Poisson bracket, deformed bracket and gauge group actions in Kontsevich deformation quantization

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Abstract : We express the difference between Poisson bracket and deformed bracket for Kontsevich deformation quantization on any Poisson manifold by means of second derivative of the formality quasi-isomorphism. The counterpart on star products of the action of formal diffeomorphisms on Poisson formal bivector fields is also investigated.

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Introduction

The existence of a star product on any Poisson manifold (M, γ) is derived from the more general formality theorem of M. Kontsevich [K], which stipulates the existence of a L_∞-quasi-isomorphism $\mathcal{U}$ (cf. § II) from the differential graded Lie algebra of polyvector fields on any manifold M (with vanishing differential an Schouten bracket) into the differential graded Lie algebra of polydifferential operators on M (with Hochschild differential and Gerstenhaber bracket).

Given such a L_∞-quasi-isomorphism there is a canonical and explicit way to produce a star product $* = *_\gamma$ from Poisson bivector field, and more generally from any formal Poisson bivector field γ. We briefly recall this construction on § II. We call *formality star product* any star product on M obtained that way. Due to the fact that $\mathcal{U}$ is a L_∞-quasi-isomorphism, any star product is gauge-equivalent to a formality star product [K § 4.4], [AMM § A.2].

Let $f, g \in C^\infty(M)[[\hbar]]$, and let $H_f = [\gamma, f]$, $H_g = [\gamma, g]$ the associated hamiltonian (formal) vector fields. We compute here the second derivative at $\hbar\gamma$ of quasi-isomorphism

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$\mathcal{U}$ evaluated at (H_f, H_g), and more generally at (Y, H_g) where Y stands for any formal vector field on M. The main result is theorem III.3, consisting of three equations, which in turn imply the following formula, relating Poisson bracket, deformed bracket, tangent map Φ at $\hbar\gamma$ of $\mathcal{U}$ and second derivative Ψ at $\hbar\gamma$ of $\mathcal{U}$:

$$\Psi(H_f.H_g) = \frac{1}{\hbar}\Big(\Phi(\{f,g\}) - \frac{\Phi(f) * \Phi(g) - \Phi(g) * \Phi(f)}{\hbar}\Big).$$

There is another consequence of theorem III.3 in terms of gauge group action : namely we try to understand the star product $*_{g.\gamma}$ obtained from formal Poisson bivector field $g.\gamma$ where g is a formal diffeomorphism of the manifold M. Formal diffeomorphisms also act naturally on star-products via action on $C^\infty(M)$, but it is quite obvious that $*_{g.\gamma}$ is not the image of $*_\gamma$ by the action of g in that sense.

Gauge group G_1 of formal diffeomorphisms however acts on formality star products in a more subtle way : considering the set of all formality star-products as a formal pointed manifold (FSP), the action we seek amounts to a nontrivial embedding of G_1 into the group $\mathcal{G}$ of formal diffeomorphisms of (FSP).

This "big" group $\mathcal{G}$ contains in particular the gauge group G_2 of formal differential operators. We know (from [K § 4.5.2]) that star products $*_\gamma$ and $*_{g.\gamma}$ are gauge-equivalent : there exists an element $\tilde{g} = I + \hbar\tilde{g}_1 + \hbar^2\tilde{g}_2 + \cdots$ of G_2 such that $*_{g.\gamma} = \tilde{g}.*_\gamma$. But this $\tilde{g}$ depends on γ, so the embedding of G_1 into $\mathcal{G}$ alluded to above is not an embedding of G_1 into G_2.

I. Super-grouplike elements in cofree cocommutative graded coalgebras

Let $V = \bigoplus V^{(n)}$ be a $\mathbb{Z}$-graded vector space over a field k with zero characteristic, and let $\mathcal{C} = S(V)$ its symmetric algebra in the graded sense. We will throughout the paper denote by π the projection of $S(V)$ onto V. Defining a coproduct on elements of V by :

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

and extending it to an algebra morphism from $S(V)$ to the tensor product $S(V) \otimes S(V)$ (in the graded sense) we endow $S(V)$ with a structure of graded bialgebra. The set of primitive elements is precisely V, and the co-unity is given by the projection on constants.

