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Renormalization in connected graded Hopf algebras: an introduction

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Abstract. We give an account of the Connes-Kreimer renormalization in the context of connected graded Hopf algebras. We first explain the Birkhoff decomposition of characters in the more general context of connected filtered Hopf algebras, then specializing down to the graded case in order to introduce the notions of locality, renormalization group and Connes-Kreimer's Beta function. The connection with Rota-Baxter and dendriform algebras will also be outlined. This introductory/survey article is based on joint work with Kurusch Ebrahimi-Fard, Li Guo and Frédéric Patras ([**19**], [**16**], [**21**], [**22**], [**23**]).

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1. Introduction

In any physical system in interaction, it is crucial to make a distinction between actually measured parameters and *bare* parameters, i.e. the value these parameters would take in absence of any interaction with the environment. *Renormalization* can be shortly defined as any device enabling us to pass from the bare parameters to the actually observed parameters, which will be called *renormalized*. We can have an idea of it by considering a spherical balloon moving in a fluid (water, the air, any gas...), as considered by G. Green as early as 1836 ([**26**], see also [**9**] and [**10**]): at very low speed (such that the friction is negligible), everything happens as if an extra mass $\frac{M}{2}$ had been added to the balloon mass m_0, where M is the mass of the fluid volume replaced by the balloon. The total force $F = mg$ acting on the balloon (with $m = m_0 + \frac{M}{2}$) splits into gravity $F_0 = m_0 g_0$ and Archimedes' force $-Mg_0$, where $g_0 \simeq 9.81\ m.s^{-2}$ is the gravity on the Earth's surface. Bare parameters are the mass m_0, the gravity force F_0 and acceleration g_0, whereas the renormalized parameters are:

$$m = m_0 + \frac{M}{2}, \qquad F = (1 - \frac{M}{m_0})F_0, \qquad g = \frac{m_0 - M}{m_0 + \frac{M}{2}} g_0. \tag{1}$$

Let us remark that the initial aceleration g decreases from g_0 to $-2g_0$ when the interaction, represented by the fluid mass M, increases from 0 to $+\infty$. An extra difficulty arises in quantum field theory, even in its perturbative approach: bare parameters are usually infinite! They are typically given by divergent integrals[1] like, for example :

$$\int_{\mathbb{R}^4} \frac{1}{1+\|p\|^2}\, dp. \qquad (2)$$

These infinite quantities illustrate the fact that switching off the interactions in quantum field theory is impossible except as a mental exercise[2]. One must then subtract another infinite quantity to the bare parameter in order to recover the (observed, hence finite) renormalised parameter. This process very often splits into two steps:

(1) *regularization*, which replaces the infinite bare parameter by a function of an auxiliary variable z, which tends to infinity when z tends to some z_0.

(2) Renormalization itself, of purely combinatorial nature. For *renormalizable* theories, it extracts a finite part from the function above when z tends to z_0.

There are a lot of ways to regularize: let us mention the cut-off regularization, which consists in considering integrals like (2) on a ball of radius z (with $z_0 = +\infty$), and dimensional regularization ([**28**], [**4**]), which "integrates on a space of complex dimension z", where z_0 is the spatial dimension d (for example $d = 4$ for the Minkowski space-time)[3]. In this case the function which appears is meromorphic in z with a pole in z_0.

Renormalization is given by the BPHZ algorithm (BPHZ for N. Bogoliubov, O. Parasiuk, K. Hepp and W. Zimmermann, [**3**], [**27**], [**51**]). The combinatoric objects here are *Feynman graphs*, classified according to their loop number L. The *Feynman rules* associate to each graph[4] some quantity to be regularized and renormalized. One has first to choose a *renormalization scheme*, i.e. the finite part for the "simplest" quantities, corresponding to one-loop graphs ($L = 1$). One can then renormalize the other quantities by induction on L. When regularized Feynman rules give meromorphic functions of one complex variable z (which is the case, for instance, for dimensional regularization), a most popular regularization scheme is the *minimal subtraction scheme*, which consists in taking the value at z_0 after removing the polar part. D. Kreimer first observed [**30**] that Feynman graphs are organized in a connected graded Hopf algebra. The BPHZ algorithm is then re-interpreted in terms of a Birkhoff decomposition for the regularized Feynman rules understood as a $\mathcal{A}$-valued character of the Hopf algebra, where $\mathcal{A}$ is some algebra of functions of the variable z (e.g. meromorphic functions of the complex variable z for dimensional regularization) [**8**].

2. A summary of Birkhoff–Connes–Kreimer factorization

We introduce the crucial property of connectedness for bialgebras. The main interest resides in the possibility to implement recursive procedures in connected bialgebras, the induction taking place with respect to a filtration (e.g. the coradical filtration) or a grading. An important example of these techniques is the recursive construction of the antipode, which then "comes for free", showing that any connected bialgebra is in fact a connected Hopf algebra. The recursive nature of Bogoliubov's formula in the BPHZ [**3, 27, 51**] approach to perturbative renormalization ultimately comes from the connectedness of the underlying Hopf algebra respectively the corresponding pro-nilpotency of the Lie algebra of infinitesimal characters.

For details on bialgebras and Hopf algebras we refer the reader to the standard references, e.g. [**48**]. The use of bialgebras and Hopf algebras in combinatorics can at least be traced back to the seminal work of Joni and Rota [**29**].

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[1] More precisely, the physical parameters are given by a *series* in the coupling constants (representing the interaction), each term of which is a divergent integral. We focus here on the renormalization of each of those terms, leaving aside the question of renormalizing the whole series.

[2] contrarily to the balloon considered above, for which the interaction can be brought very close to zero by letting it evolve in a quasi-perfect vacuum.

[3] This "space of dimension z" has been recently given a rigorous meaning in terms of type II factors and spectral triples ([**10**] § 19.2).

[4] together with an extra datum: its *external momenta*.

2.1. Connected graded bialgebras. Let k be a field with characteristic zero. A *graded Hopf algebra* on k is a graded k-vector space:

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

endowed with a product $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$, a coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, a unit $u : k \to \mathcal{H}$, a co-unit $\varepsilon : \mathcal{H} \to k$ and an antipode $S : \mathcal{H} \to \mathcal{H}$ fulfilling the usual axioms of a Hopf algebra [**48**], and such that:

$$\begin{aligned} m(\mathcal{H}_p \otimes \mathcal{H}_q) &\subset \mathcal{H}_{p+q}, \\ \Delta(\mathcal{H}_n) &\subset \bigoplus_{p+q=n} \mathcal{H}_p \otimes \mathcal{H}_q, \\ S(\mathcal{H}_n) &\subset \mathcal{H}_n. \end{aligned}$$

If we do not ask for the existence of an antipode S on $\mathcal{H}$ we get the definition of a *graded bialgebra*. In a graded bialgebra $\mathcal{H}$ we shall consider the increasing filtration:

$$\mathcal{H}^n = \bigoplus_{p=0}^{n} \mathcal{H}_p.$$

Suppose moreover that $\mathcal{H}$ is *connected*, i.e. $\mathcal{H}_0$ is one-dimensional. Then we have:

$$\operatorname{Ker} \varepsilon = \bigoplus_{n \geq 1} \mathcal{H}_n.$$

Proposition 1. *For any $x \in \mathcal{H}^n, n \geq 1$ we can write:*

$$\Delta x = x \otimes \mathbf{1} + \mathbf{1} \otimes x + \widetilde{\Delta} x, \qquad \widetilde{\Delta} x \in \bigoplus_{\substack{p+q=n, \\ p \neq 0, q \neq 0}} \mathcal{H}_p \otimes \mathcal{H}_q.$$

The map $\widetilde{\Delta}$ is coassociative on $\operatorname{Ker}\varepsilon$ *and $\widetilde{\Delta}_k := (I^{\otimes k-1} \otimes \widetilde{\Delta})(I^{\otimes k-2} \otimes \widetilde{\Delta}) \cdots \widetilde{\Delta}$ sends $\mathcal{H}^n$ into $(\mathcal{H}^{n-k})^{\otimes k+1}$.*

Proof. Thanks to connectedness we clearly can write:

$$\Delta x = a(x \otimes \mathbf{1}) + b(\mathbf{1} \otimes x) + \widetilde{\Delta} x$$

with $a, b \in k$ and $\widetilde{\Delta} x \in \operatorname{Ker}\varepsilon \otimes \operatorname{Ker}\varepsilon$. The co-unit property then tells us that, with $k \otimes \mathcal{H}$ and $\mathcal{H} \otimes k$ canonically identified with $\mathcal{H}$:

$$x = (\varepsilon \otimes I)(\Delta x) = bx, \qquad x = (I \otimes \varepsilon)(\Delta x) = ax,$$

hence $a = b = 1$. We shall use the following two variants of Sweedler's notation:

$$\Delta x = \sum_{(x)} x_1 \otimes x_2, \qquad \widetilde{\Delta} x = \sum_{(x)} x' \otimes x'',$$

the second being relevant only for $x \in \operatorname{Ker}\varepsilon$. If x is homogeneous of degree n we can suppose that the components x_1, x_2, x', and x'' in the expressions above are homogeneous as well, and we have then $|x_1| + |x_2| = n$ and $|x'| + |x''| = n$. We easily compute:

$$\begin{aligned} (\Delta \otimes I)\Delta(x) = {} & x \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes x \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes x \\ & + \sum_{(x)} x' \otimes x'' \otimes \mathbf{1} + x' \otimes \mathbf{1} \otimes x'' + \mathbf{1} \otimes x' \otimes x'' \\ & + (\widetilde{\Delta} \otimes I)\widetilde{\Delta}(x) \end{aligned}$$

