# On quantization of quadratic Poisson structures

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**Abstract :** Any classical r-matrix on the Lie algebra of linear operators on a real vector space $V$ gives rise to a quadratic Poisson structure on $V$ which admits a deformation quantization stemming from the construction of V. Drinfel'd [Dr], [Gr]. We exhibit in this article an example of quadratic Poisson structure which does not arise this way.

## I. Introduction

let $V$ be a finite-dimensional real vector space. The linear action of the Lie group $Gl(V)$ on $V$ induces by differentiation a Lie algebra isomorphism from $\mathfrak{g} = \mathfrak{gl}(V)$ to the Lie algebra of linear vector fields on $V$. Given a basis $(e_1, \ldots, e_n)$ and then identifying $\mathfrak{gl}(V)$ with the Lie algebra of real $n \times n$ matrices the isomorphism is given by :

$$J(E_{ij}) = x_i \partial_j,$$

where $E_{ij}$ is the matrix with entries all vanishing except one equal to 1 on the $\mathrm{i}^{\mathrm{th}}$ line and $\mathrm{j}^{\mathrm{th}}$ column.

There is a unique way to extend the Lie bracket of $\mathfrak{g}$ to a graded Lie bracket, called the *Schouten bracket* on the shifted exterior algebra $\Lambda(\mathfrak{g})[1]$ in a way compatible with the exterior product. The shift means that elements of $\Lambda^k(\mathfrak{g})$ are of degree $k-1$, and then the Schouten bracket maps $\Lambda^k(\mathfrak{g}) \times \Lambda^l(\mathfrak{g})$ to $\Lambda^{k+l-1}(\mathfrak{g})$. The exterior algebra $\Lambda(\mathfrak{g})$ inherits then a structure of Gerstenhaber algebra (cf. for example [V], introduction).

The space $T^{\mathrm{poly}}(V)$ of polyvector fields on $M$ is also endowed with a Gerstenhaber algebra structure, with Schouten bracket extending Lie bracket of vector fields [V]. The subalgebra (for exterior product) generated by linear vector fields is a Gerstenhaber subalgebra $\widetilde{\Lambda}(V)$ of $T^{\mathrm{poly}}(V)$. The isomorphism $J$ extends to a surjective Gerstenhaber algebra morphism :

$$J^\bullet : \Lambda^\bullet(\mathfrak{g}) \longmapsto \widetilde{\Lambda}^\bullet(V).$$

Map $J^k$ has nontrivial kernel for $k \geq 2$ as long as $V$ has dimension $\geq 2$ : for example we have :

$$J^2(E_{ij} \wedge E_{kj}) = 0.$$

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A *classical r-matrix* on $\mathfrak{g}$ is by definition an element $r$ of $\mathfrak{g}\wedge\mathfrak{g}$ such that $[r,r]=0$. According to the discussion above the bivector field $J^2(r)$ defines then a quadratic Poisson structure on $V$. A natural question arises then : can one recover this way any quadratic Poisson structure on $V$? It was claimed true in [BR] but Z.H. Liu and P. Xu discovered that the authors' argument was not correct [LX]. They brought up a positive answer in the two-dimensional case ([LX] prop. 2.1) : namely the general quadratic Poisson structure $(ax_1^2+2bx_1x_2+cx_2^2)\partial_1\wedge\partial_2$ is equal to $J^2(r)$ with :

$$r=\begin{pmatrix} b & -a \\ c & -b \end{pmatrix}\wedge\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We give here a *negative* answer to this question in general : after a somewhat lengthy but elementary computation we show in § III that bivector field $(x_1^2+x_2x_3)\partial_2\wedge\partial_3$ on $\mathbb{R}^3$ is a counterexample to this conjecture : it is outside the image of the set of r-matrices by $J^2$.