Let $\mathfrak{m}$ be a (projective limit of) commutative finite dimensional nilpotent algebra(s). We will consider the (completed) tensor product $V\widehat{\otimes}\mathfrak{m}$ as a topologically free $\mathfrak{m}$-module and we will see the topologically free $\mathfrak{m}$-module $\mathcal{C}_\mathfrak{m} = S(V)\widehat{\otimes}\mathfrak{m} \oplus k.1$ as a topological bialgebra over $\mathfrak{m}$.

Let $\tau : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ be the signed flip defined by :

$$\tau(v \otimes w) = (-1)^{|v||w|} w \otimes v.$$

A nonzero element $v \in \mathcal{C}_\mathfrak{m}$ will be called *super-grouplike* if we have :

$$\Delta v = \frac{I + \tau}{2}(v \otimes v).$$

As an example, any even grouplike element is super-grouplike, as well as any $1 + x$ where $x \in V\widehat{\otimes}\mathfrak{m}$ and x odd.

Proposition I.1.
Any super-grouplike element in the coalgebra $\mathcal{C}_\mathfrak{m}$ is of the form :

$$g = e^{\cdot v} = 1 + v + \cdots + \frac{1}{n!}(v \cdots v)_{\text{n times}} + \cdots$$

with $v \in V \widehat{\otimes} \mathfrak{m}$, and conversely any such exponential is super-grouplike.

Proof. Consider the decomposition $g = g_+ + g_-$ of our supergrouplike element into its even and odd components. We have then :

$$\frac{1+\tau}{2}\Delta(g) = \frac{1+\tau}{2}\Delta(g_+ + g_-) = g_+ \otimes g_+ + g_- \otimes g_+ + g_+ \otimes g_-,$$

hence :

$$\Delta(g_+) = g_+ \otimes g_+$$
$$\Delta(g_-) = g_- \otimes g_+ + g_+ \otimes g_-$$

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So g_+ is nonzero, grouplike in the ordinary sense, and $g_- g_+^{-1}$ is an odd primitive element v_-. So g_+ writes :

$$g_+ = e^{\cdot v_+}$$

with $v_+ \in V$ even. To see this one can write $g_+ = 1 + \varepsilon$ with $\varepsilon \in S(V)\widehat{\otimes}\mathfrak{m}$ and directly check that its logarithm is primitive. We have then :

$$g = (1 + v_-)e^{\cdot v_+} = e^{\cdot v_-} e^{\cdot v_+} = e^{v_- + v_+}.$$

The converse is straightforward.

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Let $\widetilde{\mathcal{C}} = S^+(V) = \bigoplus_{n \geq 1} S^n(V)$ be the cofree cocommutative graded coalgebra without co-unity cogenerated by V. The coproduct is given by :

$$\widetilde{\Delta}(v) = \Delta(v) - 1 \otimes v - v \otimes 1.$$

Let $\widetilde{\mathcal{C}}_\mathfrak{m} = \widetilde{\mathcal{C}}\widehat{\otimes}\mathfrak{m}$. It is easy (and left to the reader) to derive a version of the result above in that setting :

Proposition I.2.
Any super-grouplike element in the coalgebra $\widetilde{\mathcal{C}}_\mathfrak{m}$ is of the form :

$$g = e^{\cdot v} - 1 = v + \cdots + \frac{1}{n!}(v \cdots v)_{\text{n times}} + \cdots$$

with $v \in V \widehat{\otimes} \mathfrak{m}$, and conversely any such element is super-grouplike in $\widetilde{\mathcal{C}}_\mathfrak{m}$.