and

$$\begin{aligned} (I \otimes \Delta)\Delta(x) = {} & x \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes x \otimes \mathbf{1} + \mathbf{1} \otimes 1 \otimes x \\ & + \sum_{(x)} x' \otimes x'' \otimes \mathbf{1} + x' \otimes \mathbf{1} \otimes x'' + \mathbf{1} \otimes x' \otimes x'' \\ & + (I \otimes \widetilde{\Delta})\widetilde{\Delta}(x), \end{aligned}$$

hence the co-associativity of $\widetilde{\Delta}$ comes from the one of Δ. Finally it is easily seen by induction on k that for any $x \in \mathcal{H}^n$ we can write:

$$\widetilde{\Delta}_k(x) = \sum_x x^{(1)} \otimes \cdots \otimes x^{(k+1)},$$

with $|x^{(j)}| \geq 1$. The grading imposes:

$$\sum_{j=1}^{k+1} |x^{(j)}| = n,$$

so the maximum possible for any degree $|x^{(j)}|$ is $n-k$. □

2.2. Connected filtered bialgebras. A *filtered Hopf algebra* on k is a k-vector space together with an increasing $\mathbb{Z}_+$-indexed filtration:

$$\mathcal{H}^0 \subset \mathcal{H}^1 \subset \cdots \subset \mathcal{H}^n \subset \cdots, \quad \bigcup_n \mathcal{H}^n = \mathcal{H}$$

endowed with a product $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$, a coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, a unit $u : k \to \mathcal{H}$, a co-unit $\varepsilon : \mathcal{H} \to k$ and an antipode $S : \mathcal{H} \to \mathcal{H}$ fulfilling the usual axioms of a Hopf algebra, and such that:

$$m(\mathcal{H}^p \otimes \mathcal{H}^q) \subset \mathcal{H}^{p+q}, \qquad \Delta(\mathcal{H}^n) \subset \sum_{p+q=n} \mathcal{H}^p \otimes \mathcal{H}^q, \qquad \text{and } S(\mathcal{H}^n) \subset \mathcal{H}^n.$$

If we do not ask for the existence of an antipode S on $\mathcal{H}$ we get the definition of a *filtered bialgebra*. For any $x \in \mathcal{H}$ we set:

$$|x| := \min\{n \in \mathbb{N},\ x \in \mathcal{H}^n\}.$$

Any graded bialgebra or Hopf algebra is obviously filtered by the canonical filtration associated to the grading:

$$\mathcal{H}^n := \bigoplus_{i=0}^n \mathcal{H}_i,$$

and in that case, if x is an homogeneous element, x is of degree n if and only if $|x| = n$. We say that the filtered bialgebra $\mathcal{H}$ is connected if $\mathcal{H}^0$ is one-dimensional. There is an analogue of Proposition 1 in the connected filtered case, the proof of which is very similar:

Proposition 2. *For any $x \in \mathcal{H}^n, n \geq 1$ we can write:*

$$\Delta x = x \otimes \mathbf{1} + \mathbf{1} \otimes x + \widetilde{\Delta} x, \qquad \widetilde{\Delta} x \in \sum_{\substack{p+q=n,\\ p\neq 0, q\neq 0}} \mathcal{H}^p \otimes \mathcal{H}^q.$$

The map $\widetilde{\Delta}$ is coassociative on $\operatorname{Ker}\varepsilon$ and $\widetilde{\Delta}_k = (I^{\otimes k-1} \otimes \widetilde{\Delta})(I^{\otimes k-2} \otimes \widetilde{\Delta}) \cdots \widetilde{\Delta}$ sends $\mathcal{H}^n$ into $(\mathcal{H}^{n-k})^{\otimes k+1}$.

The coradical filtration endows any pointed Hopf algebra $\mathcal{H}$ with a structure of filtered Hopf algebra (S. Montgomery, [**39**] Lemma 1.1). If $\mathcal{H}$ is moreover irreducible (i.e. if the image of k under the unit map u is the unique one-dimensional simple subcoalgebra of $\mathcal{H}$) this filtered Hopf algebra is moreover connected.

2.3. The convolution product. An important result is that any connected filtered bialgebra is indeed a filtered Hopf algebra, in the sense that the antipode comes for free. We give a proof of this fact as well as a recursive formula for the antipode with the help of the *convolution product*: let $\mathcal{H}$ be a (connected filtered) bialgebra, and let $\mathcal{A}$ be any k-algebra (which will be called the *target algebra*). The convolution product on the space $\mathcal{L}(\mathcal{H}, \mathcal{A})$ of linear maps from $\mathcal{H}$ to $\mathcal{A}$ is given by:

$$\begin{aligned}\varphi * \psi(x) &= m_{\mathcal{A}}(\varphi \otimes \psi)\Delta(x)\\ &= \sum_{(x)} \varphi(x_1)\psi(x_2).\end{aligned}$$

Proposition 3. *The map $e = u_{\mathcal{A}} \circ \varepsilon$, given by $e(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}$ and $e(x) = 0$ for any $x \in \operatorname{Ker}\varepsilon$, is a unit for the convolution product. Moreover the set $G(\mathcal{A}) := \{\varphi \in \mathcal{L}(\mathcal{H}, \mathcal{A}),\ \varphi(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}\}$ endowed with the convolution product is a group.*

PROOF. The first statement is straightforward. To prove the second let us consider the formal series:

$$\begin{aligned}\varphi^{*-1}(x) &= \big(e-(e-\varphi)\big)^{*-1}(x)\\ &= \sum_{m\ge 0}(e-\varphi)^{*m}(x).\end{aligned}$$

Using $(e-\varphi)(\mathbf{1})=0$ we have immediately $(e-\varphi)^{*m}(\mathbf{1})=0$, and for any $x\in\operatorname{Ker}\varepsilon$:

$$(e-\varphi)^{*n}(x)=m_{\mathcal{A},n-1}(\underbrace{\varphi\otimes\cdots\otimes\varphi}_{n\text{ times}})\widetilde{\Delta}_{n-1}(x).$$

When $x\in\mathcal{H}^p$ this expression vanishes then for $n\ge p+1$. The formal series ends up then with a finite number of terms for any x, which proves the result. □

COROLLARY 1. *Any connected filtered bialgebra $\mathcal{H}$ is a filtered Hopf algebra. The antipode is defined by:*

$$S(x)=\sum_{m\ge 0}(u\circ\varepsilon-I)^{*m}(x). \tag{3}$$

It is given by $S(\mathbf{1})=\mathbf{1}$ and recursively by any of the two formulas for $x\in\operatorname{Ker}\varepsilon$:

$$S(x)=-x-\sum_{(x)}S(x')x'' \quad\text{and}\quad S(x)=-x-\sum_{(x)}x'S(x'').$$

PROOF. The antipode, when it exists, is the inverse of the identity for the convolution product on $\mathcal{L}(\mathcal{H},\mathcal{H})$. One just needs then to apply Proposition 3 with $\mathcal{A}=\mathcal{H}$. The two recursive formulas follow directly from the two equalities:

$$m(S\otimes I)\Delta(x)=\ 0\ =m(I\otimes S)\Delta(x)$$

fulfilled by any $x\in\operatorname{Ker}\varepsilon$. □

Let $\mathfrak{g}(\mathcal{A})$ be the subspace of $\mathcal{L}(\mathcal{H},\mathcal{A})$ formed by the elements α such that $\alpha(\mathbf{1})=0$. It is clearly a subalgebra of $\mathcal{L}(\mathcal{H},\mathcal{A})$ for the convolution product. We have:

$$G(\mathcal{A})=e+\mathfrak{g}(\mathcal{A}). \tag{4}$$

From now on we shall suppose that the ground field k is of characteristic zero. For any $x\in\mathcal{H}^n$ the exponential:

$$\exp^*(\alpha)(x)=\sum_{k\ge 0}\frac{\alpha^{*k}(x)}{k!}$$

is a finite sum (ending up at $k=n$). It is a bijection from $\mathfrak{g}(\mathcal{A})$ onto $G(\mathcal{A})$. Its inverse is given by:

$$\log^*(e+\alpha)(x)=\sum_{k\ge 1}\frac{(-1)^{k-1}}{k}\alpha^{*k}(x).$$

This sum again ends up at $k=n$ for any $x\in\mathcal{H}^n$. Let us introduce a decreasing filtration on $\mathcal{L}=\mathcal{L}(\mathcal{H},\mathcal{A})$:

$$\mathcal{L}^n:=\{\alpha\in\mathcal{L},\ \alpha_{|\mathcal{H}^{n-1}}=0\}.$$

Clearly $\mathcal{L}^0=\mathcal{L}$ and $\mathcal{L}^1=\mathfrak{g}(\mathcal{A})$. We define the valuation $\operatorname{val}\varphi$ of an element φ of $\mathcal{L}$ as the biggest integer k such that φ is in $\mathcal{L}^k$. We shall consider in the sequel the ultrametric distance on $\mathcal{L}$ induced by the filtration:

$$d(\varphi,\psi)=2^{-\operatorname{val}(\varphi-\psi)}. \tag{5}$$

For any $\alpha,\beta\in\mathfrak{g}(\mathcal{A})$ let $[\alpha,\beta]=\alpha*\beta-\beta*\alpha$.

PROPOSITION 4. *We have the inclusion:*

$$\mathcal{L}^p*\mathcal{L}^q\subset\mathcal{L}^{p+q},$$

and moreover the metric space $\mathcal{L}$ endowed with the distance defined by (5) is complete.

