# Distribution of Fourier coefficients of primitive forms


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# Presented work

[1] E. Kowalski, O. Robert & J. Wu, Small gaps in coefficients of $L$-functions and $\mathfrak{B}$-free numbers in short intervals, *Revista Mat. Iberoamericana* **23** (2007), No. 1, 281–322.

[2] Y.-K. & J. Wu, The number of Hecke eigenvalues of same signs, *Preprint 2008*.

## § 1. Motivation

- *Ramanujan's function* $\tau(n)$

  For $\Im m\, z > 0$, define

$$\Delta(z) := e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{24} =: \sum_{n=1}^{\infty} \tau(n) e^{2\pi inz}.$$

- *Ramanujan's conjecture*

$$\boxed{|\tau(n)| \leqslant d(n) n^{11/2} \quad (n \geqslant 1)}$$

where $d(n)$ is the divisor function. This conjecture has been proved by Deligne (1974).

- *Lehmer's conjecture*

$$\boxed{\tau(n) \neq 0 \quad (n \geqslant 1)}$$

This is open !

- *Two partial results on Lehmer's conjecture*

  Lehmer (1959) : $\tau(n) \neq 0$ for $n \leqslant 10^{15}$.

  Serre (1981) : $|\{n \leqslant x : \tau(n) \neq 0\}| \sim \alpha x \quad (x \to \infty,\ \alpha > 0)$.

- *Questions*

  (i) $\alpha = 1$ ? (Nonvanishing of $\tau(n)$)

  (ii) $\displaystyle\lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x,\ \tau(n)\gtreqless 0} 1 = \frac{1}{2}$ ? (Signs of $\tau(n)$)

## § 2. Modular forms

- *Notation*

$$k := \text{integer} \geqslant 2,$$
$$N := \text{squarefree integer},$$
$$\log_r := \text{the } r\text{-fold iterated logarithm},$$
$$\chi_0 := \text{trivial Dirichlet character } (\text{mod } N),$$
$$\chi := \text{Dirichlet character } (\text{mod } N) \text{ such that } \chi(-1) = (-1)^k.$$

- *Cusp forms*

Denote by $S_k(N, \chi)$ the set of all cusp forms of weight $k$ and of level $N$ with nebentypus $\chi$.

- *Decomposition of $S_k(N,\chi)$*

$S_k(N,\chi)$ equipped with the Petersson inner product $\langle\cdot,\cdot\rangle$ is a finite dimensional Hilbert space and we have

$$S_k(N,\chi) = S_k^\flat(N,\chi) \oplus S_k^\sharp(N,\chi),$$

where $S_k^\flat(N,\chi)$ is the linear subspace of $S_k(N,\chi)$ spanned by all forms of type $f(dz)$, where $d \mid N$ and $f \in S_k(N',\chi')$ with $N' < N$ and $dN' \mid N$. Here $\chi' \pmod{N'}$ is the character which induces $\chi$. $S_k^\sharp(N,\chi)$ is the linear subspace of $S_k(N,\chi)$ orthogonal to $S_k^\flat(N,\chi)$ with respect to $\langle\cdot,\cdot\rangle$.

- *Newforms*

Denote by $S_k^*(N,\chi)$ the set of all newforms in $S_k^\sharp(N,\chi)$. It constitutes a base of $S_k^\sharp(N,\chi)$.

We can prove $\Delta(z) \in S_{12}(1) := S_{12}(1,\chi_0)$.

- *Fourier development of* $f \in S_k^*(N,\chi)$ *at* $\infty$

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z} \qquad (\Im m\, z > 0),$$

where $\lambda_f(n)$ has the following properties:

(i) $\lambda_f(1) = 1$,

(ii) $T_n f = \lambda_f(n) n^{(k-1)/2} f$ for any $n \geqslant 1$,

(iii) for all integers $m \geqslant 1$ and $n \geqslant 1$,

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \chi(d) \lambda_f\left(\frac{mn}{d^2}\right),$$

where $T_n$ is the $n$th Hecke's operator.