Let us now compute the image of a super-grouplike element by a certain coderivation in the co-unityless setting :

Lemma I.3.
Let Q be a coderivation of coalgebra $\mathcal{C}$ with vanishing Taylor coefficients (cf. § II below) except Q_2, and extend it naturally by $\mathfrak{m}$-linearity to a coderivation of coalgebra $\widetilde{C}_{\mathfrak{m}}$. Let $X, Y \in V\widehat{\otimes}\mathfrak{m}$ with X even and Y odd. Then we have :

$$Q(e^{\cdot(X+Y)} - 1) = \frac{1}{2}Q_2(X.X)e^{\cdot(X+Y)} + Q_2(Y.X)e^{\cdot X}.$$

Proof. We have the following explicit formula for a coderivation in terms of its Taylor coefficients [AMM § III.2] : namely, for any n-uple of homogeneous elements in V,

$$Q(x_1 \cdots x_n) = \sum_{\substack{I \coprod J = \{1,\ldots,n\} \\ I,J \neq \emptyset}} \varepsilon_x(I,J)\big(Q_{|I|}(x_I)\big).x_J,$$

where x_I stands for $x_{i_1} \cdots x_{i_k}$ when $I = i_1, \ldots, i_k$, and $\varepsilon_x(I,J)$ is the *Quillen sign* associated with partition (I,J), i.e. the signature of the trace on odd x_i's of the shuffle associated with partition (I,J). We have then :

$$Q(e^{\cdot X} - 1) = \frac{1}{2}Q_2(X.X)e^{\cdot X}$$
$$Q(Y.e^{\cdot X}) = \frac{1}{2}Q_2(X.X)Y.e^{\cdot X} + Q_2(Y.X)e^{\cdot X}.$$

Summing up the two equalities above we get the result.

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II. Kontsevich's formality theorem

Let M be any real C^∞ manifold, let $\mathfrak{g}_1$ the differential graded Lie algebra of polyvector fields on M with zero differential and Schouten-Nijenhuis bracket, and let $\mathfrak{g}_2$ the differential graded Lie algebra of polydifferential operators on M with Gerstenhaber bracket and Hochschild diffenrential. The gradings are such that a degree n homogeneous element in $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$) is a $n+1$-vector field (resp. a $n+1$-differential operator).

We can consider the shifted spaces $\mathfrak{g}_1[1]$ and $\mathfrak{g}_2[1]$ as formal graded pointed manifolds : it means that for $i = 1, 2$ we have a coderivation Q^i of degree 1 on coalgebra without co-unity $S^+(\mathfrak{g}_i[1])$ satisfying the *master equation* :

$$[Q^i, Q^i] = 0,$$

where $\mathfrak{g}_i[1]$ is meant for space $\mathfrak{g}_i$ with grading shifted by 1 : a degree n homogeneous element in $\mathfrak{g}_1[1]$ (resp. $\mathfrak{g}_2[1]$) is now a $n+2$-vector field (resp. a $n+2$-differential operator).

Theorem II.1 (M. Kontsevich).
There exists a L_∞-quasi-isomorphism from formal graded pointed manifold $\mathfrak{g}_1[1]$ to formal graded pointed manifold $\mathfrak{g}_2[1]$: namely, there exists a coalgebra morphism :

$$\mathcal{U} : S^+(\mathfrak{g}_1[1]) \longrightarrow S^+(\mathfrak{g}_2[1])$$

such that :

$$\mathcal{U} \circ Q^1 = Q^2 \circ U,$$

and such that the restriction $\mathcal{U}_1$ of $\mathcal{U}$ to $\mathfrak{g}_1[1]$ is a quasi-isomorphism of cochain complexes.*

Let us briefly recall how formality theorem is related to deformation quantization : due to the universal property of cofree cocommutative coalgebras, coderivations Q^i and L_∞-quasi-isomorphism $\mathcal{U}$ are uniquely determined by their *Taylor coefficients* :

$$Q^i_k : S^k(\mathfrak{g}_i[1]) \longrightarrow \mathfrak{g}_i[2]$$
$$\mathcal{U}_k : S^k(\mathfrak{g}_1[1]) \longrightarrow \mathfrak{g}_2[1],$$

