Renormalised multiple zeta values which respect quasi-shuffle relations

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This note is a condensed report of a joint work with Sylvie PAYCHA [MP2], where details and proofs can be found. We propose a definition of multiple zeta values with integer arguments of any sign, which fulfill the quasi-shuffle relations. They are derived from a character Φ of some quasi-shuffle Hopf algebra with values in the meromorphic functions of a complex variable z, obtained from a suitable regularisation procedure: following Connes and Kreimer, the renormalised values are given by the positive part of the Birkhoff decomposition of Φ evaluated at $z = 0$. The multiple zeta values thus defined are moreover rational at nonpositive arguments.

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1 The quasi-shuffle Hopf algebra

We recall here the definition of the quasi-shuffle Hopf algebra and the shuffle Hopf algebra built on a commutative (not necessary unital) algebra $\mathcal{A}$, as well as the explicit construction by M. Hoffman ([H2]) of an isomorphism between these two Hopf algebras.

Definition 1 *Let $k, l, r \in \mathbb{N}$ with $k + l - r > 0$. A (k, l)-*quasi-shuffle *of type r is a surjective map π from $\{1, \ldots, k+l\}$ onto $\{1, \ldots, k+l-r\}$ such that $\pi(1) < \cdots < \pi(k)$*

and $\pi(k+1) < \cdots < \pi(k+l)$. *We shall denote by* $\text{mix sh}(k,l;r)$ *the set of* (k,l)*-quasi-shuffles of type* r. *The elements of* $\text{mix sh}(k,l;0)$ *are the ordinary* (k,l)*-shuffles. Quasi-shuffles are also called* mixable shuffles *or* stuffles. *We denote by* $\text{mix sh}(k,l)$ *the set of* (k,l)*-quasi-shuffles (of any type).*

Let $(\mathcal{A},\bullet)$ a commutative (not necessarily unital) algebra. Let Δ be the deconcatenation coproduct on $\mathcal{T}(\mathcal{A}) = \bigoplus_{k\geq 0}\mathcal{A}^{\otimes k}$, let $\star_\bullet$ the product on $\mathcal{T}(\mathcal{A})$ defined by:

$$(v_1\otimes\cdots\otimes v_k)\star_\bullet(v_{k+1}\otimes\cdots\otimes v_{k+l}) = \sum_{\pi\in\text{mix sh}(k,l)} w_1^\pi\otimes\cdots\otimes w_{k+l-r}^\pi,$$

with :

$$w_j^\pi = \prod_{i\in\{1,\ldots,k+l\},\pi(i)=j} v_i.$$

(the product above is the product $\bullet$ of $\mathcal{A}$, and contains only one or two terms).

Remark 1 *When the multiplication* $\bullet$ *of the algebra* $\mathcal{A}$ *is set to* 0, *the quasi-shuffle product* $\star_\bullet$ *reduces to the ordinary shuffle product* ш *.*

Theorem 1 (M. Hoffman, [H2] theorems 3.1 and 3.3)

- $\big(\mathcal{T}(\mathcal{A}),\star_\bullet,\Delta\big)$ *is a commutative connected filtered Hopf algebra.*
- *There is an isomorphism of Hopf algebras :*

$$\exp : \big(T(\mathcal{A}),ш,\Delta\big)\xrightarrow{\sim}\big(T(\mathcal{A}),\star_\bullet,\Delta\big).$$

M. Hoffman gives a detailed proof in a slightly more restricted context [H2], which can easily be adapted in full generality (see also [EG]). Hoffman's isomorphism is explicitly built as follows: let $\mathcal{P}(n)$ be the set of compositions of the integer n, i.e. the set of sequences $I = (i_1,\ldots,i_k)$ of positive integers such that $i_1+\cdots+i_k = n$. For any $u = v_1\otimes\cdots\otimes v_n\in T(\mathcal{A})$ and any composition $I = (i_1,\ldots,i_k)$ of n we set:

$$I[u] := (v_1\bullet\cdots\bullet v_{i_1})\otimes(v_{i_1+1}\bullet\cdots\bullet v_{i_1+i_2})\otimes\cdots\otimes(v_{i_1+\cdots+i_{k-1}+1}\bullet\cdots\bullet v_n).$$

We then further define:

$$\exp u = \sum_{I=(i_1,\cdots,i_k)\in\mathcal{P}(n)}\frac{1}{i_1!\cdots i_k!}I[u].$$

Moreover ([H2], Lemma 2.4), the inverse log of exp is given by :

$$\log u = \sum_{I=(i_1,\cdots,i_k)\in\mathcal{P}(n)}\frac{(-1)^{n-k}}{i_1\cdots i_k}I[u].$$