(iv) Further if $f \in S_k^*(N,\chi_0)$, then $\lambda_f(n) \in \mathbb{R}$ for $n \geqslant 1$.

- *Deligne's inequality*

  Deligne (1974): If $f \in S_k^*(N,\chi)$, then $\forall\, p$, $\exists\ \alpha_f(p), \beta_f(p)$ such that

$$\begin{cases} |\alpha_f(p)| = \pm p^{-1/2}, & \beta_f(p) = 0 \qquad \text{if } \ p \mid N \\ |\alpha_f(p)| = 1, & \alpha_f(p)\beta_f(p) = \chi(p) \quad \text{if } \ p \nmid N \end{cases}$$

and

$$\lambda_f(p^\nu) = \alpha_f(p)^\nu + \alpha_f(p)^{\nu-1}\beta_f(p) + \cdots + \beta_f(p)^\nu \quad (\forall\, \nu \geqslant 0).$$

In particular

$$|\lambda_f(p^\nu)| \leqslant \nu + 1 \quad (\forall\, p \text{ and } \forall\, \nu \geqslant 0).$$

More generally

$$|\lambda_f(n)| \leqslant d(n) \quad (n \geqslant 1)$$

where $d(n)$ is divisor function.

## § 3. Nonvanishing of $\lambda_f(n)$

- *Notation*

$$f \in S_k(N,\chi): \quad \mathfrak{P}_f := \{p : \lambda_f(p) = 0\}.$$

- *Forms of CM type*

Ribet (1977): If $f \in S_k^{\mathrm{cm}}(N,\chi)$ (the subspace of $S_k(N,\chi)$ spanned by all CM forms), then

$$\big|\mathfrak{P}_f \cap [1,x]\big| = \frac{x}{2\log x} + O\bigg(\frac{x}{(\log x)^2}\bigg).$$

By using Landau's method, we can prove $\exists\, \alpha > 0$ such that for $x \to \infty$,

$$\sum_{n\leqslant x,\, \lambda_f(n)\neq 0} 1 \sim \frac{\alpha x}{\sqrt{\log x}}, \qquad \sum_{n\leqslant x,\, \lambda_f(n)=0} 1 \sim x.$$

- *Lang-Trotter's conjecture*

  If $f \in S_k^*(N,\chi)\smallsetminus S_k^{\mathrm{cm}}(N,\chi)$, then

$$\big|\mathfrak{P}_f \cap [1,x]\big| \ll_f \begin{cases} \sqrt{x}/\log x & \text{if } \ k=2,\\ \log_2 x & \text{if } \ k=3,\\ 1 & \text{if } \ k\geqslant 4.\end{cases}$$

- *Forms of non-CM type*

  Serre (1981): If $f \in S_k^*(N,\chi)\smallsetminus S_k^{\mathrm{cm}}(N,\chi)$, then $\forall\,\delta < \frac{1}{2}$,

$$\big|\mathfrak{P}_f \cap [1,x]\big| \ll_{f,\delta} \frac{x}{(\log x)^{1+\delta}}. \tag{1}$$

From this, we deduce that there are constants $C > c > 0$ such that

$$cx \leqslant \sum_{n\leqslant x,\, \lambda_f(n)\neq 0} 1 \leqslant Cx \tag{2}$$

for $x \geqslant x_0(f)$.

- *Serre's function* $i_f(n)$

  For $f \in S_k(N,\chi) \smallsetminus S_k^{\mathrm{cm}}(N,\chi)$, we define

$$i_f(n) := \max\{i : \lambda_f(n+j) = 0 \ \ (0 < j \leqslant i)\}.$$

The inequalities (2) imply that

$$i_f(n) \ll_f n \quad (n \geqslant 1), \tag{3}$$

since

$$0 = \sum_{\substack{n < m \leqslant n + i_f(n) \\ \lambda_f(m) \neq 0}} 1 \geqslant c\big(n + i_f(n)\big) - Cn = c i_f(n) - (C-c)n.$$

- *Serre's question* (1981)

  Find constant $\theta < 1$ such that

$$i_f(n) \ll_{f,\theta} n^{\theta} \qquad (\forall\, n \geqslant 1). \tag{4}$$

**Remark 1**. Balog & Ono (2001) remarked that Serre's question has been resolved before proposing it ! There is two methods !

