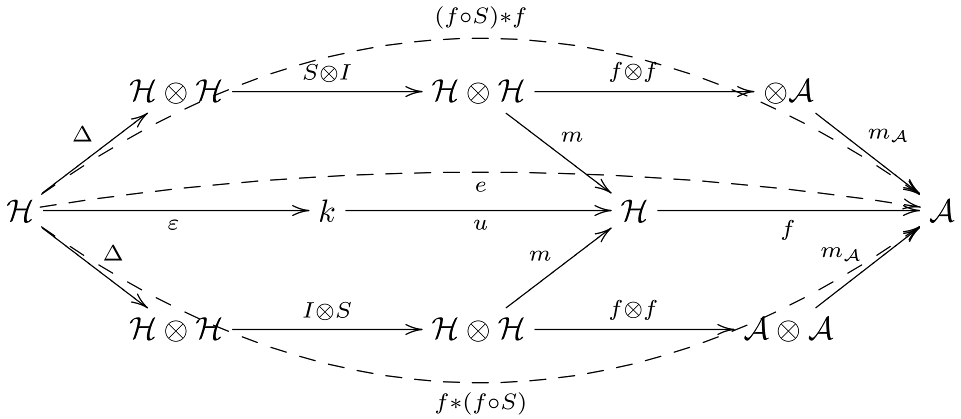
If f and g are cocycles we get :

$$\begin{aligned} f * g(xy) &= \sum_{(x)(y)} f(y_1 x_1) g(y_2 x_2) \\ &= f * g(yx). \end{aligned}$$

If $\mathcal{A}$ is commutative and if f and g are characters we get :

$$\begin{aligned} f * g(xy) &= \sum_{(x)(y)} f(x_1) f(y_1) g(x_2) g(y_2) \\ &= \sum_{(x)(y)} f(x_1) g(x_2) f(y_1) g(y_2) \\ &= (f * g)(x)(f * g)(y). \end{aligned}$$

The unit $e = u_{\mathcal{A}}\varepsilon$ is both a cocycle and an algebra morphism. The formula for the inverse of a character comes easily from the commutativity of the following diagram :



Finally the fact that the inverse of a cocycle τ such that $\tau(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}$ is a cocycle comes from 1) and from the formula :

$$\tau^{-1}(x) = \sum_{k \geq 0} (e - \tau)^{*k}(x).$$

•

We call *derivations* (or *infinitesimal characters*) *with values in the algebra* $\mathcal{A}$ those elements α of $\mathcal{L}(\mathcal{H}, \mathcal{A})$ such that :

$$\alpha(xy) = e(x)\alpha(y) + \alpha(x)e(y).$$

Proposition II.4.2.
Suppose that $\mathcal{A}$ is a commutative algebra. Let G_1 (resp. $\mathfrak{g}_1$) be the set of characters of $\mathcal{H}$ with values in $\mathcal{A}$ (resp the set of derivations of $\mathcal{H}$ with values in $\mathcal{A}$), and let G_2 (resp. $\mathfrak{g}_2$) be the set of cocycles φ from $\mathcal{H}$ to $\mathcal{A}$ such that $\varphi(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}$ (resp. $\varphi(\mathbf{1}) = 0$). Then G_1 and G_2 are subgroups of G, the exponential restricts to a bijection from $\mathfrak{g}_1$ onto G_1 (resp. from $\mathfrak{g}_2$ onto G_2), and $\mathfrak{g}_1, \mathfrak{g}_2$ are Lie subalgebras of $\mathfrak{g}$.

Proof. Part of these results are a reformulation of proposition II.4.1 and some points are straightforward. The only non-trivial point concerns $\mathfrak{g}_1$ and G_1. Take two derivations α and β with values in $\mathcal{A}$ and compute :

$$\begin{aligned}(\alpha * \beta)(xy) &= \sum_{(x)(y)} \alpha(x_1x_2)\beta(y_1y_2)\\ &= \sum_{(x)(y)} \big(\alpha(x_1)e(y_1) + e(x_1)\alpha(y_1)\big).\big(\beta(x_2)e(y_2) + e(x_2)\alpha(y_2)\big)\\ &= (\alpha * \beta)(x)e(y) + \alpha(x)\beta(y) + \beta(x)\alpha(y) + e(x)(\alpha * \beta)(y).\end{aligned}$$

Using the commutativity of $\mathcal{A}$ we immediately get :

$$[\alpha, \beta](xy) = [\alpha, \beta](x)e(y) + e(x)[\alpha, \beta](y),$$

which shows that $\mathfrak{g}_1$ is a Lie algebra. Now for $\alpha \in \mathfrak{g}_1$ we have :

$$\alpha^{*n}(xy) = \sum_{k=0}^{n} \binom{n}{k} \alpha^{*k}(x)\alpha^{*(n-k)}(y),$$

as easily see by induction on n. A straightforward computation then yields :

$$e^{*\alpha}(xy) = e^{*\alpha}(x)e^{*\alpha}(y).$$

•

II.5. Birkhoff decomposition

We consider here the situation where the algebra $\mathcal{A}$ admits a *renormalization scheme*, i.e. a splitting into two subalgebras :

$$\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$$

with $\mathbf{1} \in \mathcal{A}_+$. As an example, take $\mathcal{A}$ as the field $k[t^{-1}, t]]$ of Laurent series, $\mathcal{A}_- = t^{-1}k[t^{-1}]$ and $\mathcal{A}_+ = k[[t]]$. The projection on $\mathcal{A}_-$ parallel to $\mathcal{A}_+$ will be denoted by π.

Theorem II.5.1.
1). Let $\mathcal{H}$ be a connected filtered Hopf algebra. Let G be the group of those $\varphi \in \mathcal{L}(\mathcal{H}, \mathcal{A})$ such that $\varphi(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}$ endowed with the convolution product. Any $\varphi \in G$ admits a unique Birkhoff decomposition :

$$\varphi = \varphi_-^{*-1} * \varphi_+,$$

where φ_- sends $\mathbf{1}$ to $\mathbf{1}_{\mathcal{A}}$ and $\operatorname{Ker}\varepsilon$ into $\mathcal{A}_-$, and where φ_+ sends $\mathcal{H}$ into A_+. The maps φ_- and φ_+ are given on $\operatorname{Ker}\varepsilon$ by the following recursive formulas :

$$\varphi_-(x) = -\pi\Big(\varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'')\Big)$$
$$\varphi_+(x) = (I - \pi)\Big(\varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'')\Big).$$

2). If $\tau \in G$ is a cocycle, the components τ_- and τ_+ occurring in the Birkhoff decomposition of τ are cocycles as well.

3). If the algebra $\mathcal{A}$ is commutative and if χ is a character, the components χ_- and χ_+ occurring in the Birkhoff decomposition of χ are characters as well.

Proof. Points 1) and 3) together give an abstract counterpart of Theorem 4 of [CK], point 2) is new up to my knowledge. The proof goes along the same lines : for the first assertion it is immediate from the definition of π that φ_- sends $\operatorname{Ker}\varepsilon$ into $\mathcal{A}_-$, and that φ_+ sends $\operatorname{Ker}\varepsilon$ into $\mathcal{A}_+$. It only remains to check equality $\varphi_+ = \varphi_- * \varphi$, which is an easy computation :

$$\begin{aligned}\varphi_+(x) &= (I - \pi)\Big(\varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'')\Big).\\ &= \varphi(x) + \varphi_-(x) + \sum_{(x)} \varphi_-(x')\varphi(x'')\\ &= (\varphi_- * \varphi)(x).\end{aligned}$$

To prove the second assertion it is sufficient to prove that τ_- is a cocycle whenever τ is a cocycle. The same property for τ_+ comes then from proposition II.3.1. We prove the formula $\tau_-(xy) = \tau_-(yx)$ by induction on the integer $d = |x| + |y|$: it is true for $d \le 1$. Suppose the formula is true up to $d-1$ and take any $x, y \in \mathcal{H}$ with $|x|+|y| = d$. Decompose $\Delta(xy)$ with the second version of Sweedler's notation :

$$\begin{aligned}\Delta(xy) = xy \otimes \mathbf{1} + \mathbf{1} \otimes xy + x \otimes y + y \otimes x\\ + \sum_{(x)} (x'y \otimes x'' + x' \otimes x''y) + \sum_{(y)} (xy' \otimes y'' + y' \otimes xy'')\\ + \sum_{(x)(y)} x'y' \otimes x''y''.\end{aligned}$$

We have then :

$$\tau_-(xy) = -\pi\Big(\tau(xy)+\tau_-(x)\tau(y)+\tau_-(y)\tau(x)$$
$$+\sum_{(x)}\big(\tau_-(x'y)\tau(x'')+\tau_-(x')\tau(x''y)\big)+\sum_{(y)}\big(\tau_-(xy')\tau(y'')+\tau_-(y')\tau(xy'')\big)$$
$$+\sum_{(x)(y)}\tau_-(x'y')\tau(x''y'')\Big),$$

whereas :

$$\tau_-(yx) = -\pi\Big(\tau(yx)+\tau_-(y)\tau(x)+\tau_-(x)\tau(y)$$
$$+\sum_{(y)}\big(\tau_-(y'x)\tau(y'')+\tau_-(y')\tau(y''x)\big)+\sum_{(x)}\big(\tau_-(yx')\tau(x'')+\tau_-(x')\tau(yx'')\big)$$
$$+\sum_{(x)(y)}\tau_-(y'x')\tau(y''x'')\Big).$$

Using the cocycle property for τ and the induction hypothesis we see that the two expressions are the same.