We recall in § II the construction by V.G. Drinfel'd of a translation-invariant deformation quantization on any Lie group $G$ once given a classical r-matrix on the Lie algebra $\mathfrak{g}$ [Dr], [T]. The problem reduces to the case when $r$ is non-degenerate, and the deformation quantization is then obtained by suitable restriction and transportation of Baker-Campbell-Hausdorff deformation quantization ([Ka]) of the dual $\widetilde{\mathfrak{g}}^*$ of the central extension $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ defined by $r$. The construction works moreover for any Kontsevich-type star product [ABM] on $\widetilde{\mathfrak{g}}^*$. For $\mathfrak{g}=\mathfrak{gl}(V)$, such a star product on $\widetilde{\mathfrak{g}}^*$ gives almost immediately through this construction a deformation quantization of quadratic Poisson structure $J^2(r)$ on $V$.

Let us mention that the existence of a deformation quantization for any quadratic Poisson structure has been proved by Omori, Maeda and Yoshioka [OMY], a few years before the proof by Kontsevich of the existence of a deformation quantization for any Poisson structure on any manifold [K], [CFT]. Explicit computations for all quadratic Poisson structures in dimension 3 (then including our counterexample as well) have been performed by El Galiou and Tihami [ET], by a case-by-case method based on the classification of Dufour and Haraki [DH].

## II. Quantization of Poisson structures coming from r-matrices

Let $\mathfrak{g}$ be a Lie algebra, and let $r\in\mathfrak{g}\wedge\mathfrak{g}$ a classical r-matrix. It defines an antisymmetric operator :

$$\widetilde{r}:\mathfrak{g}^*\longrightarrow\mathfrak{g}.$$

Classical Yang-Baxter equation $[r,r]=0$ is equivalent to :

$$<\xi,\,[\widetilde{r}(\eta),\,\widetilde{r}(\zeta)]>+<\eta,\,[\widetilde{r}(\zeta),\,\widetilde{r}(\xi)]>+<\zeta,\,[\widetilde{r}(\xi),\,\widetilde{r}(\eta)]>=0 \qquad (*)$$

for any $\xi,\eta,\zeta\in\mathfrak{g}^*$. The r-matrix $r$ defines a left translation-invariant Poisson structure on any Lie group $G$ with Lie algebra $\mathfrak{g}$.

### II.1. A central extension

We can firstly suppose $r$ nondegenerate, i.e. that $\widetilde{r}$ is inversible with inverse $\widetilde{\omega}$, where $\omega$ belongs to $\mathfrak{g}^* \wedge \mathfrak{g}^*$. Classical Yang-Baxter equation is in this case equivalent to :

$$< \widetilde{\omega}X,\, [Y,Z] > + < \widetilde{\omega}Y,\, [Z,X] > + < \widetilde{\omega}Z,\, [X,Y] >= 0$$

for any $X, Y, Z \in \mathfrak{g}$, i.e is equivalent to the fact that $\omega$ is a 2-cocycle with values in the trivial representation. Let $\widetilde{\mathfrak{g}}$ the central extension of $\mathfrak{g}$ by this cocycle, defined by $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$ with bracket :

$$[X+\alpha, Y+\beta] = [X,Y] + < \omega, X \wedge Y > .$$

The cocycle condition on $\omega$ is equivalent to de Jacobi identity for this bracket. Let $X_0 = (0,1) \in \widetilde{\mathfrak{g}}$, and let $\mathcal{H}$ the hyperplane in $\widetilde{\mathfrak{g}}^*$ defined by :

$$\mathcal{H} = \{\xi \in \widetilde{\mathfrak{g}}^* / < \xi, X_0 >= 1\}.$$

It is the symplectic leaf through the point $\xi_0$ defined by $< \xi_0, \mathfrak{g} >= 0$ and $< \xi_0, X_0 >= 1$. It is then the coadjoint orbit $\mathrm{Ad}^*\, \widetilde{G}.\xi_0$ for any Lie group $\widetilde{G}$ with Lie algebra $\widetilde{\mathfrak{g}}$.