For example for $v_1,v_2,v_3\in\mathcal{A}$ we have :

$$\begin{aligned}
\exp v_1 = v_1 \quad &, \quad \log v_1 = v_1,\\
\exp(v_1\otimes v_2) = v_1\otimes v_2+\frac{1}{2}v_1\bullet v_2 \quad &, \quad \log(v_1\otimes v_2) = v_1\otimes v_2-\frac{1}{2}v_1\bullet v_2,\\
\exp(v_1\otimes v_2\otimes v_3) = v_1\otimes v_2\otimes v_3 \quad &+ \quad \frac{1}{2}(v_1\bullet v_2\otimes v_3+v_1\otimes v_2\bullet v_3)+\frac{1}{6}v_1\bullet v_2\bullet v_3,\\
\log(v_1\otimes v_2\otimes v_3) = v_1\otimes v_2\otimes v_3 \quad &- \quad \frac{1}{2}(v_1\bullet v_2\otimes v_3+v_1\otimes v_2\bullet v_3)+\frac{1}{3}v_1\bullet v_2\bullet v_3.
\end{aligned}$$

We now choose as commutative algebra $\mathcal{A}$ the space of sequences spanned by $\sigma_s : n \mapsto n^{-s},\ s \in \mathbb{C}$. This algebra can also be seen as the space of functions on $[1, +\infty[$ spanned by the functions $\sigma_s : t \mapsto t^{-s},\ s \in \mathbb{Z}$ (we identify by a small abuse of notations the function σ_s and its restriction to the positive integers). The product $\bullet$ on $\mathcal{A}$ will be chosen as the ordinary commutative product of functions, or its opposite.

2 Chen sums and multiple zeta functions

We introduce the *truncated Chen sums*, which are linear forms on $\mathcal{H} = T(\mathcal{A})$ defined by:

$$\sum_{\leq}^{N,\mathrm{Chen}} f_1 \otimes \cdots \otimes f_k := \sum_{1 \leq n_k \leq \cdots \leq n_1 \leq N} f_1(n_1) \cdots f_k(n_k) \tag{1}$$

in its "weak inequality version", and by:

$$\sum_{<}^{N,\mathrm{Chen}} f_1 \otimes \cdots \otimes f_k := \sum_{1 \leq n_k < \cdots < n_1 < N} f_1(n_1) \cdots f_k(n_k) \tag{2}$$

in its "strict inequality version"for any $f_1, \ldots f_k \in \mathcal{A}$. Provided it exists we call the limit when $N \to +\infty$ a *Chen sum*. The multiple zeta functions (in both "weak inequality" and "strict inequality" versions) are defined for Re $s_1 > 1$ and Re $s_i \geq 1$ for $i = 2, \cdots, k$ by the following convergent Chen sums:

$$\overline{\zeta}(s_1, \cdots, s_k) := \sum_{\leq}^{\mathrm{Chen}} \sigma_{s_1} \otimes \cdots \otimes \sigma_{s_k} = \sum_{1 \leq n_k \leq n_{k-1} \leq \cdots \leq n_1} n_k^{-s_k} \cdots n_1^{-s_1} \tag{3}$$

$$\zeta(s_1, \cdots, s_k) := \sum_{<}^{\mathrm{Chen}} \sigma_{s_1} \otimes \cdots \otimes \sigma_{s_k} = \sum_{1 \leq n_k < n_{k-1} < \cdots < n_1} n_k^{-s_k} \cdots n_1^{-s_1}. \tag{4}$$

We have the following relations between both versions (see [H]):

$$\begin{aligned} \overline{\zeta}(a_1, \ldots, a_k) &= \sum_{I=(i_1,\cdots,i_r) \in \mathcal{P}(k)} \zeta(b_1^I, \ldots b_r^I), \\ \zeta(a_1, \ldots, a_k) &= \sum_{I=(i_1,\cdots,i_r) \in \mathcal{P}(k)} (-1)^{k-r} \overline{\zeta}(b_1^I, \ldots b_r^I) \end{aligned} \tag{5}$$

with $b_s^I := a_{i_1+\cdots+i_{s-1}+1} + \cdots + a_{i_1+\cdots+i_s}$. The quasi-shuffle relations between multiple zeta functions can be presented as follows:

Theorem 2 *1. Consider the commutative algebra $(\mathcal{A}, \bullet)$ where $\bullet$ is the* opposite *of the ordinary product. For any $f_1, \cdots, f_k \in \mathcal{A}$, for any $N \in \mathbb{N}$, the truncated Chen sums fulfill the following relations:*

$$\sum_{\leq}^{N,\mathrm{Chen}} (f_1 \otimes \cdots \otimes f_k) \star_\bullet (g_1 \otimes \cdots \otimes g_l) = \left(\sum_{\leq}^{N,\mathrm{Chen}} f_1 \otimes \cdots \otimes f_k \right) \left(\sum_{\leq}^{N,\mathrm{Chen}} g_1 \otimes \cdots \otimes g_l \right). \tag{6}$$

2. *Whenever the Chen sums converge as $N \to \infty$, in the limit we have:*

$$\sum_{\leq}^{\text{Chen}} (f_1 \otimes \cdots \otimes f_k) \star_\bullet (g_1 \otimes \cdots \otimes g_l) = \left(\sum_{\leq}^{\text{Chen}} f_1 \otimes \cdots \otimes f_k \right) \left(\sum_{\leq}^{\text{Chen}} g_1 \otimes \cdots \otimes g_l \right). \tag{7}$$