The proof of assertion 3) goes exactly as in [CK] and relies on the following *Rota-Baxter* equality in $\mathcal{A}$:

$$\pi(a)\pi(b) = -\pi(ab)+\pi\big(\pi(a)b\big)+\pi\big(\pi(b)a\big),$$

which is easily verified by decomposing a and b into their $\mathcal{A}_\pm$-parts. Let χ be a character of $\mathcal{H}$ with values in $\mathcal{A}$. Suppose that we have $\chi_-(xy)=\chi_-(x)\chi_-(y)$ for any $x,y\in\mathcal{H}$ such that $|x|+|y|\le d-1$, and compute for x,y such that $|x|+|y|=d$:

$$\chi_-(x)\chi_-(y)=\pi(X)\pi(Y),$$

with $X=\chi(x)-\sum_{(x)}\chi_-(x')\chi(x'')$ and $Y=\chi(y)-\sum_{(y)}\chi_-(y')\chi(y'')$. Using the formula :

$$\pi(X)=-\chi_-(x),$$

we get :

$$\chi_-(x)\chi_-(y)=-\pi\big(XY+\chi_-(x)Y+X\chi_-(y)\big),$$

hence :

$$\chi_-(x)\chi_-(y) = -\pi\Big(\chi(x)\chi(y)+\chi_-(x)\chi(y)+\chi(x)\chi_-(y)$$
$$+\sum_{(x)}\chi_-(x')\chi(x'')\big(\chi(y)+\chi_-(y)\big)+\sum_{(y)}\big(\chi(x)+\chi_-(x)\big)\chi_-(y')\chi(y'')$$
$$+\sum_{(x)(y)}\chi_-(x')\chi(x'')\chi_-(y')\chi(y'')\Big).$$

We have to compare this expression with :

$$\chi_-(xy) = -\pi\Big(\chi(xy) + \chi_-(x)\chi(y) + \chi_-(y)\chi(x) + \sum_{(x)}\big(\chi_-(x'y)\chi(x'') + \chi_-(x')\chi(x''y)\big) + \sum_{(y)}\big(\chi_-(xy')\chi(y'') + \chi_-(y')\chi(xy'')\big) + \sum_{(x)(y)} \chi_-(x'y')\chi(x''y'')\Big).$$

These two expressions are easily seen to be equal using the commutativity of the algebra $\mathcal{A}$, the character property for χ and the induction hypothesis.

•

Remark : define the *Bogoliubov character* as the map $b : G \to \mathcal{L}(\mathcal{H}, \mathcal{A})$ recursively given by :

$$b(\varphi)(x) = \varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'').$$

Then the components of φ in the birkhoff decomposition read :

$$\varphi_- = -\pi \circ b(\varphi), \qquad \varphi_+ = (I - \pi) \circ b(\varphi).$$

II.6. The BCH approach to Birkhoff decomposition

Let $\mathfrak{g}$ a Lie algebra endowed with a decreasing filtration :

$$\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_n \supset \cdots$$

such that the intersection of the $\mathfrak{g}_i$'s is reduced to $\{0\}$. We ask for the inclusion :

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j},$$

so in particular $\mathfrak{g}$ is a pro-nilpotent Lie algebra. Suppose that $\mathfrak{g}$ is complete for the (metric) topology defined by this filtration. The Baker-Campbell-Hausdorff series defines then a pro-nilpotent group law on $\mathfrak{g}$:

$$X.Y = X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}([X,[X,Y]] + [Y,[Y,X]]) + \cdots$$

The following proposition is implicitly used in [EGK2] :

Proposition II.6.1.
For any linear map $R : \mathfrak{g} \to \mathfrak{g}$ preserving the filtration there exists a (usually non-linear) map $\chi_R : \mathfrak{g} \to \mathfrak{g}$ such that $(\chi_R - \mathrm{Id}_\mathfrak{g})(\mathfrak{g}_i) \subset \mathfrak{g}_{2i}$ for any $i \geq 1$, and such that, with $\widetilde{R} := \mathrm{Id}_\mathfrak{g} - R$ we have :

$$\forall X \in \mathfrak{g}, \quad X = R\big(\chi_R(X)\big).\widetilde{R}\big(\chi_R(X)\big). \qquad (*)$$

Proof. Let us introduce for any $X, Y \in \mathfrak{g}$ the following expression :

$$\delta(X,Y) = X.Y - X - Y = \tfrac{1}{2}[X,Y] + \tfrac{1}{12}([X,[X,Y]] + [Y,[Y,X]]) + \cdots$$

Then equation (*) can be rewritten as :

$$\chi_R(X) = F_X\big(\chi_R(X)\big),$$

with $F_X : \mathfrak{g} \to \mathfrak{g}$ defined by :

$$F_X(Y) = X - \delta\big(R(Y), \widetilde{R}(Y)\big).$$

This map F_X is a contraction with respect to the metric associated with the filtration : indeed if $Y, \varepsilon \in \mathfrak{g}$ with $\varepsilon \in \mathfrak{g}_n$, we have :

$$F_X(Y+\varepsilon) - F_X(Y) = \delta\big(R(Y), \widetilde{R}(Y)\big) - \delta\big(R(Y+\varepsilon), \widetilde{R}(Y+\varepsilon)\big).$$

Right-hand side is a sum of iterated commutators in each of which ε does appear at least once. So it belongs to $\mathfrak{g}_{n+1}$. So the sequence $F_X^n(Y)$ converges in $\mathfrak{g}$ to a unique fixed point $\chi_R(X)$ for F_X.

Let us remark that for any $X \in \mathfrak{g}_i$, the element $F_X(X) - X$ belongs to $\mathfrak{g}_{2i}$. Now taking X as starting point it is obvious from the expression :

$$\chi_R(X) - X = \sum_{k=1}^{+\infty} (F_X^k - F_X^{k-1})(X)$$

that for any $X \in \mathfrak{g}_i$, the element $\chi_R(X) - X$ belongs to $\mathfrak{g}_{2i}$.

•

Let $\mathcal{H}$ be a connected filtered Hopf algebra, let $\mathcal{A}$ a commutative algebra endowed with a splitting $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ as in § II.5, and let π be the projection on $\mathcal{A}_-$ parallel to $\mathcal{A}_+$. Now take for $\mathfrak{g}$ any of the Lie algebras $\mathfrak{g}, \mathfrak{g}_1, \mathfrak{g}_2$ of § II.3 and II.4, set $G = \exp_*(\mathfrak{g})$ and set $R(X) = \pi \circ X$. Of course R makes sense on $\mathcal{L}(\mathcal{H}, \mathcal{A})$. Equation (*) then yields the following equality in the group G, G_1 or G_2 :

$$e^{*X} = e^{*R\left(\chi_R(X)\right)} * e^{*\widetilde{R}\left(\chi_R(X)\right)}, \qquad (**)$$

as map R respects the decreasing filtration introduced in § II.3. Now, thanks to Rota-Baxter relation in $\mathcal{A}$:

$$\pi(a)\pi(b) = \pi\Big(\pi(a)b + \pi(b)a - ab\Big),$$

K. Ebrahimi-Fard, L. Guo and D. Kreimer derive in [EGK2] two identities involving the Bogoliubov character :

$$e^{*-R\left(\chi_R(X)\right)} = R\big(b(e^{*X})\big), \qquad e^{*\widetilde{R}\left(\chi_R(X)\right)} = -\widetilde{R}\big(b(e^{*X})\big).$$

So (**) is indeed the Birkhoff decomposition of the element $\varphi = e^{*X}$ of G, G_1 or G_2, namely :

$$\varphi_- = e^{*-R\left(\chi_R(X)\right)}, \qquad \varphi_+ = e^{*\widetilde{R}\left(\chi_R(X)\right)}.$$

Of course when $\mathcal{H}$ is cocommutative the convolution product is commutative, the Lie algebras involved are abelian and the situation simplifies greatly, as $\chi_R = \mathrm{Id}_{\mathfrak{g}}$ for any linear map R here [EGK1]. Further developments on general Rota-Baxter algebras can be found in [EGK2] and [EGK3].

Remark : Rota-Baxter identity for R just guarantees that equation (**) gives a Birkhoff decomposition. When R is idempotent (which is indeed the case in the setting of § II.5, where R is built up from projection π) this decomposition is unique and is given either by (**) or by the recursive formulas of Theorem II.5.1. On the other hand, with an idempotent R which does *not* verify Rota-Baxter identity (or, which is the same, with a direct sum decomposition of the algebra $\mathcal{A}$ on two components $\mathcal{A}_+$ and $\mathcal{A}_-$ which are *not* subalgebras of $\mathcal{A}$), we still get a unique Birkhoff decomposition along the lines of theorem II.5.1 for the groups G and G_2, which does not coincide with (**). But Rota-Baxter identity arises in an essential way, as we have seen, in order to get the Birkhoff decomposition for the groups G_1 of $\mathcal{A}$-valued characters.