$k \geq 1, i = 1, 2$, obtained by composing Q^i and $\mathcal{U}$ on the right by the canonical projection : $S^+(\mathfrak{g}_i) \twoheadrightarrow \mathfrak{g}_i$ (resp.$S^+(\mathfrak{g}_2) \twoheadrightarrow \mathfrak{g}_2$). Let $\mathfrak{m} = \hbar\mathbb{R}[[\hbar]]$ the projective limit of finite-dimensional nilpotent commutative algebras $\mathfrak{m}_r = \hbar\mathbb{R}[[\hbar]]/\hbar^r\mathbb{R}[[\hbar]]$. Let $\hbar\gamma = \hbar(\gamma_0 + \hbar\gamma_1 + \hbar^2\gamma_2 + \cdots)$ be an infinitesimal formal Poisson bivector field, i.e. a solution in $\mathfrak{g}_1^{(1)}\widehat{\otimes}\mathfrak{m}$ of Maurer-Cartan equation :

$$(\hbar d\gamma +) - \frac{1}{2}[\hbar\gamma, \hbar\gamma] = 0,$$

which amounts exactly to the more geometrical equation :

$$Q^1(e^{\cdot\hbar\gamma} - 1) = 0,$$

where $e^{\cdot\hbar\gamma} - 1$ is grouplike (in the usual sense) in coalgebra $S^+(\mathfrak{g}_1[1])\widehat{\otimes}\mathfrak{m}$. Then $\mathcal{U}(e^{\cdot\hbar\gamma} - 1)$ is grouplike in coalgebra $S^+(\mathfrak{g}_2[1])\widehat{\otimes}\mathfrak{m}$. So we have :

$$\mathcal{U}(e^{\cdot\hbar\gamma} - 1) = e^{\cdot\hbar\tilde{\gamma}} - 1$$

with :

$$\tilde{\gamma} = \sum_{k\geq 1} \frac{\hbar^k}{k!}\mathcal{U}_k(\gamma^{\cdot k}).$$

Due to the fact that Q_2 vanishes at $e^{\cdot\hbar\tilde{\gamma}} - 1$ the element $\hbar\tilde{\gamma}$ verifies Maurer-Cartan equation in $\mathfrak{g}_2\widehat{\otimes}\mathfrak{m}$:

$$\hbar d\tilde{\gamma} - \frac{1}{2}[\hbar\tilde{\gamma}, \hbar\tilde{\gamma}] = 0.$$

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* According to [AMM] one should replace Schouten bracket with minus the bracket taken in the reverse order. This bracket coincides with Schouten bracket modulo a minus sign when tho odd elements are involved so it does not matter in what follows.

We denote by m the particular bidifferential operator :$f \otimes g \mapsto fg$, and we set $* = m + \hbar\tilde{\gamma}$. Maurer-Cartan equation for $\hbar\tilde{\gamma}$ is equivalent to :

$$[*, *] = 0,$$

i.e. $*$ is an associative product on $C^\infty(M)[[\hbar]]$. Starting from a Poisson bivector field $\gamma = P$ on M we construct then explicitly a star product from P and L_∞-morphism $\mathcal{U}$.

Remark : the expression $e^{\cdot\hbar\gamma} - 1$ is nothing but an algebraic way to express "the point $\hbar\gamma$ in the formal graded pointed manifold". One can be convinced by looking at the delta distribution at $\hbar\gamma$ and expressing it at 0 by means of Taylor expansion. the expression $e^{\cdot\hbar\gamma} - 1$ is then just the difference between the delta distribution at $\hbar\gamma$ qnd the delta distribution at 0. I would like to thank Siddhartha Sahi for having brought this nice geometrical interpretation to my attention.