3. *The same statements hold with the strict inequality version provided $\bullet$ is now the ordinary product without minus sign.*

Proof:

1. The domain:

$$P_{k,l} := \{n_1 > \cdots > n_k \geq 1\} \times \{n_{k+1} > \cdots > n_{k+l} \geq 1\} \subset (\mathbb{N} - \{0\})^{k+l}$$

is partitioned into:

$$P_{k,l} = \coprod_{\pi \in \text{mix sh}(k,l)} P_\pi,$$

where the domain P_π is defined by:

$$P_\pi = \{(n_1, \ldots, n_{k+l}) \,/\, n_{\pi_m} > n_{\pi_p} \text{ if } m > p \text{ and } \pi_m \neq \pi_p, \text{ and } n_m = n_p \text{ if } \pi_m = \pi_p\}.$$

As we must replace strict inequalities by weak ones, let us consider the "closures"

$$\overline{P_\pi} := \{(n_1, \ldots, n_{k+l}) \,/\, n_{\pi_m} \geq n_{\pi_p} \text{ if } m \geq p \text{ and } n_m = n_p \text{ if } \pi_m = \pi_p\}.$$

which then overlap. By the inclusion-exclusion principle we have:

$$\overline{P_{k,l}} = \coprod_{0 \leq r \leq \min(k,l)} (-1)^r \coprod_{\pi \in \text{mix sh}(k,l;r)} \overline{P_\pi}, \tag{8}$$

where we have set:

$$\overline{P_{k,l}} := \{n_1 \geq \cdots \geq n_k \geq 1\} \times \{n_{k+1} \geq \cdots \geq n_{k+l} \geq 1\} \subset (\mathbb{N} - \{0\})^{k+l}$$

Each term in equation (8) must be added if r is even, and removed if r is odd. Considering the summation of $f_1 \otimes \cdots \otimes f_{k+l}$ over each $\overline{P_\pi}$, this decomposition immediately yields the equality:

$$\left(\sum_{1 \leq n_k \leq \cdots \leq n_1 \leq N} f_1(n_1) \cdots f_i(n_k) \right) \left(\sum_{1 \leq n_{k+l} \leq \cdots \leq n_{k+1} \leq N} f_{k+1}(n_{k+1}) \cdots f_{k+l}(n_{k+l}) \right)$$

$$= \sum_{\leq}^{N,\text{Chen}} \sum_{\pi \in \text{mix sh}(k,l)} f^\pi, \tag{9}$$

where $f^\pi = f_1^\pi \otimes \cdots \otimes f_{k+l-r}^\pi$ is the tensor product of expressions defined by:

$$f_j^\pi = \bullet_{i \in \{1, \cdots, k+l\},\, \pi(i) = j} f_i.$$

The quasi-shuffle relations (6) are then a reformulation of equality (9) using the commutative algebra $(\mathcal{A}, \bullet)$.

2. Taking the limit as $N \to \infty$ provides the last statement of the theorem.

3. The proof is similar, using the domains P_π rather than the "closures" $\overline{P_\pi}$. As there are no overlappings the signs disappear in Formula (8).

□

3 Regularised Chen sums

We introduce on the algebra $\mathcal{A}$ the *Riesz regularisation procedure* which associates to any element of $\mathcal{A}$ an holomorphic family of elements of $\mathcal{A}$. It is defined by:

$$\mathcal{R}(\sigma_s)(z) = \sigma_{s+z}. \tag{10}$$

The regularisation $\mathcal{R}$ naturally extends to a morphism on the tensor algebra $T(\mathcal{A})$ via:

$$\widetilde{\mathcal{R}}(f_1 \otimes \cdots \otimes f_k) := \mathcal{R}(f_1) \otimes \cdots \otimes \mathcal{R}(f_k). \tag{11}$$

This regularisation is also compatible with the shuffle product. Twisting it with the Hoffman isomorphism exp gives us a regularisation procedure $\widetilde{\mathcal{R}}^* = \exp \circ \widetilde{\mathcal{R}} \circ \log$ compatible with the quasi-shuffle product: indeed, using the multiplicativity of $\widetilde{\mathcal{R}}$ w.r.to the shuffle product we have:

$$\begin{aligned}
\tilde{\mathcal{R}}^\star(\sigma \star_\bullet \tau) &= \exp \circ \tilde{\mathcal{R}} \circ \log(\sigma \star_\bullet \tau) \\
&= \exp \circ \tilde{\mathcal{R}}\,(\log(\sigma) ш \log(\tau)) \\
&= \exp\Big(\tilde{\mathcal{R}} \circ \log(\sigma) ш \tilde{\mathcal{R}} \circ \log(\tau)\Big) \\
&= \Big(\exp \circ \tilde{\mathcal{R}} \circ \log(\sigma)\Big) \star_\bullet \Big(\exp \circ \tilde{\mathcal{R}} \circ \log(\tau)\Big) \\
&= \tilde{\mathcal{R}}^\star(\sigma) \star_\bullet \tilde{\mathcal{R}}^\star(\tau).
\end{aligned}$$