### II.2. Kontsevich star products (after [ABM])

The linear Poisson manifold $\widetilde{\mathfrak{g}}^*$ admits a whole bunch of equivalent $\mathrm{Ad}^*\, \widetilde{G}$-invariant deformation quantizations which can be built from the enveloping algebra $\mathcal{U}(\widetilde{\mathfrak{g}})$, for example the Baker-Campbell-Hausdorff quantization or the Kontsevich quantization [K], [ABM], [Ka], [Di]. The baker-Campbell-Hausdorff quantization is given by the following integral formula [ABM] :

$$(u \underset{BCH}{\#} v)(\xi) = \iint_{\mathfrak{g}\times\mathfrak{g}} \mathcal{F}^{-1}u(x)\mathcal{F}^{-1}v(y) e^{i<\xi,\, x \underset{\hbar}{\cdot} y>}\, dx\, dy,$$

where the inverse Fourier transform is given by :

$$\mathcal{F}^{-1}u(x) = (2\pi)^{-n} \int_{\mathfrak{g}^*} u(\eta) e^{i<x,\eta>} d\eta,$$

and $x \underset{\hbar}{\cdot} y$ stands for the Baker-Campbell-Hausdorff expansion :

$$x + y + \frac{\hbar}{2}[x,y] + \frac{\hbar^2}{12}([x,[x,y]] + [y,[y,x]]) + \cdots.$$

The Lebesgue measure $d\eta$ on $\mathfrak{g}^*$ is normalized so that it is the dual mesure of Lebesgue measure $dx$ on $\mathfrak{g}$. The quantizations we can consider here are the ones called "Kontsevich star products" in [ABM]. They are all equivalent to the BCH quantization. The equivalence is a formal series of differential operators with constant coefficients on $\mathfrak{g}^*$ precisely given by a formal series of $G$-invariant polynomials on $\mathfrak{g}$ of the following form :

$$F(x) = 1 + \sum_{k\geq 1} \hbar^{2k} \sum_{c\geq 1} \sum_{(s_1,\ldots,s_c)\in S_{2k}^c} a_{s_1,\ldots,s_c} \operatorname{Tr}(\operatorname{ad} x)^{s_1} \cdots \operatorname{Tr}(\operatorname{ad} x)^{s_c},$$

where $S^c_{2k}$ stands for those $(s_1, \ldots, s_c)$ in $\mathbb{N}^c$ such that $s_1+\cdots+s_c = 2k$, $s_1 \leq s_2 \leq \cdots \leq s_c$ and $s_j \neq 1$. The star product obtained this way admits the following integral form :

$$(u\#v)(\xi) = \iint_{\mathfrak{g}\times\mathfrak{g}} \mathcal{F}^{-1}u(x)\mathcal{F}^{-1}v(y)\frac{F(-ix)F(-iy)}{F\big(-i(x \underset{\hbar}{\cdot} y)\big)} e^{i<\xi,\, x \underset{\hbar}{\cdot} y>} \, dx\, dy.$$

### II.3. Quantization of left-invariant Poisson structures

It is easy to derive from the fact that $X_0$ is central that any of the deformation quantizations defined above does define by restriction a deformation quantization of $\mathcal{H}$. Let $G$ be the subgroup of $\widetilde{G}$ with Lie algebra $\mathfrak{g}$. We clearly have :

$$\mathrm{Ad}^*\, G.\xi_0 = \mathrm{Ad}^*\, \widetilde{G}.\xi_0 = \mathcal{H}.$$

It is moreover easy to check that the stabilizer of $\xi_0$ in $\widetilde{G}$ is the one-dimensional subgroup with Lie algebra generated by $X_0$. It is a simple consequence of the nondegeneracy of the alternate bilinear form $\omega$. The dimension of $G$ is the equal to the dimension of $\mathcal{H}$. The map :

$$\begin{aligned} \varphi : G' &\longrightarrow \mathcal{H} \\ g &\longmapsto \mathrm{Ad}^*\, g.\xi_0 \end{aligned}$$

is then a local $G$-equivariant diffeomorphism near the identity (with left translation on the left-hand side and coadjoint action on the right-hand side). We can then transport any deformation quantization of $\mathcal{H}$ and get a left translation-invariant deformation quantization of a neighbourhood of the identity in $G$. It extends by translation invariance to the whole group $G$, as well as to any Lie group $G'$ locally isomorphic to $G$.