PROOF. Take any $x \in \mathcal{H}^{p+q-1}$, and any $\alpha \in \mathcal{L}^p$ and $\beta \in \mathcal{L}^q$. We have

$$(\alpha * \beta)(x) = \sum_{(x)} \alpha(x_1)\beta(x_2).$$

Recall that we denote by $|x|$ the minimal n such that $x \in \mathcal{H}^n$. Since $|x_1| + |x_2| = |x| \le p + q - 1$, either $|x_1| \le p - 1$ or $|x_2| \le q - 1$, so the expression vanishes. Now if (ψ_n) is a Cauchy sequence in $\mathcal{L}$ it is immediate to see that this sequence is *locally stationary*, i.e. for any $x \in \mathcal{H}$ there exists $N(x) \in \mathbb{N}$ such that $\psi_n(x) = \psi_{N(x)}(x)$ for any $n \ge N(x)$. Then the limit of (ψ_n) exists and is clearly defined by:

$$\psi(x) = \psi_{N(x)}(x).$$

□

As a corollary the Lie algebra $\mathcal{L}^1 = \mathfrak{g}(\mathcal{A})$ is *pro-nilpotent*, in a sense that it is the projective limit of the Lie algebras $\mathfrak{g}(\mathcal{A})/\mathcal{L}^n$, which are nilpotent.

2.4. Characters and infinitesimal characters. Let $\mathcal{H}$ be a connected filtered Hopf algebra over k, and let $\mathcal{A}$ be a *commutative* k-algebra. We shall consider unital algebra morphisms from $\mathcal{H}$ to the target algebra $\mathcal{A}$, which we shall call slightly abusively *characters*. We recover of course the usual notion of character when the algebra $\mathcal{A}$ is the ground field k. The notion of character involves only the algebra structure of $\mathcal{H}$. On the other hand the convolution product on $\mathcal{L}(\mathcal{H}, \mathcal{A})$ involves only the *coalgebra* structure on $\mathcal{H}$. Let us consider now the full Hopf algebra structure on $\mathcal{H}$ and see what happens to characters with the convolution product:

PROPOSITION 5. *Let $\mathcal{H}$ be a connected filtered Hopf algebra over k, and let $\mathcal{A}$ be a commutative k-algebra. Then the characters from $\mathcal{H}$ to $\mathcal{A}$ form a group $G_1(\mathcal{A})$ under the convolution product, and for any $\varphi \in G_1(\mathcal{A})$ the inverse is given by:*

$$\varphi^{*-1} = \varphi \circ S.$$

We call *infinitesimal characters with values in the algebra $\mathcal{A}$* those elements α of $\mathcal{L}(\mathcal{H}, \mathcal{A})$ such that:

$$\alpha(xy) = e(x)\alpha(y) + \alpha(x)e(y).$$

PROPOSITION 6. *Let $G_1(\mathcal{A})$ (resp. $\mathfrak{g}_1(\mathcal{A})$) be the set of characters of $\mathcal{H}$ with values in $\mathcal{A}$ (resp. the set of infinitesimal characters of $\mathcal{H}$ with values in $\mathcal{A}$). Then $G_1(\mathcal{A})$ is a subgroup of $G(\mathcal{A})$, the exponential restricts to a bijection from $\mathfrak{g}_1(\mathcal{A})$ onto $G_1(\mathcal{A})$, and $\mathfrak{g}_1(\mathcal{A})$ is a Lie subalgebra of $\mathfrak{g}(\mathcal{A})$.*

PROOF. Part of these results are a reformulation of Proposition 5 and some points are straightforward. The only non-trivial point concerns $\mathfrak{g}_1(\mathcal{A})$ and $G_1(\mathcal{A})$. Take two infinitesimal characters α and β with values in $\mathcal{A}$ and compute:

$$\begin{aligned}(\alpha * \beta)(xy) &= \sum_{(x)(y)} \alpha(x_1y_1)\beta(x_2y_2) \\ &= \sum_{(x)(y)} \big(\alpha(x_1)e(y_1) + e(x_1)\alpha(y_1)\big).\big(\beta(x_2)e(y_2) + e(x_2)\alpha(y_2)\big) \\ &= (\alpha * \beta)(x)e(y) + \alpha(x)\beta(y) + \beta(x)\alpha(y) + e(x)(\alpha * \beta)(y).\end{aligned}$$

Using the commutativity of $\mathcal{A}$ we immediately get:

$$[\alpha, \beta](xy) = [\alpha, \beta](x)e(y) + e(x)[\alpha, \beta](y),$$

which shows that $\mathfrak{g}_1(\mathcal{A})$ is a Lie algebra. Now for $\alpha \in \mathfrak{g}_1(\mathcal{A})$ we have:

$$\alpha^{*n}(xy) = \sum_{k=0}^{n} \binom{n}{k} \alpha^{*k}(x)\alpha^{*(n-k)}(y),$$

as easily seen by induction on n. A straightforward computation then yields:

$$\exp^*(\alpha)(xy) = \exp^*(\alpha)(x)\exp^*(\alpha)(y).$$

□

2.5. Renormalization in connected filtered Hopf algebras. We describe in this section the renormalization à la Connes–Kreimer ([**30**], [**7**], [**8**]) in the abstract context of connected filtered Hopf algebras: the objects to be renormalised are characters with values in a commutative unital target algebra $\mathcal{A}$ endowed with a *renormalization scheme*, i.e. a splitting $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ into two subalgebras which play a symmetrical role, except that one has to decide in wich one to put the unit $\mathbf{1}$. An important example is given by the *minimal subtraction* (MS) *scheme* on the algebra $\mathcal{A}$ of meromorphic functions of one variable z, where $\mathcal{A}_+$ is the algebra of meromorphic functions which are holomorphic at $z = 0$, and where $\mathcal{A}_- = z^{-1}\mathbb{C}[z^{-1}]$ stands for the "polar parts". Any $\mathcal{A}$-valued character φ admits a unique *Birkhoff decomposition*

$$\varphi = \varphi_-^{*-1} * \varphi_+,$$

where φ_+ is an $\mathcal{A}_+$-valued character, and where $\varphi_-(\operatorname{Ker}\varepsilon) \subset \mathcal{A}_-$. In the MS scheme case described just above, the renormalised character is the scalar-valued character given by the evaluation of φ_+ at $z = 0$ (whereas the evaluation of φ at $z = 0$ does not necessarily make sense).

THEOREM 1. *Factorization of the group $G(\mathcal{A})$*

(1) Let $\mathcal{H}$ be a connected filtered Hopf algebra. Let $\mathcal{A}$ be a commutative unital algebra with a renormalization scheme such that $\mathbf{1} \in \complement\mathcal{A}_+$, and let $\pi : \mathcal{A} \to \mathcal{A}$ be the projection onto $\mathcal{A}_-$ parallel to $\mathcal{A}_+$. Let $G(\mathcal{A})$ be the group of those $\varphi \in \mathcal{L}(\mathcal{H}, \mathcal{A})$ such that $\varphi(\mathbf{1}) = \mathbf{1}_\mathcal{A}$ endowed with the convolution product. Any $\varphi \in G(\mathcal{A})$ admits a unique Birkhoff decomposition

$$\varphi = \varphi_-^{*-1} * \varphi_+, \tag{6}$$

where φ_- sends $\mathbf{1}$ to $\mathbf{1}_\mathcal{A}$ and $\operatorname{Ker}\varepsilon$ into $\mathcal{A}_-$, and where φ_+ sends $\mathcal{H}$ into $\mathcal{A}_+$. The maps φ_- and φ_+ are given on $\operatorname{Ker}\varepsilon$ by the following recursive formulas

$$\varphi_-(x) = -\pi\Big(\varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'')\Big)$$

$$\varphi_+(x) = (I - \pi)\Big(\varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'')\Big).$$

where I is the identity map.

(2) If φ is a character, the components φ_- and φ_+ occurring in the Birkhoff decomposition of φ are characters as well.

PROOF. The proof goes along the same lines as the proof of Theorem 4 of [**8**]: for the first assertion it is immediate from the definition of π that φ_- sends $\operatorname{Ker}\varepsilon$ into $\mathcal{A}_-$, and that φ_+ sends $\operatorname{Ker}\varepsilon$ into $\mathcal{A}_+$. It only remains to check equality $\varphi_+ = \varphi_- * \varphi$, which is an easy computation

$$\begin{aligned}\varphi_+(x) &= (I - \pi)\Big(\varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'')\Big).\\ &= \varphi(x) + \varphi_-(x) + \sum_{(x)} \varphi_-(x')\varphi(x'')\\ &= (\varphi_- * \varphi)(x).\end{aligned}$$

The proof of assertion 2) can be carried out exactly as in [**8**] and relies on the following *Rota–Baxter relation* in $\mathcal{A}$:

$$\pi(a)\pi(b) = \pi\big(\pi(a)b + a\pi(b)\big) - \pi(ab), \tag{7}$$

which is easily verified by decomposing a and b into their $\mathcal{A}_\pm$-parts. We will derive a more conceptual proof in Paragraph 2.7 below. □

REMARK 1. *Define the* Bogoliubov preparation map *as the map $B : G(\mathcal{A}) \to \mathcal{L}(\mathcal{H}, \mathcal{A})$ given by:*

$$B(\varphi) = \varphi_- * (\varphi - e), \tag{8}$$

such that for any $x \in \operatorname{Ker}\varepsilon$ we have:

$$B(\varphi)(x) = \varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'').$$

The components of φ in the Birkhoff decomposition read:

$$\varphi_- = e - \pi \circ B(\varphi), \qquad \varphi_+ = e + (I-\pi)\circ B(\varphi). \tag{9}$$