- *Rankin-Selberg convolution* (The first method)

  Rankin (1939) and Selberg (1940): For $f \in S_k^*(N, \chi)$, we have

$$\sum_{n \leqslant x} |\lambda_f(n)|^2 = A_f x + O_f(x^{3/5}) \quad (A_f > 0).$$

This implies trivially (4) with $\theta = \frac{3}{5}$, since

$$\begin{aligned} 0 &= \sum_{n < m \leqslant n + i_f(n)} \lambda_f(m)^2 = \sum_{m \leqslant n + i_f(n)} \lambda_f(m)^2 - \sum_{m \leqslant n} \lambda_f(m)^2 \\ &= A_f\big(n + i_f(n)\big) - A_f n + O\big((n + i_f(n))^{3/5} + n^{3/5}\big) \\ &= A_f i_f(n) + O(n^{3/5}) \quad \text{(via (3))}. \end{aligned}$$

- *Erdős' $\mathfrak{B}$-free numbers* (The second method)

Let

$$\mathfrak{B} = \{b_k \in \mathbb{N} : 1 < b_1 < \cdots < b_k < \cdots\}$$

such that

$$\sum_{i=1}^{\infty} \frac{1}{b_i} < \infty \qquad \text{and} \qquad (b_i, b_j) = 1 \quad (i \neq j). \tag{5}$$

An integer $n \geqslant 1$ is called $\mathfrak{B}$-free if $b \nmid n$ for any $b \in \mathfrak{B}$. The set of all $\mathfrak{B}$-free numbers is denoted by $\mathfrak{A} = \mathfrak{A}(\mathfrak{B})$.

Taking $\mathfrak{B} = \{p^2 : p \text{ prime}\} =: \mathfrak{P}^2$, then $\mathfrak{A}(\mathfrak{P}^2)$ is the set of all squarefree integers.

Erdős (1966): $\exists\, \theta < 1$ such that for $x \geqslant x_0(\mathfrak{B}, \theta)$,

$$\Big|\{x < n \leqslant x + x^\theta : n \text{ is } \mathfrak{B}\text{–free}\}\Big| \gg_{\mathfrak{B},\theta} x^\theta. \tag{6}$$

The records on $\theta$:

$\theta = \frac{1}{2} = 0.5 + \varepsilon$ (Szemerédi, 1973),

$\theta = \frac{9}{20} = 0.45 + \varepsilon$ (Bantle & Grupp, 1986),

$\theta = \frac{5}{12} = 0.4166 + \varepsilon$ (Wu, 1990),

$\theta = \frac{17}{41} = 0.4146 + \varepsilon$ (Wu, 1993),

$\theta = \frac{33}{80} = 0.4125 + \varepsilon$ (Wu, 1994) and (Zhai, 2000),

$\theta = \frac{40}{97} = 0.4123 + \varepsilon$ (Sargos & Wu, 2000),

$\theta = \varepsilon$ (Conjecture),

where $\varepsilon$ is an arbitrarily small positive number.

- *Application of $\mathfrak{B}$-free numbers*

  Take

$$\mathfrak{P}_f := \{p : \lambda_f(p) = 0\},$$

$$\mathfrak{B}_f := \mathfrak{P}_f \cup \{p^2 : p \notin \mathfrak{P}_f\}.$$

The multiplicativity of $\lambda_f(m)$ implies

$$\boxed{m \text{ is } \mathfrak{B}_f\text{–free} \ \Rightarrow\ \lambda_f(m) \neq 0}$$

In fact, if $m$ is $\mathfrak{B}_f$-free, we have

(i) if $p \mid m$, then $\lambda_f(p) \neq 0$,

(ii) $m = p_1 \cdots p_j$ with $\lambda_f(p_i) \neq 0$ for $1 \leqslant i \leqslant j$ and $p_1 < \cdots < p_j$,

(iii) $\lambda_f(m) = \lambda_f(p_1) \cdots \lambda_f(p_j) \neq 0$.