The deformation quantization on $G$ can be written :

$$u\#v = \sum_{k\geq 0} \hbar^k C_k(u,v),$$

where the $C_k$'s are left-invariant bidifferential operators on $G$. There exists then an element $F = \sum \hbar^k F_k$ in $\big(\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})\big)[[\hbar]]$ such that :

$$u\#v(g) = F(u \otimes v)(g,g). \qquad (**)$$

Let us now fix a basis $x_1, \ldots, x_n$ of $\mathfrak{g}$, and consider elements of $\mathcal{U}(\mathfrak{g})$ as polynomials $F(x) = F(x_1, \ldots, x_n)$ of the $n$ noncommuting variables $x_1, \ldots, x_n$, which satisfy the relations :

$$x_i x_j - x_j x_i = [x_i, x_j] = \sum_k c^k_{ij} x_k.$$

Introducing a second identical set of noncommuting variables $y = (y_1, \ldots, y_n)$ commuting with th $x_j' s$ we can write any element $A \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ as $A(x,y)$. The element $F$ defined above can then be written $F(x,y)$ as a formal series with coefficients $F_k(x,y)$.

**Proposition II.1.**
*The formal series* $F = F(x,y) \in \big(\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})\big)[[\hbar]]$ *above verifies :*

*1)* $F_0(x,y) = 0.$

*2)* $F_1(x,y) = \frac{1}{2}\sum_{i,j} r_{ij} x_i y_j.$

*3)* $F_k(x,0) = F_k(0,y) = 0$ *for* $k \geq 1$.

*4)* $F(x+y,z)F(x,y) = F(x,y+z)F(y,z).$

*Conversely any* $F(x,y)$ *endowed with those 4 properties defines by formula* $(**)$ *a left translation deformation quantization of* $G$.

*Proof.* It is well-known : see for example [Dr], [T] : first condition comes from the fact that $C_0(u,v)$ is the ordinary product $uv$. Second property comes from the expression of left-invariant Poisson bracket on $G$ defined from the r-matrix, third property expresses the fact that $1\#u = u\#1 = u$, and last property is an expression of the associativity of star product $\#$. Let us elaborate a bit on that last point : any element $X_j$ of the basis corresponds to polynomial expression $G(x) = x_j$. The Leibniz rule :

$$X_j.(\varphi\psi) = (X_j.\varphi)\psi + \varphi.(X_j.\psi)$$

can be written as :

$$G(x) \circ m = m \circ G(x+y),$$

where $m : C^\infty(G \times G) \to C^\infty(G)$ stands for multiplication, here the restriction to the diagonal. The formula above extends to any polynomial expression $G$ representing any element of the enveloping algebra. We have then :

$$\begin{aligned}
(u\#v)\#w &= \big(m \circ F(u \otimes v)\big)\#w \\
&= m \circ F\Big(\big(m \circ F(u \otimes v)\big) \otimes w\Big) \\
&= m \circ F \circ (m \otimes I) \circ (F \otimes I)(u \otimes v \otimes w) \\
&= m \circ F(x,z) \circ (m \otimes I) \circ F(x,y)(u \otimes v \otimes w) \\
&= m \circ (m \otimes I) \circ F(x+y,z)F(x,y)(u \otimes v \otimes w).
\end{aligned}$$

Similarly we have :

$$u\#(v\#w) = m \circ (I \otimes m) \circ F(x,y+z)F(y,z)(u \otimes v \otimes w).$$

The associativity condition for product $\#$ is then equivalent to property 4) of the proposition.