On $\operatorname{Ker}\varepsilon$ *they reduce to* $-\pi\circ B(\varphi)$, $(I-\pi)\circ B(\varphi)$, *respectively. Plugging equation (8) inside (9) and setting* $\alpha := e - \varphi$ *we get the following expression for* φ_-*:*

$$\begin{aligned}\varphi_- &= e + P(\varphi_- * \alpha) \\ &= e + P(\alpha) + P\big(P(\alpha)*\alpha\big) + \cdots + \underbrace{P\Big(P\big(\ldots P(}_{n \text{ times}}\alpha)*\alpha\big)\cdots *\alpha\Big) + \cdots\end{aligned} \tag{10}$$

and for φ_+ *we find:*

$$\varphi_+ = e - \tilde{P}(\varphi_- * \alpha) \tag{11}$$

$$\begin{aligned}&= e + \tilde{P}(\varphi_+ * \beta) \\ &= e + \tilde{P}(\beta) + \tilde{P}\big(\tilde{P}(\beta)*\beta\big) - \cdots + \underbrace{\tilde{P}\Big(\tilde{P}\big(\ldots \tilde{P}(}_{n \text{ times}}\beta)*\beta\big)\cdots *\beta\Big) + \cdots\end{aligned} \tag{12}$$

with $\beta := -\varphi^{-1}*\alpha = e - \varphi^{-1}$, *and where* $\tilde{P}$ *and* P *are projections on* $\mathcal{L}(\mathcal{H},\mathcal{A})$ *defined by* $\tilde{P}(\alpha) = (I-\pi)\circ\alpha$ *and* $P(\alpha) = \pi\circ\alpha$, *respectively.*

Remark 2. *Although subalgebras* $\mathcal{A}_+$ *and* $\mathcal{A}_-$ *can obviously be interchanged so that* $\mathbf{1}$ *always belongs to* $\mathcal{A}_+$, *it is better to keep the notation* $\mathcal{A}_-$ *for the counterterms and* $\mathcal{A}_+$ *for the renormalized quantities. Hence the unit* $\mathbf{1}$ *of the target algebra* $\mathcal{A}$ *can belong to* $\mathcal{A}_-$ *in some renormalization schemes: the most common example of this situation in physics is the* zero-momentum subtraction scheme *which can be briefly recast as follows (see* [**4**] *Paragraph 3.4.2): the target algebra* $\mathcal{A}$ *is the algebra of functions which are rational with respect to external* and *internal momenta (denoted by letters* p *and* k *respectively), well-defined at the origin, and polynomial with respect to an extra indeterminate* λ. *The product is given by:*

$$f.g(\lambda,k_1,k_2,p_1,p_2) := f(\lambda,k_1,p_1)g(\lambda,k_2,p_2). \tag{13}$$

The subalgebra $\mathcal{A}_+$ *is the subalgebra of functions* $f = \sum_{d=0}^{r}\lambda^d f_d$ *such that* f_d, *as a function of* external *momenta, vanishes at the origin at order* $\geq d+1$. *The subalgebra* $\mathcal{A}_-$ *is the subalgebra of functions* $f = \sum_{d=0}^{r}\lambda^d f_d$ *such that* f_d *is a polynomial in external momenta of degree* $\leq d$. *The character* φ *from the Hopf algebra of Feynman graphs (with external momenta) to* $\mathcal{A}$ *is given by* $\Gamma \mapsto \lambda^{d(\Gamma)}I_\Gamma$, *where* I_Γ *is the integrand for the Feynman rules, and* $d(\Gamma)$ *is the superficial degree of divergence of the graph. Hence the renormalization is performed before integrating with respect to internal momenta, and is based on subtracting the terms of degree* $\leq d(\Gamma)$ *w.r.t. external momenta in the taylor expansion of the integrand. This is the original setting in which the BPHZ algorithm has been first developed* [**3**], [**27**], [**51**]. *The on-shell scheme can be understood in a similar way, by considering Taylor expansions around the physical mass of the theory instead of arouand the origin (see* [**4**], *Paragraphs 3.1.3 and 3.3.1).*

2.6. The Baker–Campbell–Hausdorff recursion. Let $\mathcal{L}$ be any complete filtered Lie algebra. Thus $\mathcal{L}$ has a decreasing filtration $(\mathcal{L}_n)$ of Lie subalgebras such that $[\mathcal{L}_m, \mathcal{L}_n] \subseteq \mathcal{L}_{m+n}$ and $\mathcal{L} \cong \varprojlim \mathcal{L}/\mathcal{L}_n$ (i.e., $\mathcal{L}$ is complete with respect to the topology induced by the filtration). Let A be the completion of the enveloping algebra $\mathcal{U}(\mathcal{L})$ for the decreasing filtration naturally coming from that of $\mathcal{L}$. The functions:

$$\exp : A_1 \to 1 + A_1, \qquad \exp(a) = \sum_{n=0}^{\infty}\frac{a^n}{n!},$$

$$\log : 1 + A_1 \to A_1, \qquad \log(1+a) = -\sum_{n=1}^{\infty}\frac{(-a)^n}{n}$$

are well-defined and are the inverse of each other. The Baker–Campbell–Hausdorff (BCH) formula writes for any $x, y \in \mathcal{L}_1$ [**44, 50**]:

$$\exp(x)\exp(y) = \exp\big(C(x,y)\big) = \exp\big(x + y + \mathrm{BCH}(x,y)\big),$$

where $\mathrm{BCH}(x, y)$ is an element of $\mathcal{L}_2$ given by a Lie series the first few terms of which are:

$$\mathrm{BCH}(x, y) = \frac{1}{2}[x, y] + \frac{1}{12}[x, [x, y]] + \frac{1}{12}[y, [y, x]] - \frac{1}{24}[x, [y, [x, y]]] + \cdots$$

Now let $P : \mathcal{L} \to \mathcal{L}$ be any linear map preserving the filtration of $\mathcal{L}$. We define $\tilde{P}$ to be $\mathrm{Id}_{\mathcal{L}} - P$. For $a \in \mathcal{L}_1$, define $\chi(a) = \lim_{n\to\infty} \chi_{(n)}(a)$ where $\chi_{(n)}(a)$ is given by the *BCH*-recursion:

$$\begin{aligned} &\chi_{(0)}(a) := a, \\ &\chi_{(n+1)}(a) = a - \mathrm{BCH}\big(P(\chi_{(n)}(a)),\, (\mathrm{Id}_{\mathcal{L}} - P)(\chi_{(n)}(a))\big), \end{aligned} \tag{14}$$

and where the limit is taken with respect to the topology given by the filtration. Then the map $\chi : \mathcal{L}_1 \to \mathcal{L}_1$ satisfies:

$$\chi(a) = a - \mathrm{BCH}\big(P(\chi(a)),\, \tilde{P}(\chi(a))\big). \tag{15}$$

This map appeared in [**13**], [**12**], where more details can be found, see also [**?, 37**]. The following proposition ([**16**], [**?**]) gives further properties of the map χ.

Proposition 7. *For any linear map $P : \mathcal{L} \to \mathcal{L}$ preserving the filtration of $\mathcal{L}$ there exists a (usually non-linear) unique map $\chi : \mathcal{L}_1 \to \mathcal{L}_1$ such that $(\chi - \mathrm{Id}_{\mathcal{L}})(\mathcal{L}_i) \subset \mathcal{L}_{2i}$ for any $i \geq 1$, and such that, with $\tilde{P} := \mathrm{Id}_{\mathcal{L}} - P$ we have:*

$$\forall a \in \mathcal{L}_1, \quad a = C\Big(P\big(\chi(a)\big),\, \tilde{P}\big(\chi(a)\big)\Big). \tag{16}$$

This map is bijective, and its inverse is given by:

$$\chi^{-1}(a) = C\big(P(a),\, \tilde{P}(a)\big) = a + \mathrm{BCH}\big(P(a),\, \tilde{P}(a)\big). \tag{17}$$

Proof. Equation (16) can be rewritten as:

$$\chi(a) = F_a\big(\chi(a)\big),$$

with $F_a : \mathcal{L}_1 \to \mathcal{L}_1$ defined by:

$$F_a(b) = a - \mathrm{BCH}\big(P(b), \tilde{P}(b)\big).$$

This map F_a is a contraction with respect to the metric associated with the filtration: indeed if $b, \varepsilon \in \mathcal{L}_1$ with $\varepsilon \in \mathcal{L}_n$, we have:

$$F_a(b + \varepsilon) - F_a(b) = \mathrm{BCH}\big(P(b),\, \tilde{P}(b)\big) - \mathrm{BCH}\big(P(b + \varepsilon),\, \tilde{P}(b + \varepsilon)\big).$$

The right-hand side is a sum of iterated commutators in each of which ε does appear at least once. So it belongs to $\mathcal{L}_{n+1}$. So the sequence $F_a^n(b)$ converges in $\mathcal{L}_1$ to a unique fixed point $\chi(a)$ for F_a.

Let us remark that for any $a \in \mathcal{L}_i$, then, by a straightforward induction argument, $\chi_{(n)}(a) \in \mathcal{L}_i$ for any n, so $\chi(a) \in \mathcal{L}_i$ by taking the limit. Then the difference $\chi(a) - a = \mathrm{BCH}\big(P\big(\chi(a)\big),\, \tilde{P}\big(\chi(a)\big)\big)$ clearly belongs to $\mathcal{L}_{2i}$. Now consider the map $\psi : \mathcal{L}_1 \to \mathcal{L}_1$ defined by $\psi(a) = C\big(P(a),\, \tilde{P}(a)\big)$. It is clear from the definition of χ that $\psi \circ \chi = \mathrm{Id}_{\mathcal{L}_1}$. Then χ is injective and ψ is surjective. The injectivity of ψ will be an immediate consequence of the following lemma.

