Theoretical and numerical hierarchical control of some PDEs

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Outline

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   - The Stackelberg-Nash strategy
   - The main result

2. Numerical analysis and results
   - Computation of Nash equilibria
   - Computation of Pareto equilibria
   - Numerical solution of the Stackelberg-Nash null control problem

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CONTROL PROBLEMS

What is usual: act to get good (or the best) results for

\[
\begin{align*}
E(y) &= F(v) \\
&\quad + \ldots
\end{align*}
\]

What is easier? Solving? Controlling?

Two classical approaches:

- Optimal control
- Controllability
OPTIMAL CONTROL

A general optimal control problem

Minimize $J(v)$
Subject to $v \in V_{ad}, \ y \in Y_{ad}, \ (v, y)$ satisfies

$$E(y) = F(v) + \ldots \quad (S)$$

Main questions: $\exists$, uniqueness/multiplicity, characterization, computation, $\ldots$

We could also consider similar bi-objective optimal control:

"Minimize" $J_1(v), J_2(v)$
Subject to $v \in V_{ad}, \ldots$
CONTROLLABILITY

A null controllability problem

Find \((v, y)\)
Such that \(v \in V_{ad}\), \((v, y)\) satisfies \((ES)\), \(y(T) = 0\)

with \(y : [0, T] \mapsto H\),

\[
E(y) \equiv y_t + A(y) = F(v) + \ldots
\]

\((ES)\)

Again many interesting questions: \(\exists\), uniqueness/multiplicity, characterization, computation, \ldots

A very rich subject for PDEs, see [Russell, J.-L. Lions, Coron, Zuazua, \ldots]
Question: How can we adopt both viewpoints together?

Example: Optimal-control / controllability problem

A simplified model for the autonomous car driving problem

The system:
\[ x_t = f(x, u), \quad x(0) = x_0 \]

Constraints:
\[ \text{dist.} \ (x(t), Z(t)) \geq \varepsilon \quad \forall t \]
\[ u \in \mathcal{U}_{ad} \quad (|u(t)| \leq C) \]

\( u \) determines direction and speed

Goals (prescribed \( x_T \) and \( \hat{x} \)):
- \( x(T) = x_T \) (or \( |x(T) - x_T| \leq \varepsilon \ldots \))
- Minimize \( \sup_t |x(t) - \hat{x}(t)| \)

[Sontag, Sussman-Tang, \ldots]
Optimal control + controllability
Automatic driving

Figure: The ICARE Project, INRIA, France. Autonomous car driving. Malis-Morin-Rives-Samson, 2004

The car in the street
What was announced in 2014:
• Nissan ID 1.0 (2015), highways and traffic jams (no lane change) - OK
• ID 2.0 (2018), overtaking and lane change
• ID 3.0 (2020), complete autonomous driving in town

http://reports.nissan-global.com/EN/?p=17295
Hierarchical control

The system and the controls. Meaning

Another way to connect optimal control and controllability:

HIERARCHICAL CONTROL (Stackelberg-Nash, Stackelberg-Pareto, . . .)

The main ideas in the context of Navier-Stokes:

Three controls: one leader, two followers

\[
\begin{align*}
  y_t + (y \cdot \nabla)y - \Delta y + \nabla p &= f_1\mathcal{O} + v_1\mathcal{O}_1 + v_2\mathcal{O}_2, \quad (x, t) \in \Omega \times (0, T) \\
  \nabla \cdot y &= 0, \quad (x, t) \in \Omega \times (0, T) \\
  y &= 0, \quad (x, t) \in \partial\Omega \times (0, T) \\
  y(x, 0) &= y^0(x), \\
\end{align*}
\]

Disjoint domains \(\mathcal{O}, \mathcal{O}_i, (i = 1, 2)\)

Three objectives:

- “Simultaneously”, \(y \approx y_{i,d}\) in \(\Omega \times (0, T), i = 1, 2\), reasonable effort:

  \[
  \text{Minimize} \quad \alpha_i \iiint_{\Omega \times (0, T)} |y - y_{i,d}|^2 + \mu \iint_{\mathcal{O}_i \times (0, T)} |v_i|^2, \quad i = 1, 2
  \]

- Bi-objective optimal control - The task of the followers

  In practice, does an equilibrium \((v_1(f), v_2(f))\) exist for each \(f\)?

- Get \(y(x, T) \equiv 0\)

  Null controllability - The task of the leader

  Can we find \(f\) such that \(y(T) = 0\)?
Hierarchical control
The system and the controls. Meaning

\[
\begin{aligned}
    y_t + (y \cdot \nabla)y - \Delta y + \nabla p &= f_1\mathcal{O} + v_1 1_{\mathcal{O}_1} + v_2 1_{\mathcal{O}_2}, \\
    \nabla \cdot y &= 0, \\
    y &= 0, \\
    y(x, 0) &= y^0(x),
\end{aligned}
\]

Many applications:

- **Heating**: Controlling temperatures
  Heat sources at different locations - Heat PDE (linear, semilinear, etc.)