In view of Serre's (1), $\mathfrak{B}_f$ satisfies (5). Thus Erdős' (6) implies : $\exists\, \theta < 1$ such that for $x \geqslant x_0(f,\theta)$

$$\sum_{\substack{x<m\leqslant x+x^\theta \\ \lambda_f(m)\neq 0}} 1 \geqslant \sum_{\substack{x<m\leqslant x+x^\theta \\ m \text{ is } \mathfrak{B}_f-\text{free}}} 1 \gg_{f,\theta} x^\theta,$$

from which we deduce (taking $x = n$)

$$i_f(n) < n^\theta$$

for $n \geqslant x_0(f,\theta)$. Thus

$$i_f(n) \ll_{f,\theta} n^\theta \quad (n \geqslant 1).$$

In particular, the value $\theta = \frac{40}{97} + \varepsilon$ is admissible.

**Theorem 1 (Kowalski, Robert & Wu, 2007).** *Suppose that*

$$\big|\mathfrak{P}_f \cap [1,x]\big| \ll_f x^\rho/(\log x)^{\Theta_\rho} \qquad (x \geqslant 2) \tag{7}$$

*for any* $f \in S_k^*(N,\chi)\smallsetminus S_k^{\rm cm}(N,\chi)$, *where* $\rho \in [0,1]$ *and* $\Theta_\rho \in \mathbb{R}$ *such that* $\Theta_1 > 1$. *Then for any* $f \in S_k(N,\chi)\smallsetminus S_k^{\rm cm}(N,\chi)$ *we have*

$$i_f(n) \ll_{f,\varepsilon} n^{\theta(\rho)+\varepsilon},$$

*where*

$$\theta(\rho) = \begin{cases} 1/4 & \text{if} \quad 0 \leqslant \rho \leqslant 1/3, \\ 10\rho/(19\rho+7) & \text{if} \quad 1/3 < \rho \leqslant 9/17, \\ 3\rho/(4\rho+3) & \text{if} \quad 9/17 < \rho \leqslant 15/28, \\ 5/16 & \text{if} \quad 15/28 < \rho \leqslant 5/8, \\ (22\rho/(24\rho+29) & \text{if} \quad 5/8 < \rho \leqslant 9/10, \\ 7\rho/(9\rho+8) & \text{if} \quad 9/10 < \rho \leqslant 1. \end{cases}$$

**Corollary 2 (KRW, 2007).** *For any form* $f \in S_k(N,\chi) \smallsetminus S_k^{\mathrm{cm}}(N,\chi)$, *we have* $i_f(n) \ll_{f,\varepsilon} n^{7/17+\varepsilon}$ *for all* $n \geqslant 1$.

[(1) $\Rightarrow$ (7) with $\rho = 1$ and $\Theta_1 = 1 + \delta$]
$\theta(1) = \frac{7}{17} \approx 0.411$ improves very slightly Balog & Ono's $\frac{40}{97} \approx 0.412$.

**Corollary 3 (KRW, 2007).** *For any* $f \in S_k(N,\chi) \smallsetminus S_k^{\mathrm{cm}}(N,\chi)$, *Lang-Trotter's conjecture implies* $i_f(n) \ll_{f,\varepsilon} n^{10/33+\varepsilon}$ *for all* $n \geqslant 1$.

[Lang-Trotter's conjecture implies (7) with $\rho = \frac{1}{2}$ and $\Theta_{1/2} = 1$]
$\theta(\frac{1}{2}) = \frac{10}{33} \approx 0.303$ improves considerably Alkan's $\frac{1}{3} \approx 0.333$.