II.7. Renormalized traces and characters

Keeping the same notations we take $k = \mathbb{C}$ as ground field and we specialize to the case when $\mathcal{A}$ is the field of meromorphic functions (resp. the field of germs of meromorphic functions at z_0), $\mathcal{A}_+$ is the algebra of meromorphic functions which are holomorphic at z_0 (resp. the algebra of germs of holomorphic functions at z_0) and $\mathcal{A}_-$ is the non-unital algebra $(z - z_0)^{-1}\mathbb{C}[(z - z_0)^{-1}]$. Applying the projection π to such a meromorphic fuction amounts to "take its divergent part at z_0". This particular splitting is called the *minimal subtraction scheme*. It is by no means unique : for example for any automorphism θ of the field of (germs of) meromorphic functions, we can consider the splitting :

$$\mathcal{A} = \mathcal{A}_+^\theta \oplus \mathcal{A}_-^\theta,$$

with $\mathcal{A}_\pm^\theta = \theta(\mathcal{A}_\pm)$. The corresponding projection on $\mathcal{A}_-^\theta$ is given by $\pi^\theta = \theta \circ \pi \circ \theta^{-1}$. As an example of automorphism θ fix a constant c, consider the change of variable $z \mapsto z'$ such that :

$$\frac{1}{z' - z_0} = \frac{1}{z - z_0} + c$$

(hence $z' = z_0 + \dfrac{z - z_0}{1 + c(z - z_0)}$), and set $\theta(f)(z) = f(z')$.

Considering a linear map φ from $\mathcal{H}$ to $\mathcal{A}$ (and a particular splitting of $\mathcal{A}$) we can consider its Birkhoff decomposition $\varphi = \varphi_-^{*-1} * \varphi_+$ given by Theorem II.5.1, and evaluate $\varphi_+(x)$ at $z = z_0$ for any $x \in \mathcal{H}$. This gives a linear map $\varphi_+^{z_0}$ from $\mathcal{H}$ to $\mathbb{C}$ which we call the *renormalized value of* φ *at* $z = z_0$. According to Theorem II.5.1 the renormalized value of a cocycle at z_0 is a trace, and the renormalized value of a character at $z = z_0$ is a $\mathbb{C}$-valued character. All this procedure depends in an essential way on the choice of the renormalization scheme, i.e. the splitting of $\mathcal{A}$.

II.8. More on connected graded Hopf algebras

Let $\mathcal{H}$ be a connected graded Hopf algebra. The grading induces a biderivation Y defined on homogeneous elements by :

$$\begin{aligned} Y : \mathcal{H}_n &\longrightarrow \mathcal{H}_n \\ x &\longmapsto nx. \end{aligned}$$

Exponentiating we get a one-parameter group θ_t of automorphisms of the Hopf algebra $\mathcal{H}$, defined on $\mathcal{H}_n$ by :

$$\theta_t(x) = e^{nt}x.$$

Lemma II.8.1.
$\varphi \mapsto \varphi \circ Y$ is a derivation of $\big(\mathcal{L}(\mathcal{H},\mathcal{A}),*\big)$, and $\varphi \mapsto \varphi \circ \theta_t$ is an automorphism of $\big(\mathcal{L}(\mathcal{H},\mathcal{A}),*\big)$ for any complex t.

Proof. We compute for $\varphi, \psi \in \mathcal{L}(\mathcal{H},\mathcal{A})$ and for an homogeneous element x of $\mathcal{H}$:

$$\begin{aligned} (\varphi * \psi) \circ Y(x) &= |x| \sum_{(x)} \varphi(x_1)\psi(x_2) \\ &= \sum_{(x)} (|x_1| + |x_2|)\varphi(x_1)\psi(x_2) \\ &= \sum_{(x)} (\varphi \circ Y)(x_1)\psi(x_2) + \varphi(x_1)(\psi \circ Y)(x_2) \\ &= \big((\varphi \circ Y) * \psi + \varphi * (\psi \circ Y)\big)(x), \end{aligned}$$

which shows the first assertion. The second part of the proposition is proven similarly. •

Using the fact that $e \circ Y = 0$ we easily compute for any derivation α with values in $\mathcal{A}$:

$$\begin{aligned} (\alpha \circ Y)(xy) &= \alpha(Y(x).y + x.Y(y)) \\ &= (\alpha \circ Y)(x)e(y) + (e \circ Y)(x)\alpha(y) + \alpha(x)(e \circ Y)(y) + e(x)(\alpha \circ Y)(y) \\ &= (\alpha \circ Y)(x)e(y) + e(x)(\alpha \circ Y)(y). \end{aligned}$$

So we have proved :

Lemma II.8.2.
The map $\alpha \mapsto \alpha \circ Y$ is a linear automorphism of the space of derivations of $\mathcal{H}$ with values in $\mathcal{A}$. Its inverse is given by $\alpha \mapsto \alpha \circ Y^{-1}$, where $Y^{-1}(x) = |x|^{-1}x$ for x homogeneous of positive degree, and $Y^{-1}(\mathbf{1}) = 0$.

Remark : the notation Y^{-1} is of course slightly incorrect, as the inverse of Y does not make sense on $\mathcal{H}_0$. The convention $Y^{-1}(\mathbf{1}) = 0$ is arbitrary : any other value of $Y^{-1}(\mathbf{1})$ would give the same result, as derivations with values in $\mathcal{A}$ vanish at $\mathbf{1}$.

II.9. Examples

II.9.1. The Hopf algebra of positive integers

This example is a simplified version of the one given by D. Kreimer in [K2 § 2.1]. Consider the algebra $\mathcal{N}$ of the multiplicative semigroup $\mathbb{N}^* = \{1, 2, 3, \ldots\}$ of positive integers. As a vector space it admits a basis $(e_n)_{n\in\mathbb{N}^*}$ with product given by $e_n.e_m = e_{nm}$ and extended by linearity. We endow $\mathcal{N}$ with a structure of commutative cocommutative connected graded Hopf algebra thanks to the decomposition of any integer into a product of prime factors : namely we set $\Delta(e_1) = e_1 \otimes e_1$, and for any prime p :

$$\Delta(e_p) = e_p \otimes e_1 + e_1 \otimes e_p,$$

and we extend Δ to an algebra isomorphism. Hence,

$$\Delta(e_{p_1 \cdots p_k}) = \sum_{I \amalg J = \{1,\ldots,k\}} e_{p_I} \otimes e_{p_J},$$

where p_I denotes the product of the primes $p_j, j \in I$. The grading is clearly given by the number of prime factors (including multiplicities). The antipode is given by :

$$S(e_n) = (-1)^{|n|} e_n.$$

Suppose that the ground field is $k = \mathbb{C}$. The map $n \mapsto n^z$ defines a character φ of $\mathcal{N}$ with values into the holomorphic functions. Then the Riemann Zeta function is nothing but the evaluation of φ on the element :

$$\omega = e_1 + e_2 + e_3 + \cdots = \prod_{p \text{ prime}} \frac{1}{1 - e_p}.$$

Here $1/(1 - e_p)$ stands for the infinite sum : $e_1 + e_p + e_{p^2} + \cdots$. Of course ω is not an element of $\mathcal{N}$: it makes sense (as well as the abstract Euler product expansion on the right-hand side) only in the completion of $\mathcal{N}$ with respect to the *fine filtration* defined by the vector space grading $d(n) = n - 1$. But evaluating the character φ on both sides of this equality gives the well-known Euler product expression of the Zeta function.

II.9.2. Tensor and symmetric algebras

The tensor Hopf algebra $T(V)$ of any vector space V (cf. Example I.6.2) is obviously graded. The symmetric Hopf algebra is a particular case of enveloping Hopf algebra, with V viewed as an abelian Lie algebra. The Hopf algebra $S(V)$ is a cocommutative commutative connected graded Hopf algebra. Note that an enveloping algebra is not graded in general, since the quotienting ideal generated by $x \otimes y - y \otimes x - [x, y]$ is not homogeneous.

II.9.3. Planar decorated rooted trees

We borrow in this section some material from [F]. A *planar rooted tree* is an oriented connected contractible graph, with a finite number of vertices, together with an embedding of it into the plane, such that only one vertex has only outgoing edges (the root). We have drawn below the planar rooted trees with four vertices :

Figure 1 : the planar rooted trees with four vertices.

Let $\mathcal{T}$ be the set of planar rooted trees. Let V be a vector space on some field k, and let t be a planar rooted tree. The *space of decorations of t by V* is the vector space $V^{\otimes t}$. A planar rooted tree t together with an indecomposable element of $V^{\otimes t}$ is called a *decorated rooted tree*. Let us consider the vector space :

$$\mathcal{T}_V = \bigoplus_{t \in \mathcal{T}} V^{\otimes t},$$

let $\mathcal{H}_V$ be the (noncommutative) free algebra generated by $\mathcal{T}_V$. Products of decorated trees (decorated forests) generate $\mathcal{H}_V$ as a graded vector space, the degree of a decorated forest being given by the total number of vertices. The connected graded Hopf algebra structure on $\mathcal{H}_V$ is given by the co-unit ε sending $\mathbf{1}$ to 1 and any nonempty decorated forest to 0, and by a coproduct which we describe shortly here :

An *elementary cut* on a tree is a cut on some edge of the given tree. An admissible cut is a cut such that any path starting from the root contains at most one elementary cut. The *empty cut* is considered as elementary, as well as the *total cut*, i.e. a cut below the root. A cut on a forest is said to be admissible if its restriction to any tree factor is admissible. Any elementary cut c sends a forest F to a couple $\big(P^c(F), R^c(F)\big)$, the *crown* and the *trunk* respectively. The trunk of a tree is a tree, but the crown of a tree is a forest. Let $\mathrm{Adm}(F)$ the set of admissible cuts of the forest F, and let $\mathrm{Adm}^*(F)$ the set of elementary cuts discarding the empty cut and the total cut. The coproduct :

$$\Delta(F) = \sum_{c \in \mathrm{Adm}\, F} P^c(F) \otimes R^c(F)$$

is graded, co-associative and compatible with the product [F]. The compatibility with the product is clear (due to the definition of an admissible cut for a forest). There is a beautiful proof of the co-associativity in [F] using induction on the degree and grafting of any forest on a decorated root. We propose here a more intuitive proof : say that a couple (c_1, c_2) of cuts is *bi-admissible* if both cuts c_1, c_2 are admissible and if c_1 never bypasses c_2, i.e. if c_2 never cuts the trunk of c_1. Any bi-admissible couple $c = (c_1, c_2)$ of cuts c defines a crown $P^c(F) = P^{c_2}(F)$, a trunk $R^c(F) = R^{c_1}(F)$, and a middle $M^c(F)$:

Figure 2 : an example of bi-admissible couple of cuts. From thickest to thinnest : trunk, middle and crown.