Lemma 1. *The map ψ increases the ultrametric distance given by the filtration.*

Proof. For any $x, y \in \mathcal{L}_1$ the distance $d(x, y)$ is given by 2^{-n} where $n = \sup\{k \in \mathbb{N},\, x - y \in \mathcal{L}_k\}$. We have then to prove that $\psi(x) - \psi(y) \notin \mathcal{L}_{n+1}$. But:

$$\begin{aligned} \psi(x) - \psi(y) &= x - y + \mathrm{BCH}\big(P(x),\, \tilde{P}(x)\big) - \mathrm{BCH}\big(P(y),\, \tilde{P}(y)\big) \\ &= x - y + \Big(\mathrm{BCH}\big(P(x),\, \tilde{P}(x)\big) - \mathrm{BCH}\big(P(x) - P(x - y),\, \tilde{P}(x) - \tilde{P}(x - y)\big)\Big). \end{aligned}$$

The rightmost term inside the large brackets clearly belongs to $\mathcal{L}_{n+1}$. As $x - y \notin \mathcal{L}_{n+1}$ by hypothesis, this proves the claim. □

The map ψ is then a bijection, so χ is also bijective, which proves Proposition 7. □

Corollary 2. *For any $a \in \mathcal{L}_1$ we have the following equality taking place in $1 + A_1 \subset A$:*

$$\exp(a) = \exp\big(P(\chi(a))\big) \exp\big(\tilde{P}(\chi(a))\big). \tag{18}$$

Putting (10) and (18) together we get for any $\alpha \in \mathcal{L}_1$ the following *non-commutative Spitzer identity*:

$$e + P(\alpha) + \cdots + \underbrace{P\Big(P(\ldots P(}_{n \text{ times}}\alpha)*\alpha)\cdots*\alpha\Big) + \cdots = \exp\Big[-P\Big(\chi\big(\log(e-\alpha)\big)\Big)\Big]. \tag{19}$$

This identity is valid for any filtration-preserving Rota–Baxter operator P in a complete filtered Lie algebra (see section 4). For a detailed treatment of these aspects, see [**13**], [**12**], [**16**], [**22**].

2.7. Application to perturbative renormalization. Suppose now that $\mathcal{L} = \mathcal{L}(\mathcal{H}, \mathcal{A})$ (with the setup and notations of paragraph 2.5), and that the operator P is now the projection defined by $P(a) = \pi \circ a$. It is clear that Corollary 2 applies in this setting and that the first factor on the right-hand side of (18) is an element of $G_1(\mathcal{A})$, the group of $\mathcal{A}$-valued characters of $\mathcal{H}$, which sends $\operatorname{Ker}\varepsilon$ into $\mathcal{A}_-$, and that the second factor is an element of G_1 which sends $\mathcal{H}$ into $\mathcal{A}_+$. Going back to Theorem 1 and using uniqueness of the decomposition (6) we see then that (18) in fact is the Birkhoff–Connes–Kreimer decomposition of the element $\exp^*(a)$ in G_1. Indeed, starting with the infinitesimal character a in the Lie algebra $\mathfrak{g}_1(\mathcal{A})$ equation (18) gives the Birkhoff–Connes–Kreimer decomposition of $\varphi = \exp^*(a)$ in the group $G_1(\mathcal{A})$ of $\mathcal{A}$-valued characters of $\mathcal{H}$, i.e.:

$$\varphi_- = \exp^*\big(-P(\chi(a))\big) \text{ and } \varphi_+ = \exp^*\big(\tilde{P}(\chi(a))\big) \text{ such that } \varphi = \varphi_-^{-1} * \varphi_+,$$

thus proving the second assertion in Theorem 1. Comparing Corollary 2 and Theorem 1 the reader may wonder upon the role played by the Rota–Baxter relation (7) for the projector P. In the following section we will show that it is this identity that allows to write the exponential $\varphi_- = \exp^*\big(-P(\chi(a))\big)$ as a recursion, that is, $\varphi_- = e + P(\varphi_- * \alpha)$, with $\alpha = e - \varphi$. Equivalently, this amounts to the fact that the group $G_1(\mathcal{A})$ factorizes into two subgroups $G_1^-(\mathcal{A})$ and $G_1^+(\mathcal{A})$, such that $\varphi_\pm \in G_1^\pm(\mathcal{A})$.

3. Locality, the renormalization group and the Beta function

3.1. The Dynkin operator. Any connected graded Hopf algebra $\mathcal{H}$ admits a natural biderivation Y defined by $Y(x) = nx$ for $x \in \mathcal{H}_n$. The map $\varphi \mapsto \varphi \circ Y$ is a derivation of $\big(\mathcal{L}(\mathcal{H}, \mathcal{A}), *\big)$. When the ground field is $k = \mathbb{R}$ or $\mathbb{C}$ the biderivation Y gives rise to the one-parameter subgroup of automorphisms of $\mathcal{H}$ given by $\theta_t(x) = e^{nt}x$ for $x \in \mathcal{H}_n$, and $\varphi \mapsto \varphi \circ \theta_t$ is an automorphism of $\big(\mathcal{L}(\mathcal{H}, \mathcal{A}), *\big)$ for any $t \in k$.

The *Dynkin operator* is defined as the endomorphism $D = S * Y$ of $\mathcal{H}$ (where S is the antipode). One can show that for any commutative unital algebra $\mathcal{A}$ the correspondence $\varphi \mapsto \varphi \circ D$ gives rise to a bijection Ξ from the group of characters $G_{\mathcal{A}}$ onto the Lie algebra of infinitesimal characters $\mathfrak{g}_{\mathcal{A}}$. When $k = \mathbb{R}$ ou $\mathbb{C}$ the inverse $\Xi^{-1} = \Gamma : \mathfrak{g}_{\mathcal{A}} \to G_{\mathcal{A}}$ is given by the following formula ([**37**] § 8.2) :

$$\Gamma(\alpha) = e + \sum_{n \geq 1} \int_{0 \leq v_n \leq \cdots \leq v_1 \leq +\infty} (\alpha \circ \theta_{-v_1}) * \cdots * (\alpha \circ \theta_{-v_n})\, dv_1 \cdots dv_n. \tag{20}$$

Applying this to an element x in $\mathcal{H}$ decomposed into its homogeneous components x_k (one can suppose $x_0 = 0$), and using the equality :

$$\int_{0 \leq v_l \leq \cdots \leq v_1 \leq +\infty} e^{-k_1 v_1} \cdots e^{-k_l v_l}\, dv_1 \cdots dv_l = \frac{1}{k_1(k_1+k_2)\cdots(k_1+\cdots+k_l)}, \tag{21}$$

one easily infers the explicit formula [**17**] :

$$\Gamma(\alpha) = e + \sum_{n \geq 1} \sum_{k_1, \ldots, k_l \in \mathbb{N}^*,\ k_1 + \cdots + k_l = n} \frac{\alpha_{k_1} * \cdots * \alpha_{k_l}}{k_1(k_1+k_2)\cdots(k_1+\cdots+k_l)}, \tag{22}$$

with $\alpha_k = \alpha \circ \pi_k$, and where for any $k \geq 0$ one denotes by π_k the projection of $\mathcal{H}$ onto the homogeneous component $\mathcal{H}_k$ of degree k. One can then easily verify the same formula on the field of rational numbers, and then on any field k of characteristic zero. The Dynkin operator was introduced in the general setting of commutative or cocommutative Hopf algebras by F. Patras and Chr. Reutenauer ([**42**], see also [**17**]). Several properties, such as the explicit formula (22) above, still make sense for any connected graded Hopf algebra.

3.2. The renormalization group and the Beta function. We suppose $k=\mathbb{R}$ or $k=\mathbb{C}$ here. We will consider the one-parameter group $\varphi \mapsto \varphi\circ\theta_{tz}$ of automorphisms of the algebra $\big(\mathcal{L}(\mathcal{H},\mathcal{A}),*\big)$ i.e.:

$$\varphi^t(x)(z) := e^{tz|x|}\varphi(x)(z). \tag{23}$$

Differentiating at $t=0$ we get:

$$\frac{d}{dt}_{|t=0}\varphi^t = z(\varphi\circ Y). \tag{24}$$

Let $G_{\mathcal{A}}$ be any of the two groups $G(\mathcal{A})$ or $G_1(\mathcal{A})$ (see Paragraph 2.4). We denote by $G_{\mathcal{A}}^{\text{loc}}$ the set of *local* elements of $G_{\mathcal{A}}$, i.e. those $\varphi\in G_{\mathcal{A}}$ such that the negative part of the Birkhoff decomposition of φ^t does not depend on t, namely:

$$G_{\mathcal{A}}^{\text{loc}} = \Big\{\varphi\in G_{\mathcal{A}} \Big|\ \frac{d}{dt}(\varphi^t)_- = 0\Big\}.$$

In particular the dimensional-regularised Feynman rules verify this property: in physical terms, the counter terms do not depend on the choice of the arbitrary mass-parameter μ ('tHooft's mass) one must introduce in dimensional regularisation in order to get dimensionless expressions, which is indeed a manifestation of locality (see [**9**]). We also denote by $G_{\mathcal{A}-}^{\text{loc}}$ the elements φ of $G_{\mathcal{A}}^{\text{loc}}$ such that $\varphi=\varphi_-^{*-1}$. Since composition on the right with Y is a derivation for the convolution product, the map Ξ of the preceding paragraph verifies a cocycle property:

$$\Xi(\varphi*\psi) = \Xi(\psi) + \psi^{*-1}*\Xi(\varphi)*\psi. \tag{25}$$

We summarise some key results of [**9**] in the following proposition:

Proposition 8. *(1) For any $\varphi\in G_{\mathcal{A}}$ there is a one-parameter family h_t in $G_{\mathcal{A}}$ such that $\varphi^t = \varphi*h_t$, and we have:*

$$\dot{h}_t := \frac{d}{dt}h_t = h_t*z\Xi(h_t) + z\Xi(\varphi)*h_t. \tag{26}$$

(2) $z\Xi$ restricts to a bijection from $G_{\mathcal{A}}^{\text{loc}}$ onto $\mathfrak{g}_{\mathcal{A}}\cap\mathcal{L}(\mathcal{H},\mathcal{A}_+)$. Moreover it is a bijection from $G_{\mathcal{A}}^{\text{loc}}{}_-$ onto those elements of $\mathfrak{g}_{\mathcal{A}}$ with values in the constants, i.e.:

$$\mathfrak{g}_{\mathcal{A}}^c = \mathfrak{g}_{\mathcal{A}}\cap\mathcal{L}(\mathcal{H},\mathbb{C}).$$

(3) For $\varphi\in G_{\mathcal{A}}^{\text{loc}}$, the constant term of h_t, defined by:

$$F_t(x) = \lim_{z\to 0} h_t(x)(z) \tag{27}$$

is a one-parameter subgroup of $G_{\mathcal{A}}\cap\mathcal{L}(\mathcal{H},\mathbb{C})$, the scalar-valued characters of $\mathcal{H}$.