**Corollary 4 (KRW, 2007).** *Let* $E/\mathbb{Q}$ *be an elliptic curve without CM and* $f$ *the associated newform. Then* $i_f(n) \ll_{E,\varepsilon} n^{33/94+\varepsilon}$ *for all* $n \geqslant 1$.

[For elliptic curve, Elkies proved (7) with $\rho = \frac{3}{4}$ and $\Theta_{3/4} = 0$]
$\theta(\frac{3}{4}) = \frac{33}{94} \approx 0.351$ improves considerably Alkan's $\frac{69}{169} \approx 0.408$.

## § 4. Further improvements

- *Expected result*

$$\theta(0) = 0 \text{ in Theorem 1}$$

- *Defect of the constraint "squarefree"*

$$\theta(0) = \tfrac{1}{4} \neq 0 \text{ in Theorem 1}$$

- *Squarefree integers in short intervals*

  Filaseta & Trifonov (1992) : $\exists\, c > 0$ such that

$$\big|\{x < n \leqslant x + cx^{1/5}\log x : n \text{ is squarefree}\}\big| \gg x^{1/5}\log x.$$

**Remark 2**. With the multiplicative constraint "squarefree", we have few chance for obtaining $\theta(0) = 0$.

- *Treatment without the constraint "squarefree"*

**Lemma 1 (KRW, 2007).** *Let* $f \in S_k^*(N,\chi)$. *Then* $\exists\, \nu_f$ *such that for any* $p \nmid N$*: either* $\lambda_f(p^\nu) \neq 0$ $(\nu \geqslant 0)$ *or* $\exists\ \nu \leqslant \nu_f$ *such that* $\lambda_f(p^\nu) = 0$.

**Lemma 2 (KRW, 2007).** *For* $f \in S_k^*(N,\chi) \smallsetminus S_k^{\mathrm{cm}}(N,\chi)$*, we define*

$$\mathfrak{P}_{f,\nu} := \big\{p : \lambda_f(p^\nu) = 0 \text{ and } \lambda_f(p^j) \neq 0 \ (0 \leqslant j < \nu)\big\},$$
$$\mathfrak{B}_f^* := \mathfrak{P}_{f,1} \cup \mathfrak{P}_{f,2} \cup \cdots \cup \mathfrak{P}_{f,\nu_f}.$$

*Then for any* $\delta < \frac{1}{2}$*, we have*

$$\big|\mathfrak{B}_f^* \cap [1,x]\big| \ll_{f,\delta} \frac{x}{(\log x)^{1+\delta}}.$$

**Remark 3**. This contains Serre's (1) and leads us to propose a generalized Lang-Trotter's conjecture.

- *Generalized Lang-Trotter's conjecture*

**Conjecture 1.** *If* $f \in S_k^*(N,\chi) \smallsetminus S_k^{\mathrm{cm}}(N,\chi)$, *then*

$$\left|\mathfrak{B}_f^* \cap [1,x]\right| \ll_f \begin{cases} \sqrt{x}/\log x & \text{if } k=2, \\ \log_2 x & \text{if } k=3, \\ 1 & \text{if } k \geqslant 4. \end{cases}$$

**Lemma 3 (KRW, 2007).** *If* $f \in S_k^*(N)$ *such that* $\lambda_f(n) \in \mathbb{Z}$ *for any* $n \geqslant 1$ *(for example for elliptic curves), then Conjecture 1 is equivalent to Lang-Trotter's conjecture.*

- *Relation between* $\mathfrak{B}_f^*$*-free and* $\lambda_f(n) \neq 0$

$$\boxed{n \text{ is } \mathfrak{B}_f^*\text{-free} \Leftrightarrow \lambda_f(n) \neq 0}$$

**Theorem 5 (KRW, 2007).** *Suppose that*

$$\left|\mathfrak{B}_f^* \cap [1, x]\right| \ll_f \frac{x^\rho}{(\log x)^{\Theta_\rho}} \qquad (x \geqslant 2)$$