Let $\text{Adm}_2\, F$ the set of bi-admissible couples of cuts of the forest F. It is quite straightforward to set down the formula for the iterated coproduct :

$$(\Delta \otimes I) \circ \Delta(F) = (I \otimes \Delta) \circ \Delta(F) = \sum_{c \in \text{Adm}_2 F} P^c(F) \otimes M^c(F) \otimes R^c(F).$$

Of course the n-fold iterated coproduct admits a similar expression, involving n-admissible n-uples of admissible cuts and $n+1$ "level segments" of the forest, from the crown down to the trunk.

By corollary II.2.2 the connected graded bialgebra $\mathcal{H}_V$ thus obtained admits an antipode given on $\text{Ker}\,\varepsilon$ by any of the two recursive formulas :

$$\begin{aligned} S(F) &= -F - \sum_{c \in \text{Adm}^*(F)} S\big(P^c(F)\big).R^c(F) \\ &= -F - \sum_{c \in \text{Adm}^*(F)} P^c(F).S\big(R^c(F)\big). \end{aligned}$$

The square of the antipode does not in general coincide with the identity.

II.9.3. Decorated rooted trees

The construction is the same except that we consider rooted trees independently from any embedding into the plane, and we consider the *free commutative* algebra generated by decorated rooted trees. We thus obtain a commutative Hopf algebra $\mathcal{H}'_V$ which is clearly a quotient of $\mathcal{H}_V$. This Hopf algebra is thoroughly investigated in [F].

III. Hopf algebras of Feynman graphs

We treat this example (more exactly this family of examples) in a separate section for two main reasons : firstly the Hopf algebras appearing there are pointed but not connected, and secondly this is the very example where a link is established with quantum field theory.

The non-connectedness is not a very serious problem : as we shall see we can reason on a connected quotient and go back. The formula for the coproduct will differ slightly from that of Connes-Kreimer in order to deal with this non-connectedness problem, but both will agree on the connected quotient. We follow [K1] quite closely, with some modifications in order to allow self-loops.

III.1. Discarding exterior structures

Anyone a little bit familiar with quantum field theory knows that Feynman graphs are made of internal and external edges of different types, and that an external edge comes with a vector attached to it (an *exterior momentum*). The sum of all exterior momenta of a given graph must be equal to zero, reflecting the global conservation of momenta in an interaction. The *Feynman rules* attach to a graph together with such an external structure an integral which can be divergent. This integral can be regularized by various procedures, among them *dimensional regularization* : the idea is to "let the dimension of the space of momenta vary in the complex numbers", a procedure which has been recently given a precise geometrical contents by A. Connes and M. Marcolli ([CM2] § 15). The divergent integral is now replaced by a meromorphic function with poles at least at the entire dimensions where the original integral diverges [C Chap. 4], [E].

The approach of renormalization by A. Connes and D. Kreimer can be summarized as follows : organize Feynman graphs with their exterior structures into a graded Hopf algebra, understand the (regularized, e.g. by means of dimensional regularization) Feynman rules as a character of this Hopf algebra with values into some algebra $\mathcal{A}$ (e.g. the meromorphic functions), choose a renormalization scheme, i.e. a splitting $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$ into two subalgebras, apply the method of § II.5 and II.6 to extract a renormalized value, and finally recognize that this method agrees with algorithms already developed by physicists, such as the Bogoliubov-Parasiuk-Hepp-Zimmerman (BPHZ) algorithm.

Our first step will consist in constructing a Hopf algebra from Feynman diagrams without exterior structure (i.e. with exterior momenta nullified).

III.2. Operations on Feynman graphs

A *Feynman graph* is a (non-oriented, non-planar) graph with a finite number of vertices and edges. An *internal edge* is an edge connected at both ends to a vertex (which can be the same in case of a self-loop), an *external edge* is an edge with one open end, the other end being connected to a vertex.

A Feynman graph is called by physicists *vacuum graph, tadpole graph, self-energy graph*, resp. *interaction graph* if its number of external edges is 0,1, 2, resp. > 2.

The edges (internal or external) will be of different types labelled by a positive integer $(1, 2, 3, \ldots)$, each type being represented by the way the corresponding edge is drawn (full, dashed, wavy, various colours, etc...). Let $\tau(e) \in \mathbb{N}^*$ be the type of the edge e. For any vertex v let $\mathrm{st}(v)$ be the *star* of v, i.e. the set of all edges attached to v, *with self-loops counted twice*. Hence the valence of the vertex is given by the cardinal of $\mathrm{st}(v)$. Finally to each vertex we associate its *type*, the sequence $(n_1, \ldots, n_r)$ of positive integers where n_j stands for the number of edges of type j in $\mathrm{st}(v)$. Let $T(v)$ be the type of the vertex v

For example, in the φ^n *theory* there is only one type of edge, and two types of vertices : the bivalent vertices and the n-valent vertices. In quantum electrodynamics there are two types of edges : the fermion edges (usually drawn full), and the boson edges (usually drawn wavy), and three types of vertices : bivalent boson-boson vertices, bivalent fermion-fermion vertices, and trivalent vertices with two fermion edges and one boson edge. Most of the pictures will be drawn in φ^3 or φ^4 theory, or in quantum electrodynamics.

A *one-particle irreducible graph* (in short, 1PI graph) is a connected graph which remains connected when we cut any internal edge. A disconnected graph is said to be *locally 1PI* if any of its connected components is 1PI. The *residue* of a connected graph is the graph with only one vertex obtained by shrinking all internal edges to a point. Of course any connected graph has the same type as its residue.

Figure 3 : A QED interaction graph and its residue.

A *subgraph* of a Feynman graph is either the empty graph, or a *nonempty* (connected or disconnected) set of internal edges together with the vertices they encounter and the stars of those vertices. A *proper subgraph* of Γ is a subgraph different from the empty graph or the whole graph Γ itself. If γ is a subgraph inside a graph Γ, the *contracted graph* Γ/γ is the graph obtained by replacing all connected components of γ by their residues inside Γ. As an example the residue of a graph Γ is equal to Γ/Γ.

Figure 4 : A subgraph γ inside a graph Γ in φ^3 theory. The contracted graph Γ/γ does not belong to φ^3.

III.3. The graded Hopf algebra structure

Fix a set $\mathcal{T} = \{T_1, \ldots, T_k\}$ of finite sequences of positive integers, which will be the possible vertex types we want to deal with. Let $V_{\mathcal{T}}$ be the vector space generated by all connected 1PI Feynman graphs with vertex types in $\mathcal{T}$, and all residues of those. Let $\mathcal{B}_{\mathcal{T}} = S(V_{\mathcal{T}})$ be the free commutative algebra generated by V. We shall identify the unit $\mathbf{1}$ with the empty graph and any element of $\mathcal{B}_{\mathcal{T}}$ with a linear combination of disconnected locally 1PI graphs. The algebra structure is obvious, the co-unity is given by $\varepsilon(\mathbf{1}) = 1$ and $\varepsilon(\Gamma) = 0$ for any nonempty graph Γ. The grading (at least, one possible grading) is given on connected graphs by the *loop number* :

$$L := I - V + 1,$$

where I is the number of internal edges and V is the number of vertices of a given graph. This grading is extended to non-connected graphs in such a way that it is compatible with the algebra structure. It is important to notice that any nonempty subgraph has a non-vanishing loop number. The coproduct is given on connected 1PI graphs by the following formula :

$$\begin{aligned}\Delta(\Gamma) &= \sum_{\substack{\gamma \text{ subgraph of } \Gamma \\ \Gamma/\gamma \in V_{\mathcal{T}}}} \gamma \otimes \Gamma/\gamma \\ &= \Gamma \otimes \operatorname{res}\Gamma + \mathbf{1} \otimes \Gamma + \sum_{\substack{\gamma \text{ proper subgraph of } \Gamma \\ \Gamma/\gamma \in V_{\mathcal{T}}}} \gamma \otimes \Gamma/\gamma \qquad \text{if } L(\Gamma) \geq 1, \\ \Delta(\Gamma) &= \Gamma \otimes \Gamma \qquad \text{if } L(\Gamma) = 0,\end{aligned}$$

and extended to non-connected graphs by multiplicativity. We leave it to the reader as an easy exercice to show that the coproduct respects the loop number as well. Figure 5 below illustrates a coproduct computation in φ^3 theory. Two terms of the sum have been removed because the corresponding contracted graphs have a vertex the type of which is outside $\mathcal{T}$ (here a pentavalent and an hexavalent vertex respectively), and then does not belong to $V_{\mathcal{T}}$. On the other hand residues with any number of external edges are allowed.