Proof. For any $\varphi\in G_{\mathcal{A}}$ one can write:

$$\varphi^t = \varphi*h_t \tag{28}$$

with $h_t\in G_{\mathcal{A}}$. From (28), (24) and the definition of Ξ we immediately get:

$$\varphi*\dot{h}_t = \varphi*h_t*z\Xi(\varphi*h_t).$$

Equation (26) then follows from the cocycle property (25). This proves the first assertion. Now take any character $\varphi\in G_{\mathcal{A}}^{\text{loc}}$ with Birkhoff decomposition $\varphi=\varphi_-^{*-1}*\varphi_+$ and write the Birkhoff decomposition of φ^t:

$$\begin{aligned}\varphi^t &= (\varphi^t)_-^{*-1}*(\varphi^t)_+\\ &= (\varphi_-)^{*-1}*(\varphi^t)_+\\ &= (\varphi*\varphi_+^{*-1})*(\varphi^t)_+\\ &= \varphi*h_t,\end{aligned}$$

with h_t taking values in $\mathcal{A}_+$. Then $z\Xi(\varphi)$ also takes values in $\mathcal{A}_+$, as a consequence of equation (26) at $t=0$. Conversely, suppose that $z\Xi(\varphi)$ takes values in $\mathcal{A}_+$. We show that h_t also takes values in $\mathcal{A}_+$ for any t, which immediately implies that φ belongs to $G_{\mathcal{A}}^{\text{loc}}$.

For any $\gamma\in\mathfrak{g}_{\mathcal{A}}$, let us introduce the linear transformation U_γ of $\mathfrak{g}_{\mathcal{A}}$ defined by:

$$U_\gamma(\delta) := \gamma*\delta + z\delta\circ Y.$$

If γ belongs to $\mathfrak{g}_{\mathcal{A}}\cap\mathcal{L}(\mathcal{H},\mathcal{A}_+)$ then U_γ restricts to a linear transformation of $\mathfrak{g}_{\mathcal{A}}\cap\mathcal{L}(\mathcal{H},\mathcal{A}_+)$.

Lemma 2. *For any $\varphi \in G_\mathcal{A}$, $n \in \mathbb{N}$ we have:*

$$z^n \varphi \circ Y^n = \varphi * U^n_{z\Xi(\varphi)}(e).$$

Proof. Case $n = 0$ is obvious, $n = 1$ is just the definition of Ξ. We check thus by induction, using again the fact that composition on the right with Y is a derivation for the convolution product:

$$\begin{aligned} z^{n+1}\varphi \circ Y^{n+1} &= z(z^n\varphi \circ Y^n) \circ Y \\ &= z\big(\varphi * U^n_{z\Xi(\varphi)}(e)\big) \circ Y \\ &= z(\varphi \circ Y) * U^n_{z\Xi(\varphi)}(e) + z\varphi * \big(U^n_{z\Xi(\varphi)}(e) \circ Y\big) \\ &= \varphi * \big(z\Xi(\varphi) * U^n_{z\Xi(\varphi)}(e) + zU^n_{z\Xi(\varphi)}(e) \circ Y\big) \\ &= \varphi * U^{n+1}_{z\Xi(\varphi)}(e). \end{aligned}$$

□

Let us go back to the proof of Proposition 8. According to Lemma 2 we have for any t, at least formally:

$$\varphi^t = \varphi * \exp(tU_{z\Xi(\varphi)})(e). \tag{29}$$

We still have to fix the convergence of the exponential just above in the case when $z\Xi(\varphi)$ belongs to $\mathcal{L}(\mathcal{H}, \mathcal{A}_+)$. Let us consider the following decreasing bifiltration of $\mathcal{L}(\mathcal{H}, \mathcal{A}_+)$:

$$\mathcal{L}^{p,q}_+ = (z^q\mathcal{L}(\mathcal{H}, \mathcal{A}_+)) \cap \mathcal{L}^p,$$

where $\mathcal{L}^p$ is the set of those $\alpha \in \mathcal{L}(\mathcal{H}, \mathcal{A})$ such that $\alpha(x) = 0$ for any $x \in \mathcal{H}$ of degree $\leq p-1$. In particular $\mathcal{L}^1 = \mathfrak{g}_0$. Considering the associated filtration:

$$\mathcal{L}^n_+ = \sum_{p+q=n} \mathcal{L}^{p,q}_+,$$

we see that for any $\gamma \in \mathfrak{g}_0 \cap \mathcal{L}(\mathcal{H}, \mathcal{A}_+)$ the transformation U_γ increases the filtration by 1, i.e:

$$U_\gamma(\mathcal{L}^n_+) \subset \mathcal{L}^{n+1}_+.$$

The algebra $\mathcal{L}(\mathcal{H}, \mathcal{A}_+)$ is not complete with respect to the topology induced by this filtration, but the completion is $\mathcal{L}(\mathcal{H}, \widehat{\mathcal{A}_+})$, where $\widehat{\mathcal{A}_+} = \mathbb{C}[[z]]$ stands for the formal series. Hence the right-hand side of (29) is convergent in $\mathcal{L}(\mathcal{H}, \widehat{\mathcal{A}_+})$ with respect to this topology. Hence for any $\gamma \in \mathcal{L}(\mathcal{H}, \mathcal{A}_+)$ and for φ such that $z\Xi(\varphi) = \gamma$ we have $\varphi^t = \varphi * h_t$ with $h_t \in \mathcal{L}(\mathcal{H}, \widehat{\mathcal{A}_+})$ for any t. On the other hand we already know that h_t takes values in meromorphic functions for each t. So h_t belongs to $\mathcal{L}(\mathcal{H}, \mathcal{A}_+)$, which proves the first part of the second assertion. Equation (26) at $t = 0$ reads:

$$z\Xi(\varphi) = \dot{h}(0) = \frac{d}{dt}_{|t=0}(\varphi^t)_+. \tag{30}$$

For $\varphi \in G^{\mathrm{loc}}_{\mathcal{A}_-}$ we have, thanks to the property $\varphi(\operatorname{Ker}\varepsilon) \subset \mathcal{A}_-$:

$$\begin{aligned} h_t(x) = (\varphi^t)_+(x) &= (I-\pi)\Big(\varphi^t(x) + \sum_{(x)} \varphi^{*-1}(x')\varphi^t(x'')\Big) \\ &= t(I-\pi)\big(z|x|\varphi(x) + z\sum_{(x)} \varphi^{*-1}(x')\varphi(x'')|x''|\big) + O(t^2) \\ &= t\operatorname{Res}(\varphi \circ Y) + O(t^2), \end{aligned}$$

hence:

$$\dot{h}(0) = \operatorname{Res}\,(\varphi \circ Y). \tag{31}$$

From equations (24), (31) and the definition of Ξ we get:

$$z\Xi(\varphi) = \operatorname{Res}\,(\varphi \circ Y) \tag{32}$$

for any $\varphi \in G^{\mathrm{loc}}_{\mathcal{A}_-}$, hence $z\widetilde{R}(\varphi) \in \mathfrak{g}^c$. Conversely let β in $\mathfrak{g}^c$. Consider $\psi = \Xi^{-1}(z^{-1}\beta)$. This element of $G_\mathcal{A}$ verifies, thanks to the definition of Ξ:

$$z\psi \circ Y = \psi * \beta.$$

Hence for any $x \in \operatorname{Ker} \varepsilon$ we have:

$$z\psi(x) = \frac{1}{|x|}\Big(\beta(x) + \sum_{(x)} \psi(x')\beta(x'')\Big).$$

As $\beta(x)$ is a constant (as a function of the complex variable z) it is easily seen by induction on $|x|$ that the right-hand side evaluated at z has a limit when z tends to zero. Thus $\psi(x) \in \mathcal{A}_-$, and then:

$$\psi = \Xi^{-1}\Big(\frac{1}{z}\beta\Big) \in G^{\rm loc}_{\mathcal{A}_-},$$

which proves assertion (2).

Let us prove assertion (3): Equation $\varphi^t = \varphi * h_t$ together with $(\varphi^t)^s = \varphi^{t+s}$ yields:

$$h_{s+t} = h_s * (h_t)^s. \tag{33}$$

Taking values at $z = 0$ immediately yields the one-parameter group property:

$$F_{s+t} = F_s * F_t \tag{34}$$

thanks to the fact that the evaluation at $z = 0$ is an algebra morphism. □

We can now give a definition of the beta-function: for any $\varphi \in G^{\rm loc}_{\mathcal{A}}$, the beta-function of φ is the generator of the one-parameter group F_t defined by equation (27) in Proposition 8. It is the element of the dual $\mathcal{H}^*$ defined by:

$$\beta(\varphi) := \frac{d}{dt}_{|t=0} F_t(x) \tag{35}$$

for any $x \in \mathcal{H}$.