*for any* $f \in S_k^*(N, \chi) \smallsetminus S_k^{\mathrm{cm}}(N, \chi)$*, where* $\rho \in [0, 1]$ *and* $\Theta_\rho \in \mathbb{R}$ *such that* $\Theta_1 > 1$*. Then for any* $f \in S_k(N, \chi) \smallsetminus S_k^{\mathrm{cm}}(N, \chi)$ *and* $\varepsilon > 0$*, we have*

$$i_f(n) \ll_{f,\varepsilon} n^{\theta(\rho)+\varepsilon} \quad (n \geqslant 1),$$

*where* $\theta(\rho) = \rho/(1 + \rho)$*. In particular* $\theta(0) = 0$*.*

**Corollary 6 (KRW, 2007).** *Let* $k \geqslant 3$*. Suppose that Conjecture 1 holds for all* $f \in S_k^*(N, \chi) \smallsetminus S_k^{\mathrm{cm}}(N, \chi)$*. Then for any* $\varepsilon > 0$ *and all* $f \in S_k(N, \chi) \smallsetminus S_k^{\mathrm{cm}}(N, \chi)$*, we have*

$$i_f(n) \ll_{f,\varepsilon} n^{\varepsilon} \quad (n \geqslant 1).$$

## § 5. Katz' conjecture and nonvanishing problem

- *Question*

  Is there a cusp form $f$ such that $\lambda_f(n) \neq 0$ for any $n \geqslant 1$ ?

- *Katz' conjecture* (1972)

  Let $S(1,1;p)$ be the sum of Kloosterman. Then

$$L(s, \mathrm{Kl}) := \prod_p \left(1 - S(1,1;p)p^{-s} + p^{1-2s}\right)^{-1} =: \sum_{n \geqslant 1} \lambda_{\mathrm{Kl}}(n) n^{-s}$$

is $L$-function of a "non-holomorphic" cusp form of weight 2 over $SL_2(\mathbb{Z})$.

**Theorem 7 (KRW, 2007).** *We have* $\lambda_{\mathrm{Kl}}(n) \neq 0$ *for any* $n \geqslant 1$.

**Remark 4**. This gives an affirmative answer *conditionally*. But very probably Katz' conjecture should not hold.

## § 6. Sign changes of Hecke's eigenvalues

- *Sato-Tate's conjecture*

  For any $-2 \leqslant \alpha \leqslant \beta \leqslant 2$ and any $f \in S_k^*(N) := S_k^*(N, \chi_0)$, we have

$$|\{p \leqslant x : \alpha \leqslant \lambda_f(p) \leqslant \beta\}| \sim \frac{x}{\log x} \int_\alpha^\beta \frac{\sqrt{4-t^2}}{2\pi} \,\mathrm{d}t \quad (x \to \infty).$$

Here $\sqrt{4-t^2}/(2\pi)\,\mathrm{d}t$ is called *Sate-Tate's measure.*

- *Hecke's eigenvalues of same signs*

  For $f \in S_k^*(N)$, define

$$\mathcal{N}_f^{\pm}(x) := \sum_{\substack{n \leqslant x,\ (n,N)=1 \\ \lambda_f(n) \gtrless 0}} 1.$$

It is natural to conjecture

$$\lim_{x\to\infty} \frac{\mathcal{N}_f^{\pm}(x)}{x} = \frac{1}{2}.$$

Kohnen, Lau & Shparlinski (2007):

$$\mathcal{N}_f^{\pm}(x) \gg_f \frac{x}{(\log x)^{17}} \qquad (x \geqslant x_0(f)).$$

Wu (2008): The exponent 17 can be reduced to $1 - 1/\sqrt{3} \approx 0.4226$ and $2 - 16/(3\pi) \approx 0.3023$ if we assume Sato-Tate's conjecture.