Figure 5 : an example of coproduct in φ^3 theory.

Figure 6 below illustrates another coproduct computation in φ^3 theory, with a bivalent vertex arising in the contracted graph :

Figure 6 : another example of coproduct in φ^3 theory.

Proposition III.3.1.
$\mathcal{B}_\mathcal{T}$ is a pointed graded bialgebra.

Proof. All axioms of a pointed graded bialgebra have been already given by the construction, except coassociativity of the coproduct. But we have for any 1PI graph of positive degree :

$$(\Delta \otimes I)\Delta(\Gamma) = \sum_{\substack{\delta\subset\gamma\subset\Gamma \\ \gamma/\delta\in\mathcal{B}_\mathcal{T},\ \Gamma/\gamma\in\mathcal{B}_\mathcal{T}}} \delta \otimes \gamma/\delta \otimes \Gamma/\gamma,$$

whereas :

$$(I \otimes \Delta)\Delta(\Gamma) = \sum_{\substack{\delta\subset\Gamma,\widetilde{\gamma}\subset\Gamma/\delta \\ \Gamma/\delta\in\mathcal{B}_\mathcal{T},\ (\Gamma/\delta)/\widetilde{\gamma}\in\mathcal{B}_\mathcal{T}}} \delta \otimes \widetilde{\gamma} \otimes (\Gamma/\delta)/\widetilde{\gamma}.$$

There is an obvious bijection $\gamma \mapsto \widetilde{\gamma} = \gamma/\delta$ from subgraphs of Γ containing δ onto subgraphs of Γ/δ, given by shrinking δ. As we have the obvious "transitive shrinking property" :

$$\Gamma/\gamma = (\Gamma/\delta)/\widetilde{\gamma},$$

the two expressions coincide.

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In order to build up a graded Hopf algebra from $\mathcal{B}_\mathcal{T}$, two choices are possible : first we can add formally the inverses of the grouplike elements, i.e. the degree zero graphs : let Σ be the set of degree zero connected 1PI graphs, let Σ^{-1} be another copy of the same set, with elements labelled $\gamma^{-1}, \gamma \in \Sigma$. Let $\widetilde{V}_\mathcal{T}$ be the vector space generated by $V_\mathcal{T}$ and Σ^{-1}, and consider :

$$\widetilde{\mathcal{H}}_\mathcal{T} = S(\widetilde{V}_\mathcal{T})/J,$$

where J is the ideal generated by $\gamma\gamma^{-1} - \mathbf{1}$, $\gamma \in \Sigma$. The coproduct on $S(V_\mathcal{T})$ is extended to $S(\widetilde{V}_\mathcal{T})$ by saying that the elements of Σ^{-1} are grouplike. The ideal J is the a bi-ideal, and so $\widetilde{\mathcal{H}}_\mathcal{T}$ is a pointed graded bialgebra. An antipode is easily given inductively with respect to the degree, as any degree zero element has an antipode given by $S(\gamma) = \gamma^{-1}$, $S(\gamma^{-1}) = \gamma$ for any $\gamma \in \Sigma$.

The second option consists in killing the degree zero graphs (except the empty graph). We set :

$$\mathcal{H}_{\mathcal{T}} = \mathcal{B}_{\mathcal{T}}/K,$$

where K is the ideal generated by $\gamma - \mathbf{1}, \gamma \in \Sigma$. It is easily seen to be a bi-ideal. The quotient is then a connected graded bialgebra, hence a Hopf algebra thanks to corollary II.3.2. We can identify the quotient with $S(V'_{\mathcal{T}})$, where $V'_{\mathcal{T}}$ stands for the vector space generated by connected $1PI$ graphs with loop number ≥ 1. The coproduct is then given by Kreimer's formula :

$$\Delta(\Gamma) = \Gamma \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + \sum_{\substack{\gamma \text{ proper subgraph of } \Gamma \\ \Gamma/\gamma \in V_{\mathcal{T}}}} \gamma \otimes \Gamma/\gamma.$$

III.4. External structures

We shall be very sketchy here. Let W be a finite-dimensional vector space (the *momentum space*). Keeping the notations of § III.3 and following [CK 2] and [K 1], a *specified graph* will be a couple (Γ, σ) where Γ is a connected graph in $V_{\mathcal{T}}$ with E external lines, and σ is a distribution on the vector subspace $M_\Gamma = M_E \subset W^E$ defined by :

$$M_E = \{(p_1, \ldots, p_E), \sum_{k=1}^{E} p_k = 0\}.$$

In order to get a Hopf algebra structure for specified graphs we must discriminate further the type of a vertex : once the number of edges of each type is fixed for a vertex, we add an extra nonzero natural number, so that there are "several kinds of vertices of the same type". This comes from the lagrangian of the given quantum field theory we are dealing with : each monomial of degree n_i with respect to the field ϕ_i ($i \in \{1, \ldots, k\}$) gives rise to vertices of type $T = (n_1, \ldots, n_k)$, and there are as many kinds of vertices of type T as monomials of "field degree" T inside the lagrangian. For example, in φ^3 theory with mass, terms $(m^2/2)\varphi^2$ and $(\partial\varphi)^2/2$ give rise to two different kinds of bivalent vertices. When taking residues we must specify the kind for the unique remaining vertex : then when considering a contracted graph Γ/γ we must consider the kind of every contracted vertex (corresponding to a connected component of the subgraph γ). This gives rise to contracted graphs $\Gamma/\gamma(i)$ where i is a multi-index.

To any vertex of kind (T, i) corresponds a specific distribution $\sigma_{T,i}$ on M_Γ, where Γ is any graph whose residue gives a vertex of type T. This extends to non-connected graphs by considering multi-indices i. Now $\mathcal{T}$ stands for the set of all *kinds* (T, i) of vertices we can encounter, $V_{\mathcal{T}}$ stands for the space generated by all connected 1PI graphs with vertex kinds in $\mathcal{T}$, and all residues of those. Let $V'_{\mathcal{T}}$ the space generated by all connected 1PI Feynman graphs with vertex kinds in $\mathcal{T}$ and nonzero loop number, let $(V'_{\mathcal{T}})_E$ the subspace of $V'_{\mathcal{T}}$ of graphs with E external edges, and finally let $W'_{\mathcal{T}}$ the corresponding space of specified graphs :

$$W'_{\mathcal{T}} = \sum_{E=0}^{\infty} (V'_{\mathcal{T}})_E \otimes \mathcal{D}'(M_E).$$

We directly give the connected version of the Hopf algebra : it is given by $\mathcal{H}_{\mathcal{T}} = S(W'_{\mathcal{T}})$, and the coproduct is given on connected specified graphs by :

$$\Delta(\Gamma,\sigma) = (\Gamma,\sigma)\otimes \mathbf{1} + \mathbf{1}\otimes(\Gamma,\sigma) + \sum_{\gamma \text{ proper subgraph of } \Gamma} \ \sum_{i,\ \Gamma/\gamma(i)\in V_{\mathcal{T}}} (\gamma,\sigma_{T,i})\otimes(\Gamma/\gamma(i),\sigma).$$

IV. An approach to the renormalization group

Keeping the notations of paragraph II, we denote by $\mathcal{H}$ a connected graded Hopf algebra, and by $\mathcal{A}$ the algebra of germs of meromorphic functions at some $z_0 \in \mathbb{C}$. The algebra $\mathcal{A}$ admits a splitting into two subalgebras :

$$\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-,$$

where $\mathcal{A}_+$ is the algebra of germs of holomorphic functions at z_0, and $\mathcal{A}_- = (z-z_0)^{-1}\mathbb{C}[(z-z_0)^{-1}]$. We denote by Y (resp. θ_t) the biderivation (resp. the one-parameter group of automorphisms) of the Hopf algebra $\mathcal{H}$ induced by the graduation (cf. § II.6). As in paragraph II we denote by G the group of the elements $\varphi \in \mathcal{L}(\mathcal{H},\mathcal{A})$ such that $\varphi(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}$ (with the convolution products), and by $\mathfrak{g}$ the subalgebra of $\mathcal{L}(\mathcal{H},\mathcal{A})$ of the elements $\varphi \in \mathcal{L}(\mathcal{H},\mathcal{A})$ such that $\varphi(\mathbf{1}) = 0$.

Recall that $G = \exp \mathfrak{g}$. As in § II we shall consider the subgroups G_1 (resp. G_2) of G formed by the characters of $\mathcal{H}$ with values in $\mathcal{A}$ (resp. by the elements of G which enjoy the cocycle property), as well as the Lie subalgebras $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$) of derivations of $\mathcal{H}$ with values in $\mathcal{A}$ (resp. of $\mathfrak{g}$ which enjoy the cocycle property). We have $G_1 = \exp \mathfrak{g}_1$ and $G_2 = \exp \mathfrak{g}_2$.