PROPOSITION 9. *For any $\varphi \in G^{\rm loc}_{\mathcal{A}}$ the beta-function of φ coincides with the one of the negative part φ_-^{*-1} in the Birkhoff decomposition. It is given by any of the three expressions:*

$$\begin{aligned}\beta(\varphi) &= \text{Res } \Xi(\varphi)\\ &= \text{Res } (\varphi_-^{*-1} \circ Y)\\ &= -\text{Res } (\varphi_- \circ Y).\end{aligned}$$

PROOF. The third equality will be derived from the second by taking residues on both sides of the equation:

$$0 = \Xi(e) = \Xi(\varphi_-) + \varphi_-^{*-1} * \Xi(\varphi_-^{*-1}) * \varphi_-,$$

which is a special instance of the cocycle formula (25). Suppose first $\varphi \in G^{\rm loc}_{\mathcal{A}_-}$, hence $\varphi_-^{*-1} = \varphi$. Then $z\Xi(\varphi)$ is a constant according to assertion 2 of Proposition 8. The proposition then follows from equation (31) evaluated at $z = 0$, and equation (32). Suppose now $\varphi \in G^{\rm loc}_{\mathcal{A}}$, and consider its Birkhoff decomposition. As both components belong to $G^{\rm loc}_{\mathcal{A}}$ we apply Proposition 8 to them. In particular we have:

$$\begin{aligned}\varphi^t &= \varphi * h_t,\\ (\varphi_-^{*-1})^t &= \varphi_-^{*-1} * v_t,\\ (\varphi_+)^t &= \varphi_+ * w_t,\end{aligned}$$

and equality $\varphi^t = (\varphi_-^{*-1})^t * (\varphi_+)^t$ yields:

$$h_t = (\varphi_+)^{*-1} * v_t * \varphi_+ * w_t. \tag{36}$$

We denote by F_t, V_t, W_t the one-parameter groups obtained from h_t, v_t, w_t, respectively, by letting the complex variable z go to zero. It is clear that $\varphi^+{}_{|z=0} = e$, and similarly that W_t is the constant one-parameter group reduced to the co-unit ε. Hence equation (36) at $z = 0$ reduces to:

$$F_t = V_t, \tag{37}$$

hence the first assertion. the cocycle equation (25) applied to the Birkhoff decomposition reads:

$$\Xi(\varphi) = \Xi(\varphi_+) + (\varphi_+)^{*-1} * \Xi(\varphi_-^{*-1}) * \varphi_+.$$

Taking residues of both sides yields:

$$\text{Res } \Xi(\varphi) = \text{Res } \Xi(\varphi_-^{*-1}),$$

which ends the proof. □

The one-parameter group $F_t = V_t$ above is the *renormalization group* of φ [**9**].

REMARK 3. *As it is possible to reconstruct φ_- from $\beta(\varphi)$ using the explicit formula* (22) *above for Ξ^{-1}, the term φ_- (i.e. the divergence structure of φ) is uniquely determined by its residue.*

REMARK 4. *It would be interesting to define renormalization group and beta function for other renormalization schemes and other target algebras $\mathcal{A}$. A first step in that direction can be found in* [**17**].

4. Rota–Baxter and dendriform algebras

We are interested in abstract versions of identities (10) and (19) fulfilled by the counterterm character φ_-. The general algebraic context is given by Rota–Baxter (associative) algebras of weight θ, which are themselves dendriform algebras. We first briefly recall the definition of Rota–Baxter (RB) algebra and its most important properties. For more details we refer the reader to the classical papers [**1, 2, 6, 45, 46**], as well as for instance to the references [**15, 16**].

4.1. From Rota-Baxter to dendriform. Let A be an associative not necessarily unital nor commutative algebra with $R \in \mathrm{End}(A)$. We call a tuple (A, R) a *Rota–Baxter algebra* of weight $\theta \in k$ if R satisfies the *Rota–Baxter relation*

$$R(x)R(y) = R\big(R(x)y + xR(y) + \theta xy\big). \tag{38}$$

Note that the operator P of paragraph 2.5 is an idempotent Rota–Baxter operator. Its weight is thus $\theta = -1$. Changing R to $R' := \mu R$, $\mu \in k$, gives rise to a RB algebra of weight $\theta' := \mu\theta$, so that a change in the θ parameter can always be achieved, at least as long as weight non-zero RB algebras are considered.

Let us recall some classical examples of RB algebras. First, consider the integration by parts rule for the Riemann integral map. Let $A := C(\mathbb{R})$ be the ring of real continuous functions with pointwise product. The indefinite Riemann integral can be seen as a linear map on A:

$$I : A \to A, \qquad I(f)(x) := \int_0^x f(t)\, dt. \tag{39}$$

Then, integration by parts for the Riemann integral can be written compactly as:

$$I(f)(x)I(g)(x) = I\big(I(f)g\big)(x) + I\big(fI(g)\big)(x), \tag{40}$$

dually to the classical Leibniz rule for derivations. Hence, we found our first example of a weight zero Rota–Baxter map. Correspondingly, on a suitable class of functions, we define the following Riemann summation operators:

$$R_\theta(f)(x) := \sum_{n=1}^{[x/\theta]-1} \theta f(n\theta) \qquad \text{and} \qquad R'_\theta(f)(x) := \sum_{n=1}^{[x/\theta]} \theta f(n\theta). \tag{41}$$

We observe readily that:

$$\begin{aligned}
\Big(\sum_{n=1}^{[x/\theta]} \theta f(n\theta)\Big)\Big(\sum_{m=1}^{[x/\theta]} \theta g(m\theta)\Big) &= \Big(\sum_{n>m=1}^{[x/\theta]} + \sum_{m>n=1}^{[x/\theta]} + \sum_{m=n=1}^{[x/\theta]}\Big)\theta^2 f(n\theta)g(m\theta)\\
&= \sum_{m=1}^{[x/\theta]} \theta^2\Big(\sum_{k=1}^{m} f\big(k\theta\big)\Big)g(m\theta) + \sum_{n=1}^{[x/\theta]} \theta^2\Big(\sum_{k=1}^{n} g\big(k\theta\big)\Big)f(n\theta) - \sum_{n=1}^{[x/\theta]} \theta^2 f(n\theta)g(n\theta)\\
&= R'_\theta\big(R'_\theta(f)g\big)(x) + R'_\theta\big(fR'_\theta(g)\big)(x) + \theta R'_\theta(fg)(x).
\end{aligned} \tag{42}$$

Similarly for the map R_θ except that the diagonal, omitted, must be added instead of subtracted. Hence, the Riemann summation maps R_θ and R'_θ satisfy the weight θ and the weight $-\theta$ Rota–Baxter relation, respectively.

PROPOSITION 10. *Let (A, R) be a Rota–Baxter algebra. The map $\tilde{R} = -\theta id_A - R$ is a Rota–Baxter map of weight θ on A. The images of R and $\tilde{R}$, $A_\mp \subseteq A$, respectively are subalgebras in A.*

The following Proposition follows directly from the Rota–Baxter relation:

PROPOSITION 11. *The vector space underlying A equipped with the product:*

$$x *_\theta y := R(x)y + xR(y) + \theta xy \tag{43}$$

is again a Rota–Baxter algebra of weight θ with Rota–Baxter map R.

We denote it by (A_θ, R) and call it *double Rota–Baxter algebra*. The Rota–Baxter map R becomes a (not necessarily unital even if A is unital) algebra homomorphism from the algebra A_θ to A. The result in Proposition 11 is best understood in the dendriform setting which we introduce now. A *dendriform algebra* [**32**] over a field k is a k-vector space A endowed with two bilinear operations $\prec$ and $\succ$ subject to the three axioms below:

$$(a \prec b) \prec c = a \prec (b * c), \quad (a \succ b) \prec c = a \succ (b \prec c), \quad a \succ (b \succ c) = (a * b) \succ c,$$

where $a * b$ stands for $a \prec b + a \succ b$. These axioms easily yield associativity for the law $*$. The bilinear operations $\rhd$ and $\lhd$ defined by:

$$a \rhd b := a \succ b - b \prec a, \qquad a \lhd b := a \prec b - b \succ a \tag{44}$$

are left pre-Lie and right pre-Lie, respectively, which means that we have:

$$(a \rhd b) \rhd c - a \rhd (b \rhd c) = (b \rhd a) \rhd c - b \rhd (a \rhd c), \tag{45}$$

$$(a \lhd b) \lhd c - a \lhd (b \lhd c) = (a \lhd c) \lhd b - a \lhd (c \lhd b). \tag{46}$$

The associative operation $*$ and the pre-Lie operations $\rhd$, $\lhd$ all define the same Lie bracket:

$$[a, b] := a * b - b * a = a \rhd b - b \rhd a = a \lhd b - b \lhd a. \tag{47}$$

PROPOSITION 12. [**11**] *Any Rota–Baxter algebra gives rise to two dendriform algebra structures given by:*

$$a \prec b := aR(b) + \theta ab = -a\tilde{R}(b), \qquad a \succ b := R(a)b, \tag{48}$$

$$a \prec' b := aR(b), \qquad a \succ' b := R(a)b + \theta ab = -\tilde{R}(a)b. \tag{49}$$

The associated associative product $*$ is given for both structures by $a * b = aR(b) + R(a)b + \theta ab$ and thus coincides with the double Rota–Baxter product (43).