III. On particular super-grouplike elements

Let γ be a formal Poisson 2-tensor on manifold M, and let $* = m + \hbar\tilde{\gamma}$ the star-product constructed from these data with Kontsevich's L_∞-quasi-isomorphism $\mathcal{U}$ following the formula recalled in previous paragraph. Let us consider for any $g \in C^\infty(M)[[\hbar]]$ and for any formal vector field Y the super-grouplike element $e^{\cdot\hbar(\gamma+Y+g)} - 1$. we will denote by H_g the hamiltonian formal vector field $[\gamma, g]$. As a straightforward application of lemma I.3 we get the following result :

Lemma III.1.
With the same notations as in § II we have :

$$Q^1(e^{\cdot\hbar(\gamma+Y+g)} - 1) = \hbar^2(H_g.e^{\cdot\hbar(\gamma+Y+g)} + [Y, \gamma + g]e^{\cdot\hbar(\gamma+g)}).$$

Any morphism of graded coalgebras, in particular L_∞-quasi-isomorphism $\mathcal{U}$, preserves super-grouplike elements. Due to this fact and to proposition I.1 we have then :

Proposition III.2.
There exists a formal differential operator $\Phi(Y)$ and a formal series $\Phi(g) \in C^\infty(M)[[\hbar]]$ such that :

$$\mathcal{U}(e^{\cdot\hbar(\gamma+Y+g)} - 1) = e^{\cdot * - m + \hbar\Phi(Y) + \hbar\Phi(g)} - 1,$$

with :

$$\Phi(Y) = \mathcal{U}_1(Y) + \hbar\mathcal{U}_2(Y.\gamma) + \frac{\hbar^2}{2}\mathcal{U}_3(Y.\gamma.\gamma) + \cdots$$
$$\Phi(g) = \mathcal{U}_1(g) + \hbar\mathcal{U}_2(g.\gamma) + \frac{\hbar^2}{2}\mathcal{U}_3(g.\gamma.\gamma) + \cdots$$

Correspondence Φ is precisely the tangent map at $\hbar\gamma$ of quasi-isomorphism $\mathcal{U}$ [K§ 8.1].

We will now compute both terms $\pi\mathcal{U}Q^1(e^{\cdot\hbar(\gamma+Y+g)}-1)$ and $\pi Q^2\mathcal{U}(e^{\cdot\hbar(\gamma+Y+g)}-1)$, and try to get some information from the fact that they coincide, by the very definition of a L_∞-morphism. Let us introduce for any pair (Y,Z) of polyvector fields the second derivative term :

$$\Psi(Y,Z)=\sum_{k\geq 0}\frac{\hbar^k}{k!}\mathcal{U}_{k+2}(Y.Z.\gamma^{\cdot k}).$$

This expression is symmetric (in the graded sense) in $Y,Z\in\mathfrak{g}_1[1]$ and is of degree $|Y|+|Z|-2$ in $\mathfrak{g}_2[1]$, so it belongs to $C^\infty(M)[[\hbar]]$ when $|Y|+|Z|=0$ in $\mathfrak{g}_1$. The expression is skew-symmetric in (Y,Z) when Y and Z are both vector fields, and symmetric when Y is a function and Z is a bivector field. We easily compute :

$$\begin{aligned}\mathcal{U}Q^1(e^{\cdot\hbar(\gamma+Y+g)}-1)&=\hbar^2\mathcal{U}(H_g.e^{\cdot\hbar(\gamma+Y+g)}+[Y,\gamma+g]e^{\cdot\hbar(\gamma+g)})\\&=\hbar^2\mathcal{U}\big((H_g+\hbar H_g.Y+[Y,g]+[Y,\gamma])e^{\cdot\hbar(\gamma+g)}\big).\end{aligned}$$

From degree considerations we easily derive :

$$\pi\mathcal{U}Q^1(e^{\cdot\hbar(\gamma+Y+g)}-1)=\hbar^2\pi\mathcal{U}\big((H_g+\hbar H_g.Y+[Y,g]+[Y,\gamma]+\hbar[Y,\gamma].g)e^{\cdot\hbar\gamma}\big),$$

so that we finally get :

$$\pi\mathcal{U}Q^1(e^{\cdot\hbar(\gamma+Y+g)}-1)=\hbar^2\big(\Phi(H_g)+\Phi([Y,g])+\hbar\Psi(H_g,Y)+\Phi([Y,\gamma])+\hbar\Psi([Y,\gamma].g)\big).$$