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let us now look at the case when $r$ is degenerate. Then the image $\mathfrak{g}_0$ of $\widetilde{r}$ is a subspace strictly contained in $\mathfrak{g}$. By skew-symmetry $\mathfrak{g}_0$ is also the orthogonal of the kernel of $\widetilde{r}$, and classical Yang-Baxter equation $[r,r]=0$ ensures thanks to $(*)$ that $\mathfrak{g}_0$ is a Lie subalgebra of $\mathfrak{g}$. We get this way a nondegenerate $r_0 \in \mathfrak{g}_0 \wedge \mathfrak{g}_0$ such that $[r_0, r_0] = 0$. Applying the procedure above we get an $F = \sum \hbar^k F_k$ in $\big(\mathcal{U}(\mathfrak{g}_0) \otimes \mathcal{U}(\mathfrak{g}_0)\big)[[\hbar]]$ which can be seen as an element of $\big(\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})\big)[[\hbar]]$.

### II.4. A class of easily quantizable Poisson structures

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Let $F(x,y)$ a formal series in $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})[[\hbar]]$ satisfying properties 1-4 of proposition II.1 (for example that one constructed from an r-matrix along the lines above). Let $M$ be any differentiable manifold endowed with an action of $G$. The differentiation of this action induces a Lie algebra morphism from $\mathfrak{g}$ to the vector fields on $M$, which extends to an algebra morphism from $\mathcal{U}(\mathfrak{g})$ to the algebra of differential operators on $M$. Similarly it induces an algebra morphism from $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ to the algebra of differential operators on $M \times M$. The formal series of bidifferential operators defined by the formula :

$$* = m \circ F(x,y)$$

(where $m : C^\infty(M \times M) \to C^\infty(M)$ stands for ordinary multiplication of functions on $M$) defines then a star product on $M$, the associated Poisson bivector being defined by $F_1(x,y) - F_1(y,x)$. The proof of this fact goes the same way as that of proposition II.1. It is easily seen that if $F(x,y)$ comes from a classical r-matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$ then the Poisson structure on $M$ is $J^2(r)$ where $J^\bullet$ is the Gerstenhaber algebra morphism from $\Lambda(\mathfrak{g})$ to multivector fields on $M$ extending the action of $\mathfrak{g}$.

We will be interested in the sequel by the following particular situation : the manifold $M$ is a vector space $V$, the action of $G$ is linear, and there is a classical r-matrix $r$ on $\mathfrak{g}$. We can as in the introduction view $J$ as a Lie algebra morphism from $\mathfrak{g}$ to the space of linear vector fields on $V$, and extend $J$ to a morphism $J^\bullet$ of Gerstenhaber algebras from $\Lambda(\mathfrak{g})$ to $\widetilde{\Lambda}(V)$. In particular $J^2(r)$ defines a quadratic Poisson structure on $V$, and formula just above gives a quantization of this particular quadratic Poisson structure.

## III. Quadratic Poisson structures and r-matrices

### III.1. Some definitions

We keep the notations of the introduction. The Gerstenhaber algebra $\widetilde{\Lambda}(V)$ can be written as :

$$\widetilde{\Lambda}(V) = \bigoplus_{n \geq 0} \big(S^n(V) \otimes \Lambda^n(V)\big)[1] = \bigoplus_{n \geq 0} \widetilde{\Lambda}^n(V)[1].$$

A quadratic Poisson structure on $V$ can be defined as a bivector field $\Lambda$ in $\widetilde{\Lambda}^2(V)$ such that :

$$[\Lambda, \Lambda] = 0.$$

Let $\Lambda$ be an element of $\widetilde{\Lambda}^2(V)$, and let $r$ an element of $\Lambda^2(\mathfrak{g})$ such that $J^2(r) = \Lambda$. It is then obvious that $[\Lambda, \Lambda] = 0$ if and only if $J^3([r,r]) = 0$. If $n \geq 2$ then $J^2$ and $J^3$ have nontrivial kernels : Precisely we have :

$$\dim\ker J^2 = \frac{n^2(n^2-1)}{4} \qquad \text{and} \qquad \dim\ker J^3 = \frac{n^2(n^2-1)(5n^2-8)}{36}.$$