REMARK 5. [**11**] *In fact, by splitting again the binary operation $\prec$ (or alternatively $\succ'$), any Rota–Baxter algebra is* tri-dendriform [**34**]*, in the sense that the Rota–Baxter structure yields three binary operations $<, \diamond$ and $>$ subject to axioms refining the axioms of dendriform algebras. The three binary operations are defined by $a < b = aR(b)$, $a \diamond b = \theta ab$ and $a > b = R(a)b$. Choosing to put the operation $\diamond$ to the $<$ or $>$ side gives rise to the two dendriform structures above.*

Let $\overline{A} = A \oplus k.\mathbf{1}$ be our dendriform algebra augmented by a unit $\mathbf{1}$:

$$a \prec \mathbf{1} := a =: \mathbf{1} \succ a \qquad \mathbf{1} \prec a := 0 =: a \succ \mathbf{1}, \tag{50}$$

implying $a * \mathbf{1} = \mathbf{1} * a = a$. Note that $\mathbf{1} * \mathbf{1} = \mathbf{1}$, but that $\mathbf{1} \prec \mathbf{1}$ and $\mathbf{1} \succ \mathbf{1}$ are not defined [**44**], [**4**]. We recursively define the following set of elements of $\overline{A}[[t]]$ for a fixed $x \in A$:

$$\begin{aligned} w_\prec^{(0)}(x) &= w_\succ^{(0)}(x) = \mathbf{1}, \\ w_\prec^{(n)}(x) &:= x \prec \big(w_\prec^{(n-1)}(x)\big), \\ w_\succ^{(n)}(x) &:= \big(w_\succ^{(n-1)}(x)\big) \succ x. \end{aligned}$$

We also define the following set of iterated left and right pre-Lie products (44). For $n > 0$, let $a_1, \ldots, a_n \in A$:

$$\ell^{(n)}(a_1, \ldots, a_n) := \Big(\cdots\big((a_1 \rhd a_2) \rhd a_3\big) \cdots \rhd a_{n-1}\Big) \rhd a_n \tag{51}$$

$$r^{(n)}(a_1, \ldots, a_n) := a_1 \lhd \Big(a_2 \lhd \big(a_3 \lhd \cdots (a_{n-1} \lhd a_n)\big) \cdots\Big). \tag{52}$$

For a fixed single element $a \in A$ we can write more compactly for $n > 0$:

$$\ell^{(n+1)}(a) = \big(\ell^{(n)}(a)\big) \rhd a \quad \text{and} \quad r^{(n+1)}(a) = a \lhd \big(r^{(n)}(a)\big) \tag{53}$$

and $\ell^{(1)}(a) := a =: r^{(1)}(a)$. We have the following theorem [**23, 17**].

THEOREM 2. *We have:*

$$w_{\succ}^{(n)}(a) = \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\ldots,i_k>0}} \frac{\ell^{(i_1)}(a) * \cdots * \ell^{(i_k)}(a)}{i_1(i_1+i_2)\cdots(i_1+\cdots+i_k)},$$

$$w_{\prec}^{(n)}(a) = \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\ldots,i_k>0}} \frac{r^{(i_k)}(a) * \cdots * r^{(i_1)}(a)}{i_1(i_1+i_2)\cdots(i_1+\cdots+i_k)}.$$

PROOF. The free unital dendriform algebra with one generator a is naturally endowed with a connected graded cocommutative Hopf algebra structure. It has been shown in [**23**] that the associated Dynkin operator D verifies:

$$D\big(w_{\succ}^{(n)}(a)\big) = \ell^{(n)}(a), \qquad D\big(w_{\prec}^{(n)}(a)\big) = r^{(n)}(a). \tag{54}$$

This result comes then from the formula (22). □

These identities nicely show how the dendriform pre-Lie and associative products fit together. This will become even more evident in the following: we are interested in the solutions X and Y in $\overline{A}[[t]]$ of the following two equations:

$$X = \mathbf{1} + ta \prec X, \qquad Y = \mathbf{1} - Y \succ ta. \tag{55}$$

Formal solutions to (55) are given by:

$$X = \sum_{n\geq 0} t^n w_{\prec}^{(n)}(a) \qquad \text{resp.} \qquad Y = \sum_{n\geq 0} (-t)^n w_{\succ}^{(n)}(a).$$

Let us introduce the following operators in A, where a is any element of A:

$$L_{\prec}[a](b) := a \prec b \quad L_{\succ}[a](b) := a \succ b \quad R_{\prec}[a](b) := b \prec a \quad R_{\succ}[a](b) := b \succ a$$
$$L_{\lhd}[a](b) := a \lhd b \quad L_{\rhd}[a](b) := a \rhd b \quad R_{\lhd}[a](b) := b \lhd a \quad R_{\rhd}[a](b) := b \rhd a.$$

We have recently obtained the following *pre-Lie Magnus expansion* [**19**]:

THEOREM 3. *Let $\Omega' := \Omega'(ta)$, $a \in A$, be the element of $tA[[t]]$ such that $X = \exp^*(\Omega')$ and $Y = \exp^*(-\Omega')$, where X and Y are the solutions of the two equations (55), respectively. This element obeys the following recursive equation:*

$$\Omega'(ta) = \frac{R_{\lhd}[\Omega']}{1-\exp(-R_{\lhd}[\Omega'])}(ta) = \sum_{m\geq 0} (-1)^m \frac{B_m}{m!} R_{\lhd}[\Omega']^m(ta), \tag{56}$$

or alternatively:

$$\Omega'(ta) = \frac{L_{\rhd}[\Omega']}{\exp(L_{\rhd}[\Omega'])-1}(ta) = \sum_{m\geq 0} \frac{B_m}{m!} L_{\rhd}[\Omega']^m(ta), \tag{57}$$

where the B_l's are the Bernoulli numbers.

Recall that the Bernoulli numbers are defined via the generating series:

$$\frac{z}{\exp(z)-1} = \sum_{m\geq 0} \frac{B_m}{m!} z^m = 1 - \frac{1}{2}z + \frac{1}{12}z^2 - \frac{1}{720}z^4 + \cdots,$$

and observe that $B_{2m+3} = 0$, $m \geq 0$.

4.2. Non-commutative Bohnenblust-Spitzer formulas. Let n be a positive integer, and let $\mathcal{OP}_n$ be the set of ordered partitions of $\{1,\ldots,n\}$, i.e. sequences $(\pi_1,\ldots,\pi_k)$ of disjoint subsets (*blocks*) whose union is $\{1,\ldots,n\}$. We denote by $\mathcal{OP}_n^k$ the set of ordered partitions of $\{1,\ldots,n\}$ with k blocks. Let us introduce for any $\pi \in \mathcal{OP}_n^k$ the coefficient:

$$\omega(\pi) = \frac{1}{|\pi_1|(|\pi_1|+|\pi_2|)\cdots(|\pi_1|+|\pi_2|+\cdots+|\pi_k|)}.$$

THEOREM 4. *Let $a_1, \ldots, a_n$ be elements in a dendriform algebra A. For any subset $E = \{j_1, \ldots, j_m\}$ of $\{1, \ldots, n\}$ let $\mathfrak{l}(E) \in A$ defined by:*

$$\mathfrak{l}(E) := \sum_{\sigma \in S_m} \mathfrak{l}^{(m)}(a_{j_{\sigma_1}}, \ldots, a_{j_{\sigma_m}}).$$

we have:

$$\sum_{\sigma \in S_n} \Big(...(a_{\sigma_1} \succ a_{\sigma_2}) \succ \cdots a_{\sigma_{n-1}} \Big) \succ a_{\sigma_n} = \sum_{k \geq 1} \sum_{\pi \in \mathcal{OP}_n^k} \omega(\pi) \mathfrak{l}(\pi_1) * \cdots * \mathfrak{l}(\pi_k).$$

See [**22**] where this identity is settled in the Rota–Baxter setting, see also [**18**]. The proof in the dendriform context is entirely similar. Another expression for the left-hand side can be obtained [**23**]: For any permutation $\sigma \in S_n$ we define the element $T_\sigma(a_1, \ldots, a_n)$ as follows: define first the subset $E_\sigma \subset \{1, \ldots, n\}$ by $k \in E_\sigma$ if and only if $\sigma_{k+1} > \sigma_j$ for any $j \leq k$. We write E_σ in the increasing order:

$$1 \leq k_1 < \cdots < k_p \leq n - 1.$$

Then we set:

$$T_\sigma(a_1, \ldots, a_n) := \ell^{(k_1)}(a_{\sigma_1}, \ldots, a_{\sigma_{k_1}}) * \cdots * \ell^{(n-k_p)}(a_{\sigma_{k_p+1}}, \ldots, a_{\sigma_n}) \tag{58}$$

There are $p+1$ packets separated by p stars in the right-hand side of the expression (58) above, and the parentheses are set to the left inside each packet. Following [**31**] it is convenient to write a permutation by putting a vertical bar after each element of E_σ. For example for the permutation $\sigma = (3261457)$ inside S_7 we have $E_\sigma = \{2, 6\}$. Putting the vertical bars:

$$\sigma = (32|6145|7)$$

we see that the corresponding element in A will then be:

$$\begin{aligned} T_\sigma(a_1, \ldots, a_7) &= \ell^{(2)}(a_3, a_2) * \ell^{(4)}(a_6, a_1, a_4, a_5) * \ell^{(1)}(a_7) \\ &= (a_3 \rhd a_2) * \Big(\big((a_6 \rhd a_1) \rhd a_4 \big) \rhd a_5 \Big) * a_7. \end{aligned}$$

THEOREM 5. *For any $a_1, \ldots, a_n$ in the dendriform algebra A the following identity holds:*

$$\sum_{\sigma \in S_n} \big(...(a_{\sigma_1} \succ a_{\sigma_2}) \succ \cdots \big) \succ a_{\sigma_n} = \sum_{\sigma \in S_n} T_\sigma(a_1, \ldots, a_n). \tag{59}$$

A q-analog of this identity has been proved by J-C. Novelli and J-Y. Thibon [**41**].

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