Theorem 3 *For any $\sigma \in \mathcal{H} = T(\mathcal{A})$ the Chen sums $\sum_{\leq}^{\text{Chen}} \widetilde{\mathcal{R}}^*(\sigma)(z)$ and $\sum_{<}^{\text{Chen}} \widetilde{\mathcal{R}}^*(\sigma)(z)$ are holomorphic for $\operatorname{Re} z$ sufficiently large, and extend to the whole complex plane as meromorphic functions of the variable z.*

Sketch of proof: The proof is carried out by induction on the depth of the iterated sum, i.e. the degree of σ in the tensor algebra. The degree 1 case is nothing more than the analytic continuation of the Riemann Zeta function, which can be set up by means of the Euler-MacLaurin formula (see [MP2]). The result in any degree relies on similar properties for *Chen integrals* (see [MP2] and also [MP]), the bridge between Chen integrals and Chen sums being provided by an iterated use of the Euler-MacLaurin formula.

In fact this result is proven in [MP2] in the more general context of classical pseudo-differential symbols on $\mathbb{R}^d$, allowing higher-dimensional analogues of multiple zeta functions. We briefly report on this point in Section 7. □

The space of meromorphic functions will be denoted by $\mathcal{M}(\mathbb{C})$. We refer to the meromorphic functions obtained by means of Theorem 3 as *regularised Chen sums*, which we denote by $\sum_{\leq}^{\text{Chen}} \widetilde{\mathcal{R}}^*(\sigma)(z)$ and $\sum_{<}^{\text{Chen}} \widetilde{\mathcal{R}}^*(\sigma)(z)$ respectively. By virtue of the compatibility of the regularisation procedure $\widetilde{\mathcal{R}}^*$ with the quasi-shuffle product, and by analytic continuation of the quasi-shuffle relations, we have:

Proposition 1 *: The map*

$$\begin{array}{rcl}\mathcal{A} & \to & \mathcal{M}(\mathbb{C}) \\ \sigma & \mapsto & \sum \mathcal{R}(\sigma)(z) := \underset{N\to+\infty}{\mathrm{fp}} \sum_{k=1}^{N} \sigma_z(k)\end{array}$$

extends to multiplicative maps

$$\begin{array}{rcl}\Psi^{\mathcal{R}}(\textit{ resp. } \Psi'^{\mathcal{R}}) : (\mathcal{T}(\mathcal{A}), \star_\bullet) & \to & \mathcal{M}(\mathbb{C}) \\ \sigma & \mapsto & \sum_{\leq}^{\mathrm{Chen}} \tilde{\mathcal{R}}^\star(\sigma)(z) \qquad \big(\textit{ resp. } \sum_{<}^{\mathrm{Chen}} \tilde{\mathcal{R}}^\star(\sigma)(z)\big)\end{array}$$

where $\bullet$ stands for the ordinary product $\cdot$ for $\Psi'^{\mathcal{R}}$, and stands for the opposite of the ordinary product for $\Psi^{\mathcal{R}}$. In other words,

$$\begin{array}{rcl}\Psi^{\mathcal{R}}(\sigma_1 \otimes \cdots \otimes \sigma_k)(z) & := & \sum_{\leq}^{\mathrm{Chen}} \tilde{\mathcal{R}}^\star(\sigma_1 \otimes \cdots \otimes \sigma_k)(z) \\ \Psi'^{\mathcal{R}}(\sigma_1 \otimes \cdots \otimes \sigma_k)(z) & := & \sum_{<}^{\mathrm{Chen}} \tilde{\mathcal{R}}^\star(\sigma_1 \otimes \cdots \otimes \sigma_k)(z)\end{array}$$

satisfy the quasi-shuffle relations:

$$\begin{array}{rcl}\Psi^{\mathcal{R}}(\sigma \star_\bullet \tau) & = & \Psi^{\mathcal{R}}(\sigma) \cdot \Psi^{\mathcal{R}}(\tau), \\ \Psi'^{\mathcal{R}}(\sigma \star_\bullet \tau) & = & \Psi'^{\mathcal{R}}(\sigma) \cdot \Psi'^{\mathcal{R}}(\tau)\end{array}$$

which hold as equalities of meromorphic functions, and where $\bullet$ stands for $\mp\cdot$ as above.