On the other hand we have to compute :

$$\begin{aligned}\pi Q^2\mathcal{U}(e^{\cdot\hbar(\gamma+Y+g)}-1)&=\pi Q^2(e^\delta-1)\\&=[\delta,m]+\frac{1}{2}Q_2^2(\delta.\delta),\end{aligned}$$

with $\delta=*-m+\hbar(\Phi(Y)+\Phi(g))$, according to proposition III.2. We have then :

$$\begin{aligned}\pi Q^2\mathcal{U}(e^{\cdot\hbar(\gamma+Y+g)}-1)&=[*-m+\hbar\Phi(Y)+\hbar\Phi(g),m]\\&\quad+\frac{1}{2}Q_2^2\Big(\big(*-m+\hbar\Phi(Y)+\hbar\Phi(g)\big).\big(*-m+\hbar\Phi(Y)+\hbar\Phi(g)\big)\Big)\\&=[*,m]+\hbar[\Phi(Y),m]+\hbar[\Phi(g),m]\\&\quad-\frac{1}{2}[*,m]-\frac{\hbar}{2}[*,\Phi(Y)]+\frac{\hbar}{2}[*,\Phi(g)]\\&\quad-\frac{1}{2}[m,*]+\frac{\hbar}{2}[m,\Phi(Y)]-\frac{\hbar}{2}[m,\Phi(g)]\\&\quad+\frac{\hbar}{2}[\Phi(Y),*]-\frac{\hbar}{2}[\Phi(Y),m]+\frac{\hbar^2}{2}[\Phi(Y),\Phi(g)]\\&\quad+\frac{\hbar}{2}[\Phi(g),*]-\frac{\hbar}{2}[\Phi(g),m]-\frac{\hbar^2}{2}[\Phi(Y),\Phi(g)]\\&=-\hbar[*,\Phi(Y)]+\hbar[*,\Phi(g)]+\hbar^2[\Phi(Y),\Phi(g)].\end{aligned}$$

In the computation above the relation between the second Taylor coefficient Q_2^2 and Gerstenhaber bracket is the following :

$$Q_2^2(x.y) = (-1)^{|x|(|y|-1)}[x,y].$$

This extra sign one must take care of comes from the identification of $S^k(\mathfrak{g}_2[1])$ with $\Lambda^k(\mathfrak{g}_2)[k]$ which goes as follows :

$$x_1 \cdots x_k \longmapsto \varepsilon . x_1 \wedge \cdots \wedge x_k,$$

where ε is the signature of the unshuffle storing even elements on the left and odd elements on the right [AMM § II.4], [K § 4.2].

We now identify the homogeneous components of degrees 1, 0 and -1 in the two expressions, so we get the following three equations :

Theorem III.3.
1) $[*, \Phi(Y)] = \hbar\Phi([\gamma, Y])$
2) $[*, \Phi(g)] = \hbar\Phi(H_g)$
3) $[\Phi(Y), \Phi(g)] = \Phi([Y,g]) + \hbar\big(\Psi(H_g, Y) - \Psi([\gamma, Y], g)\big)$.

Equation 1) implies the following : tangent map Φ sends derivations of $C^\infty(M)[[\hbar]]$ with commutative product leaving γ invariant to derivations of $C^\infty(M)[[\hbar]]$ with deformed product. From equation 2) we see that hamiltonian formal vector fields are sent to inner derivations of the deformed algebra : these two facts proceed more directly from the fact that the tangent map is a morphism of cochain complexes [K § 8.1]. Equation 3) rewrites as follows :

$$R(Y,g) = \frac{1}{\hbar}\big(\Phi([Y,g]) - [\Phi(Y), \Phi(g)]\big),$$

where skew-symmetric bilinear term :

$$R(Y,g) = \Psi([\gamma, g], Y) - \Psi([\gamma, Y], g)$$

can be seen as a kind of curvature. As a particular case we can take for Y the hamiltonian vector field H_f for any $f \in C^\infty(M)[[\hbar]]$. From equations 2) and 3) we immediately get :

Theorem III.4.
For any f, g in $C^\infty(M)$ the following formula holds true :

$$\Psi(H_f, H_g) = \frac{1}{\hbar}\Big(\Phi(\{f,g\}) - \frac{\Phi(f) * \Phi(g) - \Phi(g) * \Phi(f)}{\hbar}\Big).$$