**III.2. A counterexample in dimension 3**

With the notations of § I, an element of $\mathfrak{g}\wedge\mathfrak{g}$ can be written as :

$$r = \sum_{i,j,k,l=1}^{n} r_{ik}^{jl} E_{ij}\wedge E_{kl}.$$

Whe shall need for further calculations the following result :

**Proposition III.1.**
*Let* $r = \sum_{i,j,k,l=1}^{n} r_{ik}^{jl} E_{ij}\wedge E_{kl}$ *be an element of* $\mathfrak{g}\wedge\mathfrak{g}$ *then* $[r,r]=0$ *if and only if for any* $i,j,k,l,m,p \in \{1,\dots,n\}$ *such that* $(i,j) < (k,l) < (m,p)$ *according to lexicographical order we have :*

$$\sum_{d=1}^{n} r_{ik}^{dl} r_{dm}^{jp} - r_{mk}^{dl} r_{di}^{pj} + r_{km}^{dp} r_{di}^{lj} - r_{im}^{dp} r_{dk}^{jl} + r_{mi}^{dj} r_{dk}^{pl} - r_{ki}^{dj} r_{dm}^{lp} = 0.$$

*Proof.* This proposition is a direct consequence of formula :

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}$$

and the following lemma :

**Lemma III.1.**
*Let* $\mathfrak{h}$ *be a finite-dimensional Lie algebra and let* $X_1,\dots,X_N$ *be a basis of* $\mathfrak{h}$.
*If* $r = \sum_{I,J=1}^{N} r^{IJ} X_I\wedge X_J$ *is an element of* $\mathfrak{h}\wedge\mathfrak{h}$ *(* $r^{IJ} = -r^{JI}$*) then*

$$[r,r] = 4 \sum_{I,J,K,L=1}^{N} r^{IJ} r^{KL} [X_I, X_K]\wedge X_J \wedge X_L.$$

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*Remark :* We can directly show proposition III.1 using relation (∗) of beginning of § II applied to elements of the dual basis of $X_1,\dots,X_n$.

**Proposition III.2.**
*The Poisson structure* $\Lambda$ *on* $\mathbb{R}^3$ *given by*

$$\Lambda = (x_1^2 + \alpha x_2 x_3)\partial_2 \wedge \partial_3$$

*with* $\alpha \neq 0$ *is not the image of a classical r-matrix by* $J^2$.

*Proof.* An element $r = \sum_{i,j,k,l=1}^{n} r_{ik}^{jl} E_{ij} \wedge E_{kl}$ of $\mathfrak{g} \wedge \mathfrak{g}$ is parametrized by 36 coefficients $r_{ik}^{jl}$. It has image $\Lambda$ if and only if the following 18 equations are satisfied :

$$\begin{array}{llll} r_{11}^{12} = r_{11}^{13} = r_{22}^{12} = r_{22}^{13} = r_{22}^{23} = r_{33}^{12} = r_{33}^{13} = r_{33}^{23} = 0, & & & r_{11}^{23} = 1 \\ r_{12}^{12} = r_{12}^{21}, & r_{12}^{13} = r_{12}^{31}, & r_{13}^{12} = r_{13}^{21} & \\ r_{13}^{13} = r_{13}^{31}, & r_{12}^{23} = r_{12}^{32}, & r_{13}^{23} = r_{13}^{32} & \\ r_{23}^{12} = r_{23}^{21}, & r_{23}^{13} = r_{23}^{31}, & r_{23}^{32} = r_{23}^{23} - \alpha & \end{array}$$