Remark 2 *The results of this paragraph remain valid modulo a small restriction if we replace the Riesz regularisation procedure by more general regularisations: for example, for any holomorphic function $\lambda : \mathbb{C} \to \mathbb{C}$ one can use the regularisation $\mathcal{R}_\lambda$ defined by:*

$$\mathcal{R}_\lambda(\sigma_s)(z) = \sigma_{s+\lambda(z)}. \tag{12}$$

One can show that the corresponding regularised Chen sums admit an analytic continuation to $\{z \in \mathbb{C}, \lambda'(z) \neq 0\}$. A particular example of this situation is given by $\lambda(z) = \lambda_\mu := z + \mu z^2$ where μ is a constant. The corresponding regularisation $\mathcal{R}_{\lambda_\mu}$ will be denoted by $\mathcal{R}_\mu$. The corresponding regularised Chen sums are then meromorphic on $\mathbb{C} - \{-1/2\mu\}$. In particular they are meromorphic in a neighbourhood of 0.

4 Renormalised Chen sums

The regularised Chen sums introduced in the previous paragraph define a character of the quasi-shuffle Hopf algebra $\mathcal{H} = (T(\mathcal{A}), \star_\bullet)$ (where $\bullet = \mp.$ is minus or plus the ordinary product of functions). We can now apply the Birkhoff-Connes-Kreimer decomposition to it:

Theorem 4 *The map*

$$\begin{array}{rcl} \mathcal{A} & \to & \mathbb{C} \\ \sigma & \mapsto & \mathrm{fp}_{z=0} \sum \mathcal{R}(\sigma)(z), \end{array}$$

where $\mathrm{fp}_{z=0}$ *stands for the constant term in the Laurent expansion at* $z=0$*, extends to a multiplicative map:*

$$\begin{array}{rcl} \psi^{\mathcal{R}}(\text{ resp. } \psi'^{\mathcal{R}}) : (\mathcal{T}(\mathcal{A}), \star_\bullet), & \longrightarrow & \mathbb{C} \\ \sigma & \mapsto & \Psi_+^{\mathcal{R}}(\sigma)(0) \qquad \left(\text{ resp. } \Psi'^{\mathcal{R}}_+(\sigma)(0)\right) \end{array}$$

defined from the Birkhoff decomposition of $\Psi^{\mathcal{R}}(\sigma)(z) = \sum_{\leq}^{\mathrm{Chen}} \tilde{\mathcal{R}}^\star(\sigma)(z)$ *(resp.* $\Psi'^{\mathcal{R}}(\sigma)(z) = \sum_{<}^{\mathrm{Chen}} \tilde{\mathcal{R}}^\star(\sigma)(z)$*) with respect to the minimal subtraction scheme. It coincides with the ordinary Chen sums* $\sum_{\leq}^{\mathrm{Chen}}$ *(resp.* $\sum_{<}^{\mathrm{Chen}}$*) when the latter converge. Here* $\bullet$ *stands for the product* $\mp\cdot$ *as in Proposition 1.*

Proof: Let us prove the weak inequality case: The Birkhoff decomposition [CK], [M] for the minimal substraction scheme reads:

$$\Psi^{\mathcal{R}} = \left(\Psi_-^{\mathcal{R}}\right)^{*-1} * \Psi_+^{\mathcal{R}}$$

with $\bullet = -\cdot$. Since $\Psi_+^{\mathcal{R}}$ is multiplicative on $(\mathcal{T}(\mathcal{A}), \star_\bullet)$, it obeys the quasi-shuffle relation:

$$\Psi_+^{\mathcal{R}}(\sigma \star_\bullet \tau)(z) = \Psi_+^{\mathcal{R}}(\sigma)(z)\, \Psi_+^{\mathcal{R}}(\tau)(z) \tag{13}$$

which holds as an equality of meromorphic functions holomorphic at $z=0$. Setting

$$\psi^{\mathcal{R}} := \Psi_+^{\mathcal{R}}(0),$$

and applying the quasi-shuffle relations (13) at $z=0$ yields

$$\psi^{\mathcal{R}}(\sigma \star_\bullet \tau) = \psi^{\mathcal{R}}(\sigma)\, \psi^{\mathcal{R}}(\tau). \tag{14}$$

The tensor products $\sigma = \sigma_{s_1} \otimes \cdots \otimes \sigma_{s_k}$ where $\mathrm{Re}\ s_1 > 1$ and $\mathrm{Re}\ s_i \geq 1$, $i = 2, \cdots, k$ span a right co-ideal of $\mathcal{T}(\mathcal{A})$. The restriction of $\Psi^{\mathcal{R}}$ to this right co-ideal takes values in functions which are holomorphic at $z=0$. In that case it follows by construction that:

$$\psi^{\mathcal{R}}(\sigma) = \Psi_+^{\mathcal{R}}(\sigma)(0) = \Psi^{\mathcal{R}}(\sigma)(0) = \sum_{\leq}^{\mathrm{Chen}} \sigma.$$

The strict inequality case can be derived similarly setting $\bullet = \cdot$. $\square$

On the grounds of this result we set the following definition:

Definition 2 *For any* $\sigma \in \mathcal{T}(\mathcal{A})$*, the* renormalised Chen sums *of* σ *(in both weak and strict inequality versions) are defined by:*

$$\sum_{\leq}^{\mathrm{Chen},\mathcal{R}} \sigma := \psi^{\mathcal{R}}(\sigma), \qquad \sum_{<}^{\mathrm{Chen},\mathcal{R}} \sigma := \psi'^{\mathcal{R}}(\sigma).$$