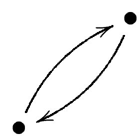
Remark 1 : Introducing the new star-product :

$$f\#g = \Phi^{-1}\big(\Phi(f) * \Phi(g)\big)$$

the formula of theorem III.4 can be rewritten as follows :

$$\Phi^{-1} \circ \Psi(H_f, H_g) = \frac{1}{\hbar}\Big(\{f,g\} - \frac{f\#g - g\#f}{\hbar}\Big).$$

Remark 2 : It is immediate to see from symmetry properties of star products that the right-hand side is $O(\hbar)$. In the flat case $M = \mathbb{R}^d$ one can use quasi-isomorphism of [K § 6], and recover this fact from the left-hand side by computing the constant term : it involves only the following graph :



the weight of which is zero [K § 7.3.1].

IV. Gauge transformations

Recall (from [K § 3.2]) that the *gauge group* associated with any differential graded Lie algebra $\mathfrak{g}$ is by definition the pro-nilpotent group G associated with pro-nilpotent Lie algebra $\mathfrak{g}^{(0)}\widehat{\otimes}\mathfrak{m}$. The gauge group acts on $\mathfrak{g}^{(1)}\widehat{\otimes}\mathfrak{m}$ by affine transformations, and the action of G is defined by exponentiation of the infinitesimal action of $\mathfrak{g}^{(0)}\widehat{\otimes}\mathfrak{m}$:

$$\alpha \otimes \gamma \in \mathfrak{g}^{(0)} \otimes \mathfrak{g}^{(1)} \mapsto \alpha.\gamma := d\alpha + [\alpha, \gamma].$$

It is easy to check that gauge group G acts on the set of solutions of Maurer-Cartan equation in $\mathfrak{g}^{(1)}\widehat{\otimes}\mathfrak{m}$.

We try in this last paragraph to give a geometrical meaning to equation 1) of theorem III.3. We let $\hbar\gamma$ run inside the set $(MC)_1$ of infinitesimal formal Poisson 2-tensors on manifold M, i.e. the subset of solutions of Maurer-Cartan equation in $\mathfrak{g}_1^{(1)}\widehat{\otimes}\mathfrak{m}$. We denote by $*_\gamma$ the star-product constructed from γ along the lines of § II. We will use the notation $\mathcal{U}'_{\hbar\gamma}$ instead of Φ to emphasize the dependance on γ. Equation 1) of theorem III.3 is then rewritten as follows :

$$[\mathcal{U}'_{\hbar\gamma}(Y), *_\gamma] = \hbar\,\mathcal{U}'_{\hbar\gamma}([Y,\gamma])$$

for any vector field Y on manifold M. Let $\mathcal{V}_Y^1$ be the vector field (of degree 0) on formal pointed manifold $\mathfrak{g}_1^{(1)}\widehat{\otimes}\mathfrak{m}$ equal to $[Y, \hbar\gamma]$ at point $\hbar\gamma$. It is the coderivation of coalgebra $S^+(\mathfrak{g}_1^{(1)})\widehat{\otimes}\mathfrak{m}$ such that :

$$\mathcal{V}_Y^1(e^{\cdot\hbar\gamma} - 1) = [Y, \hbar\gamma]e^{\cdot\hbar\gamma}.$$

It restricts to the submanifold $(MC)_1$.

The L_∞-quasi-isomorphism $\mathcal{U}$ is injective, by injectivity of first Taylor coefficient $\mathcal{U}_1$. Let $(MC)_2$ be the set of solutions of Maurer-Cartan equation in $\mathfrak{g}_2^{(1)}\widehat{\otimes}\mathfrak{m}$. Let $\mathcal{V}_Y^2$ be the vector field (of degree 0) on formal pointed manifold $\mathcal{U}(\mathfrak{g}_1^{(1)}\widehat{\otimes}\mathfrak{m})$ equal to $[\mathcal{U}'_{\hbar\gamma}(Y),*_\gamma]$ at point $\hbar\tilde{\gamma}$. It is the coderivation of the image coalgebra $\mathcal{U}\big(S^+(\mathfrak{g}_1^{(1)})\widehat{\otimes}\mathfrak{m}\big)$ such that :

$$\mathcal{V}_Y^2(e^{\cdot\hbar\tilde{\gamma}}-1)=[\mathcal{U}'_{\hbar\gamma}(Y),*_\gamma]e^{\cdot\hbar\tilde{\gamma}}.$$