To lighten writing we rename the 18 remaining unknowns as follows :

$$\begin{array}{llllll} r_{12}^{11} = a, & r_{12}^{12} = b, & r_{12}^{13} = c, & r_{13}^{11} = d, & r_{13}^{12} = e, & r_{13}^{13} = f, \\ r_{12}^{22} = g, & r_{12}^{23} = h, & r_{13}^{22} = i, & r_{13}^{23} = j, & r_{12}^{33} = k, & r_{13}^{33} = l, \\ r_{23}^{11} = m, & r_{23}^{12} = n, & r_{23}^{13} = p, & r_{23}^{22} = q, & r_{23}^{23} = r, & r_{23}^{33} = s. \end{array}$$

The unknown $r$ must not be confused with the classical $r$-matrix on the whole. The context will not lead to any confusion. We have then :

$$\begin{array}{lll} r_{11}^{23} = 1, & & \\ r_{12}^{21} = b, & r_{13}^{21} = e, & r_{12}^{31} = c, \\ r_{12}^{32} = h, & r_{13}^{31} = f, & r_{13}^{32} = j, \\ r_{23}^{21} = n, & r_{23}^{31} = p, & r_{23}^{32} = r - \alpha, \end{array}$$

and other $r_{ik}^{jl}$ are equal to 0.

If $r$ is such an element the equation $[r, r] = 0$ develops according to Proposition III.2. into a system of 84 equations involving our 18 unknowns $a, b, \ldots, s$, given by the vanishing of the 84 coefficients of elements of the basis $E_{ij} \wedge E_{kl} \wedge E_{mn}$ of $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$. The 84 equations reduce to 66 thanks to the fact that we already have $[\Lambda, \Lambda] = 0$. But we shall only consider 20 of them, which will be sufficient for exhibiting the counterexample :

Let us order the $E_{ij}$'s lexicographically from first to 9th, rename them accordingly ($A_1 = E_{11}, A_2 = E_{12}, \ldots, A_9 = E_{33}$), and labelize by $(x, y, z)$ the equation obtained by the vanishing of the coefficient of $A_x \wedge A_y \wedge A_z$. We shall consider precisely the following equations :

$(1,2,5)$ $ci - eh + n = 0$ | $(1,2,9)$ $eh - ci + d = 0$

$(1,3,5)$ $cj - ek - a = 0$ | $(1,3,7)$ $ce + f^2 - ja - ld + m = 0$

$(1,3,9)$ $ek - cj + p = 0$ | $(1,4,5)$ $mh - nc = 0$

$(1,4,6)$ $mk - pc = 0$ | $(1,5,6)$ $\alpha c + nk - ph = 0$

$(1,7,8)$ $en - im = 0$ | $(1,7,9)$ $ep - jm = 0$

$(1,8,9)$ $\alpha e + ip - jn = 0$ | $(2,3,9)$ $-3f + r + ik - jh = 0$

$(2,4,5)$ $ag - b^2 + nh - qc = 0$ | $(2,5,6)$ $2\alpha h + bh - rh + qk - cg = 0$

$(2,8,9)$ $ej + ir + \alpha i - jq - fi = 0$ | $(3,8,9)$ $is + 2\alpha j + el - fj - jr = 0$

$(4,5,7)$ $pn - rm - bm + na = 0$ | $(5,6,9)$ $-nk - rs + hp + s(r-\alpha) = 0$

$(5,8,9)$ $nj - ip + \alpha q = 0$ | $(6,8,9)$ $ln - pj + qs - (r-\alpha)^2 = 0.$

Consider the following two sums :

$$\begin{aligned} (1,2,5) + (1,2,9) : &\quad n + d = 0 \\ (1,3,5) + (1,3,9) : &\quad p - a = 0. \end{aligned}$$

Hence $n = -d$ and $p = a$. We will discuss the four cases $a = d = 0$, $a = 0$ and $d \neq 0$, $a \neq 0$ and $d = 0$, $a \neq 0$ and $d \neq 0$.