By (14) we have:

$$\sum_{\leq}^{\text{Chen},\mathcal{R}} \sigma *_\bullet \tau = \sum_{\leq}^{\text{Chen},\mathcal{R}} \sigma \sum_{\leq}^{\text{Chen},\mathcal{R}} \tau \tag{15}$$

for $\bullet = -\cdot$, and similarly for the strict inequality versions with. $\bullet = \cdot$

5 Quasi-shuffle relations for renormalised multiple zeta functions

Recall that multiple zeta functions $\overline{\zeta}(s_1, \cdots, s_k)$ and $\zeta(s_1, \cdots, s_k)$ defined above converge whenever Re $s_1 > 1$ and Re $s_2 \geq 1, \cdots,$ Re $s_k \geq 1$ in which case they obey quasi-shuffle relations. In this section, we implement the renormalisation procedure described in the previous paragraph to extend them to other integer values of s_i while preserving the quasi-shuffle relations.

Let W be the $\mathbb{R}$-vector space freely spanned by sequences $(u_1, \ldots, u_k)$ of real numbers. Let us define the quasi-shuffle product on W by:

$$(u_1, \ldots, u_k)\star(u_{k+1}, \ldots, u_{k+l}) = \sum_{0\leq r\leq \min(k,l)} (-1)^r \sum_{\pi\in \text{mix sh}(k,l;r)} (u_1^\pi, \ldots, u_{k+l-r}^\pi), \tag{16}$$

with:

$$u_j^\pi = \sum_{i\in\{1,\ldots,k+l\},\pi(i)=j} u_i.$$

(the sum above contains only one or two terms). Define a map $u \mapsto \sigma_u$ from W to $\mathcal{T}(\mathcal{A})$ by:

$$\sigma_{(u_1,\ldots,u_k)} := \sigma_{u_1} \otimes \cdots \otimes \sigma_{u_k}.$$

Then

$$\sigma_u \star_\bullet \sigma_v = \sigma_{u\star v}.$$

The same holds with $\bullet = \cdot$ provided we drop the signs $(-1)^r$ in equation (16) defining the quasi-shuffle product on W.

We now apply the results of the previous section to extend the usual quasi-shuffle relations to non converging sums. Given a real number μ, we consider the holomorphic regularisation $\mathcal{R}_\mu$ defined in Remark 2.

Theorem 5 *For any real number μ, and any $s_1, \cdots, s_k \in \mathbb{C}$ the renormalised multiple zeta values*

$$\overline{\zeta}^\mu(s_1, \cdots, s_k) := \psi^{\mathcal{R}_\mu}(s_1, \cdots, s_k), \qquad \zeta^\mu(s_1, \cdots, s_k) := {\psi'}^{\mathcal{R}_\mu}(s_1, \cdots, s_k)$$

with $\psi^{\mathcal{R}_\mu}, {\psi'}^{\mathcal{R}_\mu}$ as in Definition 2, have the following properties:

1. *They verify the quasi-shuffle relations:*

$$\overline{\zeta}^\mu(u \star v) = \overline{\zeta}^\mu(u)\,\overline{\zeta}^\mu(v) \tag{17}$$

when the quasi-shuffle product $\star$ is defined by (16), and:

$$\zeta^\mu(u \star v) = \zeta^\mu(u)\,\zeta^\mu(v) \tag{18}$$

when the quasi-shuffle product $\star$ is defined by (16) with signs $(-1)^r$ removed.

2. $\overline{\zeta}^{\mu}(s) = \zeta^{\mu}(s) = \underset{z=0}{\mathrm{fp}} \sum k^{-s-f_\mu(z)}$ *for any* $s \in \mathbb{C}$,

3. *for any positive integer* k, *and whenever* $\mathrm{Re}\ s_1 > 1$ *and* $\mathrm{Re}\ s_j \geq 1$ *for* $2 \leq j \leq k$,

$$\begin{aligned}\overline{\zeta}^{\mu}(s_1, \cdots, s_k) &= \sum_{0<s_1\leq\cdots\leq n_k} n_1^{-s_1} \cdots n_k^{-s_k} = \overline{\zeta}(s_1, \cdots, s_k),\\ \zeta^{\mu}(s_1, \cdots, s_k) &= \sum_{0<s_1<\cdots<n_k} n_1^{-s_1} \cdots n_k^{-s_k} = \zeta(s_1, \cdots, s_k)\end{aligned}$$

independently of μ.