IV.1. The renormalization map

We settle here a bijection $R : \mathfrak{g} \to \mathfrak{g}$ thanks to the biderivation Y :

Proposition IV.1.1.
The equation :

$$\varphi \circ Y = \varphi * \gamma \qquad (E)$$

defines a bijective correspondence :

$$\begin{aligned} \widetilde{R} : G &\longrightarrow \mathfrak{g} \\ \varphi &\longmapsto \gamma. \end{aligned}$$

Equivalently the equation :

$$e^{*\alpha} \circ Y = e^{*\alpha} * \gamma \qquad (E')$$

defines a (non-linear) bijective correspondence :

$$\begin{aligned} R : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ \alpha &\longmapsto \gamma, \end{aligned}$$

and $R = \widetilde{R} \circ \exp$.

Proof. Equation (E) yields for any homogeneous $x \in \mathcal{H}$:

$$|x|\varphi(x) = \gamma(x) + \sum_{(x)} \varphi(x')\gamma(x''),$$

which determines γ (recursively in $|x|$) from φ and vice-versa, starting from $\varphi(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}$ and $\gamma(\mathbf{1}) = 0$. In other words equation (E) determines a bijection $\widetilde{R}$ from G to $\mathfrak{g}$ such that $\gamma = \widetilde{R}(\varphi)$. The remainder of prop. IV.1.1 follows then immediately.

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Equation (E') yields the following explicit expression for R :

$$R(\alpha) = e^{*-\alpha} * (e^{*\alpha} \circ Y).$$

There is another explicit formula :

Proposition IV.1.2.

$$R(\alpha) = \int_0^1 e^{*-s\alpha} * (\alpha \circ Y) * e^{*s\alpha}\, ds.$$

Proof. for any $u \in \mathbb{R}$ we have :

$$e^{*u\alpha} \circ Y = e^{*u\alpha} * R(u\alpha).$$

Setting $u = t + s$ and using the group property $e^{*(t+s)}\alpha = e^{*t\alpha} * e^{*s\alpha}$ as well as the derivation property :

$$(e^{*t\alpha} * e^{*s\alpha}) \circ Y = (e^{*t\alpha} \circ Y) * e^{*s\alpha} + e^{*t\alpha} * (e^{*s\alpha} \circ Y),$$

we get :

$$e^{*(t+s)\alpha} \circ Y = e^{*(t+s)\alpha} * \big(R(s\alpha) + e^{*-s\alpha} * R(t\alpha) * e^{*s\alpha}\big).$$

Set $\gamma(t) = R(t\alpha)$: the above equation reads :

$$\gamma(t+s) = \gamma(s) + e^{*-s\alpha} * \gamma(t) * e^{*s\alpha}.$$

We have $\gamma(0) = 0$, and differentiating this equation with respect to s at $s = 0$ yields :

$$\dot\gamma(t) = \dot\gamma(0) + [\gamma(t), \alpha].$$

Differentiating once again with respect to t gives then :

$$\ddot\gamma(t) = [\dot\gamma(t), \alpha].$$

The solution of this first order differential equation is given by :

$$\dot\gamma(t) = e^{*-t\alpha} * \dot\gamma(0) * e^{*t\alpha}.$$

Expanding the equation $e^{*t\alpha} \circ Y = e^{*t\alpha} * \gamma(t)$ up to order 1 in $t = 0$ yields immediately :

$$\dot\gamma(0) = \alpha \circ Y.$$

Integrating and setting $t = 1$ establishes then proposition IV.1.2.

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Corollary IV.1.3.
Correspondence R sends $\mathcal{A}$-valued derivations to $\mathcal{A}$-valued derivations and cocycles to cocycles.

Proof. First assertion follows immediately from propositions IV.1.2, II.4.2 and II.8.2. Second assertion follows directly from proposition IV.1.2.

•

Remark : If the Hopf algebra $\mathcal{H}$ is cocommutative, then thanks to the commutativity of $\mathcal{A}$, the convolution product is commutative. The correspondence R becomes then linear and we simply have :

$$R(\alpha) = \alpha \circ Y.$$

IV.2. Inverting $\widetilde{R}$: the scattering map

We shall give an explicit expression of the map $\widetilde{R}^{-1} : \mathfrak{g} \to G$. It takes the form :

$$\widetilde{R}^{-1}(\gamma) = \lim_{t \to +\infty} \exp -tA \exp tB,$$

(cf. theorem IV.2.1 below), where A and B live in a semi-direct product Lie algebra $\widetilde{\mathfrak{g}} = \mathfrak{g} \rtimes \mathbb{C}$. We have to describe this semi-direct product and the corresponding semi-direct product group $\widetilde{G} = G \rtimes \mathbb{C}$, and then we must endow $\widetilde{G}$ with a topology so that the above limit makes sense. We adapt here the proof of Theorem 2 in [CK2]. To be precise, we define the Lie algebra :

$$\widetilde{g} = \mathfrak{g} \rtimes \mathbb{C}.Z_0,$$

where the action of Z_0 on $\mathfrak{g}$ is given by the derivation :

$$Z_0(\gamma) = \gamma \circ Y$$

(see lemma II.7.1). The corresponding group is $\widetilde{G} = G \rtimes \mathbb{C}$, where the right action of $\mathbb{C}$ on G is given by :

$$\varphi.t = \varphi \circ \theta_t,$$

so that the product is given by $(\varphi, t)(\psi, s) = \big(\varphi * (\psi \circ \theta_t), t + s\big)$. We shall not dig out a Lie group structure for $\widetilde{G}$ here, but we shall define the exponential map $\exp : \widetilde{\mathfrak{g}} \to \widetilde{G}$. It should of course coincide with the exponential already defined on G, and should verify :

$$\exp tZ_0 = (e, t)$$

so that $\exp tZ_0$ indeed acts on G by composition with $\theta_t = \exp tY$ on the right. We should be able in principle to define $\exp(tZ_0 + \gamma)$ by means of the Baker-Campbell-Hausdorff formula as long as convergence problems can be handled here. We prefer, like in [CK2], to give an alternative definition based on Araki's expansion formula [Ar] :

$$\exp(tZ_0 + \gamma) = \sum_{n=0}^{\infty} \int_{\sum_{j=0}^{n} u_j = 1,\, u_j \geq 0} \exp(u_0 tZ_0)\gamma \exp(u_1 tZ_0)\gamma \cdots \gamma \exp(u_n tZ_0)\, du_1 \cdots du_n.$$

Let us check that the sum above makes sense in our particular context : setting $v_j = u_j + u_{j+1} + \cdots + u_n$ we get :

$$\exp(-tZ_0)\exp(tZ_0+\gamma) = \exp(-tZ_0).$$

$$\exp(tZ_0).\sum_{n=0}^{\infty}\int_{0\le v_n\le\cdots\le v_1\le 1}\exp(-tv_1Z_0)\gamma\exp(tv_1Z_0)\cdots\exp(-tv_nZ_0)\gamma\exp(tv_nZ_0)\,dv_1\cdots dv_n$$

$$= \sum_{n=0}^{\infty}\int_{0\le v_n\le\cdots\le v_1\le 1}(\gamma\circ\theta_{-tv_1})*\cdots*(\gamma\circ\theta_{-tv_n})\,dv_1\cdots dv_n.$$

The sum here is well defined as a locally finite sum, as it ends up at $n = n_0$ when evaluated at any $x = \mathcal{H}^{n_0}$. It remains to check that the exponential thus defined enjoys the one-parameter group property. Indeed, for any s, t real we have :

$$\exp t(Z_0+\gamma)\exp s(Z_0+\gamma) = e^{tZ_0}\Big(\sum_{p=0}^{\infty}t^p\int_{0\le v_p\le\cdots\le v_1\le 1}(\gamma\circ\theta_{-tv_1})*\cdots*(\gamma\circ\theta_{-tv_p})\,dv_1\cdots dv_p\Big).$$

$$e^{sZ_0}\Big(\sum_{p=0}^{\infty}s^q\int_{0\le w_q\le\cdots\le w_1\le 1}(\gamma\circ\theta_{-sw_1})*\cdots*(\gamma\circ\theta_{-sw_q})\,dw_1\cdots dw_q\Big)$$

$$= e^{(t+s)Z_0}\sum_{p,q=0}^{\infty}t^ps^q\iint_{0\le v_p\le\cdots\le v_1\le 1,\,0\le w_q\le\cdots\le w_1\le 1}$$

$$(\gamma\circ\theta_{-s-tv_1})*\cdots*(\gamma\circ\theta_{-s-tv_p})*(\gamma\circ\theta_{-sw_1})*\cdots*(\gamma\circ\theta_{-sw_q})\,dv_1\cdots dv_p dw_1\cdots dw_q$$

$$= e^{(t+s)Z_0}\sum_{p,q=0}^{\infty}\iint_{0\le v_p\le\cdots\le v_1\le t,\,0\le w_q\le\cdots\le w_1\le s}$$

$$(\gamma\circ\theta_{-s-v_1})*\cdots*(\gamma\circ\theta_{-s-v_p})*(\gamma\circ\theta_{-w_1})*\cdots*(\gamma\circ\theta_{-w_q})\,dv_1\cdots dv_p dw_1\cdots dw_q$$

$$= e^{(t+s)Z_0}\sum_{n=0}^{\infty}\sum_{p+q=n}\iint_{s\le v_p\le\cdots\le v_1\le t+s,\,0\le w_q\le\cdots\le w_1\le s}$$

$$(\gamma\circ\theta_{-v_1})*\cdots*(\gamma\circ\theta_{-v_p})*(\gamma\circ\theta_{-w_1})*\cdots*(\gamma\circ\theta_{-w_q})\,dv_1\cdots dv_p dw_1\cdots dw_q$$

$$= e^{(t+s)Z_0}\sum_{n=0}^{\infty}\int_{s\le u_p\le\cdots\le u_1\le t+s}$$

$$(\gamma\circ\theta_{-u_1})*\cdots*(\gamma\circ\theta_{-u_n})\,du_1\cdots du_n$$

$$= \exp(t+s)(Z_0+\gamma).$$

We can now state the main theorem of this section :

Theorem IV.2.1.
Let $\gamma \in \mathfrak{g}$. Then :

1) For any real t the product $\exp -tZ_0 \exp t(Z_0+\gamma)$ belongs to G.