**Theorem 8 (Lau & Wu, 2008).** *For any* $f \in S_k^*(N)$, *we have*

$$\mathcal{N}_f^{\pm}(x) \gg_f x$$

*for all* $x \geqslant x_0(f)$.

- *Sign changes of Hecke's eigenvalues*

Kohnen, Lau & Shparlinski (2007): there are absolute constants $\eta < 1$ and $A > 0$ such that for any $f \in S_k^*(N)$ we have

$$\mathcal{N}_f^{\pm}(x + x^{\eta}) - \mathcal{N}_f^{\pm}(x) > 0 \quad (x \geqslant (kN)^A).$$

**Theorem 9 (Lau & Wu, 2008).** *Let $f \in S_k^*(N)$. There is an absolute constant $c > 0$ such that for any $\varepsilon > 0$ and $x \geqslant cN^{1+\varepsilon} x_0(k)$, we have*

$$\mathcal{N}_f^{\pm}(x + cN^{1/2+\varepsilon} x^{1/2}) - \mathcal{N}_f^{\pm}(x) \gg_{\varepsilon} x^{1/4-\varepsilon},$$

*where $x_0(k)$ is a suitably large constant depending on $k$ and the implied constant in $\gg_{\varepsilon}$ depends only on $\varepsilon$.*

**Remark 5**. $\lambda_f(n)$ has a sign-change in interval $\left[x, x + cN^{1/2+\varepsilon} x^{1/2}\right]$ for all sufficiently large $x$.

## § 7. Ideas of the proof of Theorem 1

Recall

$$\mathfrak{B}_f := \mathfrak{P}_f \cup \{p^2 : p \notin \mathfrak{P}_f\}.$$

The existence of $\mathfrak{B}_f$-free numbers in short intervals is a problem of sieve.

(a) System of weights for detecting $\mathfrak{B}_f$-free numbers;

(b) Exponential sum for controlling the error terms in the sieve.

- *Alkan's condition on* $\theta$

$$\delta_1(\theta) + \theta + \theta/\rho > 1,$$

where $\delta_1(\theta)$ is increasing and given by exponential sums of type II.

- *Our condition on* $\theta$

$$\delta_2(\theta) + 2\theta/\rho > 1.$$

where $\delta_2(\theta)$ is increasing and given by exponential sums of type I.

- *Two improvements*

(i) By exploiting $\{p^2 : p \notin \mathfrak{P}_f\}$, we improve $\theta + \theta/\rho$ into $2\theta/\rho$.

(ii) Our system of weights allows us to bring back to estimate bilinear forms of type I :

$$\sum_{M \leqslant m < 2M} \sum_{N \leqslant n < 2N} \psi_n r_{mn}(x, x^\theta),$$

instead of type II (as in the work of Alkan)

$$\sum_{M \leqslant m < 2M} \sum_{N \leqslant n < 2N} \phi_m \psi_n r_{mn}(x, x^\theta),$$

where $|\phi_m| \leqslant 1, |\psi_n| \leqslant 1$ and

$$r_d(x, y) := |\{x < n \leqslant x + y : d \mid n\}| - y/d.$$

Thus we have $\delta_2(\theta) > \delta_1(\theta)$.

- *Multiple exponential sums* :

By the Fourier analyse, the estimate of bilinear forms can be transformed into estimate of multiple exponential sums

$$\sum_{H\leqslant h<2H}\ \sum_{M\leqslant m<2M}\ \sum_{N\leqslant n<2N} \phi_h\psi_n e\Big(X\frac{h^\alpha m^\beta n^\gamma}{H^\alpha M^\beta N^\gamma}\Big),$$

where $e(t) := e^{2\pi it}$, $|\phi_h|\leqslant 1$, $|\psi_n|\leqslant 1$, $\alpha,\beta,\gamma\in\mathbb{R}$.

According the size of $\rho$, we use the methods of Fouvry–Iwaniec (with Robert–Sargos' refinement) and of Heath-Brown to estimate this sum.