Proof: The proof follows from Theorem 4 applied to $\mathcal{R}^\mu$ instead of the Riesz regularisation $\mathcal{R}$. □

This is the unique extension of ordinary multiple zeta functions to integer arguments $s_i \geq 1$ which verifies the three properties of Theorem 5. Indeed, the expressions $\overline{\zeta}^{\mu}(s_1, \cdots, s_k)$ and $\zeta^{\mu}(s_1, \cdots, s_k)$ converge whenever $s_1 > 1$ since by assumption, all the s_i are no smaller than 1. When they converge, they obey the quasi-shuffle relations (17) and (18) respectively. The uniqueness of the extension to the case $s_1 = 1$ then follows by induction on the length k from the quasi-shuffle relations (17) which "push" the leading term $s_1 = 1$ whenever it arises, away from the first position and therefore expresses divergent expressions in terms of convergent expressions. This provides the uniqueness of the extension of Riemann multiple zeta functions to regularised multiple zeta functions satisfying quasi-shuffle relations, once the value θ at the argument 1 is imposed; see [H], [W], [Z]. Here it is the parameter μ that plays the role of this constant θ.

More precisely we can easily compute, e.g. by means of the Euler-MacLaurin formula:

$$\begin{aligned}\theta = \zeta^{\mu}(1) &= \underset{z=0}{\mathrm{fp}}\ \underset{N\to+\infty}{\mathrm{fp}} \sum_{k=1}^{N} k^{-1-z-\mu z^2}\\ &= -\underset{z=0}{\mathrm{fp}}\ \frac{1}{z}\frac{1}{1+\mu z}(N^{-z-\mu z^2} - 1)\\ &= -\mu.\end{aligned}$$

The other mutiple zeta values with 1 on the left are then computed by means of the quasi-shuffle relations, for example:

$$\zeta^{\mu}(1, 2) = -\mu\zeta(2) - \zeta(2, 1) - \zeta(3).$$

6 Multiple zeta values at nonpositive arguments

Let us sum up the steps of the procedure we implemented to get renormalised values of multiple zeta functions $\overline{\zeta}^{\mu}(s_1, \ldots, s_k)$ which obey quasi-shuffle relations.

1. Using a holomorphic regularisation $\mathcal{R}^\mu$ we built meromorphic maps $z \mapsto \sum_{\leq}^{\mathrm{Chen}} \widetilde{\mathcal{R}^{\mu}}^{*} (\sigma_{s_1} \otimes \cdots \otimes \sigma_{s_k})$ which obey the quasi-shuffle relations as identities between meromorphic functions.

2. The poles of these meromorphic functions might add superfluous contributions to the finite part and thereby spoil the quasi-shuffle relations when taking finite parts. A Birkhoff factorisation at $z=0$ was therefore implemented to take care of these extra terms by introducing counterterms. The resulting renormalised multiple zeta values $\overline{\zeta}^{\mu}(s_1,\cdots,s_k)$ indeed obey the quasi-shuffle relations.

Since the meromorphic maps $\sum_{\leq}^{\mathrm{Chen}} \widetilde{\mathcal{R}^{\mu}}^{*}(\sigma_{s_1}\otimes\cdots\otimes\sigma_{s_k})$ obtained from the holomorphic perturbation turn out to be holomorphic for non negative integers s_i, Birkhoff factorisation is superfluous[1] and will only lead to the same finite parts corresponding to ordinary limits as $z\to 0$:

$$\overline{\zeta}^{\mu}(s_1,\ldots,s_k)=\mathrm{fp}_{z=0}\widetilde{\mathcal{R}^{\mu}}^{*}(\sigma_{s_1}\otimes\cdots\otimes\sigma_{s_k})=\lim_{z\to 0}\widetilde{\mathcal{R}^{\mu}}^{*}(\sigma_{s_1}\otimes\cdots\otimes\sigma_{s_k}).$$

We show in [MP2] that for any nonpositive integers $s_1,\ldots,s_k$ the multiple zeta values $\overline{\zeta}(s_1,\ldots,s_k)$ obtained from Riesz regularisation $\mathcal{R}$ (i.e. for $\mu=0$) are rational numbers.

The treatment of the strict inequality version $\zeta(s_1,\ldots,s_k)$ is completely similar, but we stick to the weak inequality version as it is more easy to handle for an iterated use of the Euler-MacLaurin formula, which is the key ingredient to prove the rationality by induction on the depth. Along the same lines we also outline an algorithm to compute effectively the renormalised multiple zeta values at nonpositive arguments obtained with the Riesz regularisation. For double zeta values at nonpositive arguments we obtain the following formula:

$$\zeta(-a,-b)=\frac{1}{b+1}\sum_{s=0}^{b+1}\binom{b+1}{s}B_s\zeta(-a-b+s-1)+\zeta(-a)\zeta(-b)+(-1)^{a+1}\frac{a!b!}{2(a+b+2)!}B_{a+b+2}. \tag{19}$$

In terms of Bernoulli numbers, this is equivalent to:

$$\begin{aligned}\zeta(-a,-b) &= \frac{1}{b+1}\sum_{s=0}^{b+1}\frac{(-1)^{a+b-s+1}}{a+b-s}\binom{b+1}{s}B_sB_{a+b+2-s}+\frac{(-1)^{a+b}}{(a+1)(b+1)}B_{a+1}B_{b+1}\\ &+ (-1)^{a+1}\frac{a!b!}{2(a+b+2)!}B_{a+b+2}.\end{aligned} \tag{20}$$

[1] We refer to [M]; here the tensor algebra is built over the algebra $\mathcal{A}^{+}:=\{t\mapsto t^{s},s\geq 0\}$.