2) The product above admits a limit when $t \to +\infty$ for the topology on G induced by the simple convergence topology on $\mathcal{L}(\mathcal{H}, \mathcal{A})$.
3) The inverse of the renormalization map is given by :

$$\widetilde{R}^{-1}(\gamma) = \lim_{t \to +\infty} \exp -tZ_0 \exp t(Z_0 + \gamma).$$

4) $\widetilde{R}^{-1}$ sends $\mathfrak{g}_1$ into G_1 and $\mathfrak{g}_2$ into G_2.

Proof. The first assertion comes directly from the expression :

$$\exp(-tZ_0)\exp(tZ_0 + t\gamma) = \sum_{n=0}^{\infty} \int_{0 \le v_n \le \cdots \le v_1 \le 1} (t\gamma \circ \theta_{-tv_1}) * \cdots * (t\gamma \circ \theta_{-tv_n})\, dv_1 \cdots dv_n.$$

The right-hand side belongs manifestly to G. Change of variables $v_j \to tv_j$ yields :

$$\exp(-tZ_0)\exp(tZ_0 + t\gamma) = \sum_{n=0}^{\infty} \int_{0 \le v_n \le \cdots \le v_1 \le t} (\gamma \circ \theta_{-v_1}) * \cdots * (\gamma \circ \theta_{-v_n})\, dv_1 \cdots dv_n.$$

To prove the second assertion it suffices to prove that the integrals :

$$I_n := \int_{0 \le v_n \le \cdots \le v_1 \le +\infty} (\gamma \circ \theta_{-v_1}) * \cdots * (\gamma \circ \theta_{-v_n})\, dv_1 \cdots dv_n$$

converge, as the sum $I_0 + I_1 + I_2 + \cdots$ is locally finite. The convergence is easily seen by induction on n : indeed we have $I_0 = e$ and the crucial equality valid for any $x \in \operatorname{Ker} \varepsilon$:

$$Y^{-1}(x) = \int_0^\infty \theta_{-t}(x)\, dt.$$

It follows that we have for any $a \in \mathfrak{g}$:

$$\int_0^\infty a \circ \theta_{-t}\, dt = a \circ Y^{-1}.$$

A simple computation then gives :

$$\begin{aligned} I_n &= \int_0^\infty (I_{n-1} * \gamma) \circ \theta_{-v_n}\, dv_n \\ &= (I_{n-1} * \gamma) \circ Y^{-1}, \end{aligned}$$

which inductively shows the convergence of the integrals I_n. Now equation (E) can be rewritten as :

$$\begin{aligned} &\varphi(x) = (\varphi * \gamma) \circ Y^{-1}(x) \quad \forall x \in \operatorname{Ker} \varepsilon \\ &\varphi(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}. \end{aligned} \tag{E''}$$

As $\gamma = \widetilde{R}(\varphi)$ it means that :

$$\widetilde{R}^{-1}(\gamma) = e + T\big(\widetilde{R}^{-1}(\gamma)\big),$$

where T is the transformation of $\mathcal{L} = \mathcal{L}(\mathcal{H}, \mathcal{A})$ defined by :

$$\begin{aligned} T(\psi) &= (\psi * \gamma) \circ Y^{-1} \\ &= \int_0^\infty (\psi * \gamma) \circ \theta_{-t}\, dt. \end{aligned}$$

Transformation T is a contraction on $\mathcal{L}$ for the distance associated with the filtration. $\widetilde{R}^{-1}(\gamma)$ is then the limit of the sequence (φ_n) defined by $\varphi_0 = e$ and $\varphi_{n+1} = e + T(\varphi_n)$. A straightforward computation yields :

$$\varphi_n = \sum_{k=0}^{n} I_k.$$

Hence we have :

$$\widetilde{R}^{-1}(\gamma) = \sum_{k=0}^{\infty} I_k,$$

which proves assertion 3). Finally assertion 4) comes from the fact that derivation Z_0 acts on $\mathfrak{g}_1$ and $\mathfrak{g}_2$. We can then consider semi-direct products :

$$\begin{aligned} \widetilde{\mathfrak{g}}_1 &= \mathfrak{g}_1 \rtimes \mathbb{C}.Z_0, \qquad & \widetilde{G}_1 &= G_1 \rtimes \mathbb{C}, \\ \widetilde{\mathfrak{g}}_2 &= \mathfrak{g}_2 \rtimes \mathbb{C}.Z_0, \qquad & \widetilde{G}_2 &= G_2 \rtimes \mathbb{C}, \end{aligned}$$

and thus replace the group G by any of the two groups G_1, G_2 in assertions 1), 2) and 3), which proves assertion 4) and ends the proof of the theorem.

•

Corollary IV.2.2.
The inverse of $R : \mathfrak{g} \to \mathfrak{g}$ is given by :

$$R^{-1}(\gamma) = \lim_{t \to +\infty} \operatorname{Log}\big(\exp -tZ_0 \exp t(Z_0 + \gamma)\big),$$

and R^{-1} sends $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$) into $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$).

IV.3. The residue

We keep the same notations as before except that we set $z_0 = 0$ for notational simplicity. To any $\psi \in \mathcal{L}$ we associate a linear form $\operatorname{Res}\psi$ on $\mathcal{H}$ by extracting the z^{-1} term : more precisely if we have for any $x \in \mathcal{H}$ and for any z in some pointed neighbourhood of 0 :

$$\psi(x)(z) = \sum_{n=-N}^{+\infty} \psi_n(x) z^n$$

with $\psi_n(x) \in \mathbb{C}$, then :

$$\operatorname{Res}\psi(x) := \psi_{-1}(x).$$

IV.4. Renormalization map and Birkhoff decomposition

Following more closely A. Connes and D. Kreimer [CK2] we shall focus our attention on elements of G the Birkhoff decomposition of which shares an invariance property with respect to the action of the graduation. More precisely we define first a new action of $\mathbb{C}$ on the group G by :

$$\psi^t(x)(z) = \psi(\theta_{tz}(x))(z) = e^{tz|x|}\psi(x)(z).$$

We shall consider the elements ψ of G such that in the Birkhoff decomposition :

$$\psi^t = (\psi^t)_-^{*-1} * (\psi^t)_+$$

the polar part $(\psi^t)_-$ is independent of t. The motivation to look at this very specific property is that it is fulfilled by the characters of Hopf algebras of Feynman graphs obtained via Feynman rules and dimensional regularization. In physical terms this corresponds to the well-known fact that the counterterms do not depend on the choice of the arbitrary mass μ one must introduce in order to perform dimensional regularization. Let us say that $\psi \in G$ enjoys property (Φ) if the above condition on the polar part of its Birkhoff decomposition is fulfilled. G^Φ will denote the subset of elements of G enjoying property (Φ).

For any $\psi \in \mathcal{L}$ and any $f \in \mathcal{A}$, we shall denote by $f\psi$ the element of $\mathcal{L}$ defined by :

$$\forall x \in \mathcal{H},\ f\psi(x)(z) = f(z)\psi(x)(z).$$

In particular, $(z\psi)$ is defined by $(z\psi)(x)(z) = z.\psi(x)(z)$.

Theorem IV.4.1.
The map :

$$z\widetilde{R} : G \longrightarrow \mathfrak{g}$$
$$\psi \longmapsto z\widetilde{R}(\psi)$$

restricts to a bijection from G^Φ onto $\mathfrak{g} \cap \mathcal{L}(\mathcal{H}, \mathcal{A}_+)$.

Proof. For any $\beta \in \mathfrak{g}$ introduce the linear transformation U_β of $\mathfrak{g}$ defined by :

$$U_\beta(A) = \beta * A + zA \circ Y.$$

If β belongs to $\mathfrak{g} \cap \mathcal{L}(\mathcal{H}, \mathcal{A}_+)$ then U_β restricts to a linear transformation of $\mathfrak{g} \cap \mathcal{L}(\mathcal{H}, \mathcal{A}_+)$.