Clearly vector field $\mathcal{V}_Y^2$ restricts to $\mathcal{U}\big((MC)_1\big)\subset(MC)_2$. Adding multiplication m we identify $(MC)_2$ with the set of all star products, and $\mathcal{U}\big((MC)_1\big)$ with the set (FSP) of formality star products.

Proposition IV.1.

$$\mathcal{U}\circ\mathcal{V}_Y^1=\mathcal{V}_Y^2\circ\mathcal{U}.$$

Proof. We have :

$$\mathcal{U}\circ\mathcal{V}_Y^1(e^{\cdot\hbar\gamma}-1)=\mathcal{U}'_\gamma([Y,\hbar\gamma])e^{\cdot\hbar\tilde{\gamma}}$$

and :

$$\mathcal{V}_Y^2\circ\mathcal{U}(e^{\cdot\hbar\gamma}-1)=[\mathcal{U}'_{\hbar\gamma}(Y),*_\gamma]e^{\cdot\hbar\tilde{\gamma}}.$$

The result follows then immediately from equation 1) of theorem III.3.

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It is clear that we have $\mathcal{V}_Y^2(\tau)=O(\hbar)$ for any $\tau\in\mathcal{U}\big((MC)_1\big)$. We have then :

Corollary IV.2.
Let $\gamma_Y=e^{\mathrm{ad}\,\hbar Y}\gamma$ be the transformation of formal Poisson 2-tensor γ under the formal diffeomorphism $e^{\hbar Y}$. It clearly belongs to $(MC)_1$ and the star-product $*_{\gamma_Y}$ constructed from γ_Y by means of L_∞-morphism $\mathcal{U}$ is the transformation of star-product $*_\gamma$ under the formal diffeomorphism $e^{\mathcal{V}_Y^2}$ of formal pointed manifold (FSP).

Of course differential operator $\mathcal{U}'_\gamma(Y)$ is not a vector field on M in general, so diffeomorphism $e^{\mathcal{V}_Y^2}$ of formal pointed manifold (FSP) does not come from a formal diffeomorphism of M. Correspondences $Y\mapsto\mathcal{V}_Y^1$ ans $Y\mapsto\mathcal{V}_Y^2$ are injective and respect brackets, i.e.:

$$\mathcal{V}_{[Y,Z]}^1=[\mathcal{V}_Y^1,\mathcal{V}_Z^1],$$

and, as a consequence of proposition IV.1 :

$$\mathcal{V}_{[Y,Z]}^2=[\mathcal{V}_Y^2,\mathcal{V}_Z^2].$$

By exponentiation we get then the following result :

Theorem IV.3.
Let G_1 and G_2 denote the gauge groups of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively, and let $\mathcal{G}$ be the group of formal diffeomorphisms of (FSP). The correspondence :

$$\iota : G_1 \longrightarrow \mathcal{G}$$
$$e^Y \longmapsto e^{\mathcal{V}_Y^2}$$

is an embedding of G_1 into the group $\mathcal{G}$ of formal diffeomorphisms of (FSP) such that :

$$*_{g.\gamma} = \iota(g). *_\gamma \ .$$

Remark : On one hand we have natural embeddings :

$$G_1 \subset G_2 \subset \mathcal{G},$$

but we must stress on the other hand that vector fields $\mathcal{V}_Y^1$ are linear whereas vector fields $\mathcal{V}_Y^2$ are not, and not even affine (this is due to the non-linearity of $\mathcal{U}$). The embedding of G_1 constructed above is then nontrivial, in the sense that the image of G_1 is not even contained in the second gauge group G_2.

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