One can then establish a table of values $\zeta(-a,-b)$ for $a, b \in \{0, \cdots, 6\}$:

$\zeta(-a,-b)$	$a=0$	$a=1$	$a=2$	$a=3$	$a=4$	$a=5$	$a=6$
$b=0$	$\frac{3}{8}$	$\frac{1}{12}$	$\frac{7}{720}$	$-\frac{1}{120}$	$-\frac{11}{2\,520}$	$\frac{1}{252}$	$\frac{1}{224}$
$b=1$	$\frac{1}{24}$	$\frac{1}{288}$	$-\frac{1}{240}$	$-\frac{19}{10\,080}$	$\frac{1}{504}$	$\frac{41}{20\,160}$	$-\frac{1}{480}$
$b=2$	$-\frac{7}{720}$	$-\frac{1}{240}$	0	$\frac{1}{504}$	$\frac{113}{151\,200}$	$-\frac{1}{480}$	$-\frac{307}{166\,320}$
$b=3$	$-\frac{1}{240}$	$\frac{1}{840}$	$\frac{1}{504}$	$\frac{1}{28\,800}$	$-\frac{1}{480}$	$-\frac{281}{332\,640}$	$\frac{1}{264}$
$b=4$	$\frac{11}{2\,520}$	$\frac{1}{504}$	$-\frac{113}{151\,200}$	$-\frac{1}{480}$	0	$\frac{1}{264}$	$\frac{117\,977}{75\,675\,600}$
$b=5$	$\frac{1}{504}$	$-\frac{103}{60\,480}$	$-\frac{1}{480}$	$\frac{1}{1232}$	$\frac{1}{264}$	$\frac{1}{127\,008}$	$-\frac{691}{65\,520}$
$b=6$	$-\frac{1}{224}$	$-\frac{1}{480}$	$\frac{307}{166\,320}$	$\frac{1}{264}$	$-\frac{117\,977}{75\,675\,600}$	$-\frac{691}{65\,520}$	0

Remark 3 *The formula for $\zeta(-a,-b)$ coincides with the proposal of [AET] at the end of their paper (denoted by $\zeta^*(-b,-a)$ with their notations). This coincidence does not hold in depth ≥ 3 because of the incidence of the Hoffman isomorphism* Exp *[MP2]. If $a+b$ is odd and $b \neq 0$, all terms in equation (19) vanish except the one with $s=1$. This yields:*

$$\zeta(-a,-b) = -\frac{1}{2}\zeta(-a-b), \tag{21}$$

a fact also established by L. Guo and B. Zhang by different means [GZ]. We also have for odd a :

$$\zeta(-a,0) = -\zeta(-a). \tag{22}$$

The values for $\zeta(-a,-a)$ are forced by the quasi-shuffle relations, and hence coincide with those computed in [GZ]. Values of $\zeta(-a,-b)$ disagree however for $a+b$ even and a,b distinct. This shows that the quasi-shuffle relations are not sufficient to fully determine multiple zeta values at nonpositive arguments.

7 A higher-dimensional analogue

Considering the d-dimensional vector space $\mathbb{R}^d$ endowed with the supremum norm $|(x_1, \ldots x_d)| := \sup_i |x_i|$ one can consider the d-dimensional analogue of multiple zeta functions:

$$\zeta_d(s_1,\dots,s_k) := \sum_{p_1,\dots,p_k\in\mathbb{Z}^d,\ 0<|p_k|<\dots<|p_1|} |p_1|^{-s_1}\cdots|p_k|^{-s_k}. \tag{23}$$

The sum above converges whenever Re $s_1 > d$ and Re $s_j \geq d$, $j = 2,\dots,k$. We proved in [MP2], using as commutative algebra $\mathcal{A}$ an algebra of radial pseudo-differential symbols endowed with a suitable product $\bullet$, that:

1. Higher-dimensional multiple zeta functions ζ_d can be renormalised along the same lines as above. The higher-dimensional analogues of the quasi-shuffle relations are reflected in the fact that this procedure gives a character of the associated quasi-shuffle Hopf algebra.
2. Riesz-Regularised renormalised higher-dimensional multiple zeta values at non-positive arguments are again rational numbers.
3. Higher-dimensional renormalised multiple zeta fuctions are linear combinations (with positive integer coefficients) of "usual" renormalised multiple zeta fuctions. As an example we have:

$$\begin{aligned}\zeta_1(s) &= 2\zeta(s),\\ \zeta_2(s) &= 8\zeta(s,0)+8\zeta(s),\\ \zeta_3(s) &= 48\zeta(s,0,0)+56\zeta(s,0)+26\zeta(s).\end{aligned}$$

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