**First case** : $a = d = 0$. Then looking successively at the following equations we get :

$$\begin{array}{lll} (5,8,9) \Longrightarrow q = 0 & (6,8,9) \Longrightarrow r = \alpha & (1,5,6) \Longrightarrow c = 0 \\ (1,8,9) \Longrightarrow e = 0 & (2,4,5) \Longrightarrow b = 0 & (2,5,6) \Longrightarrow h = 0 \\ (4,5,7) \Longrightarrow m = 0 & (5,6,9) \Longrightarrow s = 0 & (1,3,7) \Longrightarrow f = 0 \\ (3,8,9) \Longrightarrow j = 0 & (2,8,9) \Longrightarrow i = 0 & (2,3,9) \Longrightarrow \alpha = 0, \end{array}$$

hence a contradiction to the hypothesis $\alpha \neq 0$.

**Second case** : $a = 0$ and $d \neq 0$ (hence $n \neq 0$).

$$(1,4,6) \Longrightarrow mk = 0.$$

*First subcase* : $m = 0$. Then :

$$(1,4,5) \Longrightarrow c = 0 \qquad (1,7,8) \Longrightarrow e = 0 \qquad (1,2,5) \Longrightarrow n = 0,$$

hence a contradiction.

*Second subcase* : $m \neq 0$, hence $k = 0$.

$$(1,5,6) \Longrightarrow c = 0 \qquad (1,4,5) \Longrightarrow h = 0 \qquad (1,2,5) \Longrightarrow n = 0,$$

hence a contradiction again.

**Third case** : $a \neq 0$ and $d = 0$ (hence $p \neq 0$).

$$(1,4,5) \Longrightarrow mh = 0.$$

*First subcase* : $m = 0$. Then :

$$(1,4,6) \Longrightarrow c = 0 \qquad (1,7,9) \Longrightarrow e = 0 \qquad (1,3,5) \Longrightarrow a = 0,$$

hence a contradiction.

*Second subcase* : $m \neq 0$, hence $h = 0$.

$$(1,5,6) \Longrightarrow c = 0 \qquad (1,4,6) \Longrightarrow k = 0 \qquad (1,3,5) \Longrightarrow a = 0,$$

hence a contradiction again.

**Fourth case** : $a \neq 0$ and $d \neq 0$.

*First subcase* : $m = 0$.

$$(1,4,5) \Longrightarrow c = 0 \qquad (1,7,8) \Longrightarrow e = 0 \qquad (1,3,5) \Longrightarrow a = 0,$$

contradiction.

*Second subcase* : $m \neq 0$.

$$(1,4,6) \Longrightarrow k = \frac{a}{m}c \qquad (1,7,9) \Longrightarrow j = \frac{a}{m}e \qquad (1,3,5) \Longrightarrow a = 0,$$

contradiction.

This proves proposition III.2.

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### III.3. Cartan-type quadratic Poisson structures

Recall from [DH] that the curl of a Poisson stucture $\Lambda = \sum_{i,j} \Lambda^{ij}\partial_i \wedge \partial_j$ is defined by :

$$\operatorname{rot}\Lambda = \sum_{i,j} \partial_j \Lambda^{ij}.\partial_i.$$

It is a linear vector field (and hence can be viewed as an $n \times n$ matrix) when $\Lambda$ is quadratic. A quadratic Poisson structure is *of Cartan type* if it can be written for some choice of coordinates as :

$$\Lambda = \sum_{i,j=1}^{n} c_{ij} x_i x_j \partial_i \wedge \partial_j$$

with $c_{ji} = -c_{ij}$. J.P. Dufour and A. Haraki proved the following result :

**Theorem III.3** (Dufour - Haraki).
*Any quadratic Poisson structure the curl of which has eigenvalues $\lambda_i$ such that $\lambda_i + \lambda_j \neq \lambda_r + \lambda_s$ for any $(i,j,r,s)$ with $r \neq s$ and $\{i,j\} \neq \{r,s\}$ is of Cartan type.*

Such a Cartan-type Poisson structure is image by $J^{(2)}$, of a classical $r$-matrix, namely :

$$r = \sum_{i,j} c_{ij} E_{ii} \wedge E_{jj}.$$

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