Lemma IV.4.2.
For any $\psi \in G, n \in \mathbb{N}$ we have :

$$z^n \psi \circ Y^n = \psi * U^n_{z\widetilde{R}(\psi)}(e).$$

Proof. Case $n = 0$ is obvious, $n = 1$ is just the definition of $\widetilde{R}$. We check thus by induction :

$$\begin{aligned} z^{n+1}\psi \circ Y^{n+1} &= z(\psi \circ Y^n) \circ Y \\ &= z\big(\psi * U^n_{z\widetilde{R}(\psi)}(e)\big) \circ Y \\ &= z(\psi \circ Y) * U^n_{z\widetilde{R}(\psi)}(e) + z\psi * \big(U^n_{z\widetilde{R}(\psi)}(e) \circ Y\big) \\ &= \psi * \big(z\widetilde{R}(\psi) * U^n_{z\widetilde{R}(\psi)}(e) + zU^n_{z\widetilde{R}(\psi)}(e) \circ Y\big) \\ &= \psi * U^{n+1}_{z\widetilde{R}(\psi)}(e). \end{aligned}$$

•

Let us finish the proof of Theorem IV.4.1 : according to Lemma IV.4.2 we have for any real t, at least formally :

$$\psi^t = \psi * \exp(tU_{z\widetilde{R}(\psi)})(e). \qquad (*)$$

We have still to fix the convergence of the exponential just above in the case when $z\widetilde{R}(\psi)$ belongs to $L(\mathcal{H}, \mathcal{A}_+)$. Let us consider the following decreasing bifiltration of $\mathcal{L}(\mathcal{H}, \mathcal{A}_+)$:

$$\mathcal{L}^{p,q}_+ = \big(z^q \mathcal{L}(\mathcal{H}, \mathcal{A}_+)\big) \cap \mathcal{L}^p.$$

Considering the associated filtration :

$$\mathcal{L}^n_+ = \bigcup_{p+q=n} \mathcal{L}^{p,q}_+,$$

we see that for any $\beta \in \mathfrak{g} \cap \mathcal{L}(\mathcal{H}, \mathcal{A}_+)$ transformation U_β increases filtration by 1, i.e :

$$U_\beta(\mathcal{L}^n_+) \subset \mathcal{L}^{n+1}_+.$$

The algebra $\mathcal{L}(\mathcal{H}, \mathcal{A}_+)$ is *not* complete with respect to the topology induced by this filtration, but the completion is $\mathcal{L}(\mathcal{H}, \widehat{\mathcal{A}_+})$, where $\widehat{\mathcal{A}_+} = \mathbb{C}[[z]]$ stands for the formal series. Hence the right-hand side of (*) is convergent in $\mathcal{L}(\mathcal{H}, \widehat{\mathcal{A}_+})$ with respect to this topology. Hence for any $\gamma \in \mathcal{L}(\mathcal{H}, \mathcal{A}_+)$ and for ψ such that $z\widetilde{R}(\psi) = \gamma$ we have $\psi^t = \psi * h_t$

with $h_t \in \mathcal{L}(\mathcal{H}, \widehat{\mathcal{A}_+})$ for any t. On another hand we already know that h_t takes values in meromorphic functions for each t. So h_t belongs to $\mathcal{L}(\mathcal{H}, \mathcal{A}_+)$.

Then for any $\beta \in \mathcal{L}(\mathcal{H}, \mathcal{A}_+)$ and for ψ such that $z\widetilde{R}(\psi) = \beta$ we have $\psi^t = \psi * h(t)$ with $h(t) \in \mathcal{L}(\mathcal{H}, \mathcal{A}_+)$ for any t. This is equivalent to the fact that ψ belongs to G^Φ. Conversely take any ψ in G^Φ. Then there exists $h_t \in \mathcal{L}(\mathcal{H}, \mathcal{A}_+)$ such that $\psi^t = \psi * h_t$. Hence :

$$\frac{d}{dt}_{|t=0}\psi_t = z(\psi \circ Y) = \psi \dot{h_t}_{|_{t=0}}.$$

So $z\widetilde{R}(\psi) = \dot{h_t}_{|_{t=0}}$ belongs to $\mathcal{L}(\mathcal{H}, \mathcal{A}_+)$. This proves theorem IV.4.1.

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Lemma IV.4.3.
1) Let $\psi \in G^\Phi$, and let :

$$\psi = (\psi_-)^{*-1} * \psi_+$$

its Birkhoff decomposition. Then $\widetilde{\psi} = (\psi_-)^{*-1}$ enjoys property (Φ) as well.

2) Let $\psi \in G^\Phi$ and let h be any element of G without polar part. Then $\psi * h$ enjoys property (Φ) as well.

Proof. Assertion 1) is a consequence of the equality :

$$\psi^t = (\psi^t)_-^{*-1} * (\psi^t)_+ = \widetilde{\psi}^t * (\psi_+)^t,$$

which implies immediately :

$$\widetilde{\psi}^t = (\psi^t)_-^{*-1} * (\psi^t)_+ * \big((\psi_+)^t\big)^{*-1}.$$

This is the Birkhoff decomposition of $\widetilde{\psi}^t$: its polar part is $(\psi^t)_-$, which by hypothesis is independant of t. Assertion 2) comes from the equality :

$$(\psi * h)^t = \psi^t * h^t = (\psi^t)_-^{*-1} * \big((\psi^t)_+ * h^t\big),$$

and from the fact that h_t does not have any polar part. Thus the right-hand side is the Birkhoff decomposition of $(\psi * h)^t$, the polar part of which is clearly independant of t.

•

Thanks to Lemma IV.4.3 we can now focus our attention on elements ψ of G which enjoy property (Φ) and such that ψ sends $\operatorname{Ker}\varepsilon$ into $\mathcal{A}_-$. Let us denote by G_-^Φ the subset of G formed by these elements.

Theorem IV.4.4.
Let $\mathfrak{g}^c = \{\beta \in \mathcal{L}(\mathcal{H}, \mathbb{C}),\ \beta(\mathbf{1}) = 0\}$. *Let* $\mathfrak{g}_1^c = \mathfrak{g}^c \cap \mathfrak{g}_1$, *and* $\mathfrak{g}_2^c = \mathfrak{g}^c \cap \mathfrak{g}_2$. *Let* $G_{1,-}^\Phi = G_-^\Phi \cap G_1$ *and* $G_{2,-}^\Phi = G_-^\Phi \cap G_2$. *Then the map :*

$$\begin{aligned} G &\longrightarrow \mathfrak{g} \\ \psi &\longmapsto z\widetilde{R}(\psi) \end{aligned}$$

restricts to a bijection from G_-^Φ *(resp.* $G_{1,-}^\Phi$, $G_{2,-}^\Phi$*) to* $\mathfrak{g}^c$ *(resp* $\mathfrak{g}_1^c$, $\mathfrak{g}_2^c$*) , and we have explicitly for any* $\psi \in G_-^\Phi$ *:*

$$\widetilde{R}(\psi) = \frac{1}{z}\big((\operatorname{Res}\psi) \circ Y\big).$$

Proof. We shall need the following key lemma :

Lemma IV.4.5.
For any $\psi \in G_-^\Phi$ *we have :*

$$\frac{d}{dt}_{|t=0}\big((\psi^t)_+\big) = \operatorname{Res}(\psi \circ Y).$$

Proof. Using property (Φ), the fact that $\psi(\operatorname{Ker}\varepsilon) \subset \mathcal{A}_-$ and the explicit expression of $(\psi^t)_+$ given by theorem II.4.1 we have for any $x \in \mathcal{H}$:

$$\begin{aligned} (\psi^t)_+(x) &= (I - \pi)\Big(\psi^t(x) + \sum_{(x)} \psi^{*-1}(x')\psi^t(x'')\Big) \\ &= t(I-\pi)\big(z|x|\psi(x) + z\sum_{(x)} \psi^{*-1}(x')\psi(x'')|x''|\Big) + O(t^2) \\ &= t\operatorname{Res}\big(\psi \circ Y\big) + O(t^2). \end{aligned}$$

•

Now for any $\psi \in G_-^\Phi$ the Birkhoff decomposition of ψ^t reads :

$$\psi^t = \psi * (\psi^t)_+.$$

Differentiating with respect to t at $t = 0$ we get according to Lemma IV.4.5 :

$$z\psi \circ Y = \psi * \operatorname{Res}(\psi \circ Y).$$

We deduce then :

$$\psi \circ Y = \psi * \frac{1}{z}\operatorname{Res}(\psi \circ Y),$$

which proves the equality $\widetilde{R}(\psi) = \frac{1}{z}\operatorname{Res}(\psi \circ Y)$. As a consequence correspondence $z\widetilde{R}$ sends G_-^Φ into $\mathfrak{g}^c$. Conversely let β in $\mathfrak{g}^c$. Consider $\psi = \widetilde{R}^{-1}(z^{-1}\beta)$. This element of G verifies by definition :

$$z\psi \circ Y = \psi * \beta.$$

Hence for any $x \in \operatorname{Ker}\varepsilon$ we have :

$$z\psi(x) = \frac{1}{|x|}\Big(\beta(x) + \sum_{(x)} \psi(x')\beta(x'')\Big).$$

As $\beta(x)$ is a constant (as a fuction of the complex variable z) it is easily seen by induction on $|x|$ that the right-hand side evaluated at z has a limit when z tends to infinity. Thus $\psi(x) \in \mathcal{A}_-$, and then :

$$\psi = \widetilde{R}^{-1}(\frac{1}{z}\beta) \in G_-^\Phi.$$

The proof of the variants of theorem IV.4.4 with G_1 and G_2 is then immediate thanks to corollary IV.1.3 and theorem IV.2.1.

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Remark : the "G_1" version of Theorem IV.4.4 recovers the result of A. Connes and D. Kreimer in [CK2 § 2] : when ψ is a character, then the $\mathbb{C}$-valued derivation $z\widetilde{R}(\psi)$ is denoted by β in *loc. cit.* Theorem IV.4.4 appears then as an interpretation of the result of Connes and Kreimer in other groups than the group of $\mathcal{A}$-valued characters.

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