- **Tumor growth**: Controlling tumor cell densities
  Radiotherapy strategies - Reaction-diffusion PDEs
  **bilinear control**

- **Fluid mechanics**: Controlling fluid velocity fields
  Several mechanical actions - Stokes, Navier-Stokes or similar

- **Finances**: Controlling the price of an option
  Agents at different stock prices, etc. - Backwards in time heat-like PDE
  **Degenerate coefficients**

**Contributions**: Lions, Díaz-Lions, Glowinski-Periaux-Ramos, Guillén, . . .
**Optimal control + AC**
A SIMPLIFIED PROBLEM FOR THE 1D HEAT PDE

Again three controls: one leader, two followers

\[
\begin{aligned}
    y_t - y_{xx} &= f1_{\mathcal{O}} + v_11_{\mathcal{O}_1} + v_21_{\mathcal{O}_2}, & \quad (x, t) \in (0, 1) \times (0, T) \\
    y(0, t) &= y(1, t) = 0, & \quad t \in (0, T) \\
    y(x, 0) &= y^0(x), & \quad x \in (0, 1)
\end{aligned}
\]  

Different intervals \( \mathcal{O}, \mathcal{O}_i \)

Again three objectives:

- Simultaneously, \( y \approx y_{i,d} \) in \( \Omega \times (0, T) \), \( i = 1, 2 \), reasonable effort:

\[
\text{Minimize } \alpha_i \iint_{\Omega \times (0, T)} |y - y_{i,d}|^2 + \mu \iint_{\mathcal{O}_i \times (0, T)} |v_i|^2, \quad i = 1, 2
\]

Bi-objective optimal control - Followers’ task

- Get \( y(T) = 0 \)

Null controllability - Leader’s task

What can we do?
Hierarchical control  
The Stackelberg-Nash strategy

THE STACKELBERG-NASH STRATEGY

Step 1: $f$ is fixed

$$J_i(v_1, v_2) := \alpha_i \int_0^T \int_{\Omega} |y - y_i|^2 + \mu \int_0^T \int_{\Omega_i} |v_i|^2, \quad i = 1, 2$$

Find a Nash equilibrium $(v_1(f), v_2(f))$ with $v_i(f) \in L^2(\Omega_i \times (0, T))$:

$$J_1(v_1(f), v_2(f)) \leq J_1(v_1, v_2(f)) \quad \forall v_1 \in L^2(\Omega_1 \times (0, T))$$

$$J_2(v_1(f), v_2(f)) \leq J_2(v_1(f), v_2) \quad \forall v_2 \in L^2(\Omega_2 \times (0, T))$$

Equivalent to an optimality system:

$$\begin{cases}
y_t - y_{xx} = f \mathbf{1}_1 - \frac{1}{\mu} \phi_1 \mathbf{1}_1 - \frac{1}{\mu} \phi_2 \mathbf{1}_2 \\
-\phi_i, t - \phi_i, xx = \alpha_i (y - y_i, d), \quad i = 1, 2 \\
\phi_i(0, t) = \phi_i(1, t) = 0, \quad y(0, t) = y(1, t) = 0, \quad t \in (0, T) \\
y(x, 0) = y^0(x), \quad \phi_i(x, T) = 0, \quad x \in (0, 1) \\
v_i(f) = -\frac{1}{\mu} \phi_i|_{\Omega_i \times (0, 1)}
\end{cases}$$

$\exists (v_1(f), v_2(f))? \text{ Uniqueness?}$
Hierarchical control
The Stackelberg-Nash strategy

THE STACKELBERG-NASH STRATEGY
Step 2: Find $f$ such that

$$(HSN)_1 \begin{cases} y_t - y_{xx} = f \mathcal{O} - \frac{1}{\mu} \phi_1 \mathcal{O}_1 - \frac{1}{\mu} \phi_2 \mathcal{O}_2 \\ -\phi_i, t - \phi_i, xx = \alpha_i (y - y_i, d), \quad i = 1, 2 \\ y|_{t=0} = y^0(x), \quad \phi_i|_{t=T} = 0, \quad \text{etc.} \end{cases}$$

$$(HSN)_2 \quad y(x, T) = 0, \quad x \in (0, 1)$$

with $\|f\|_{L^2(\mathcal{O} \times (0, T))} \leq C \|y^0\|_{L^2}$

Equivalent to

$$\|\psi|_{t=0}\|^2 + \sum_{i=1}^{2} \iint_{\Omega \times (0, T)} \hat{\rho}^{-2} |\gamma^i|^2 \, dx \, dt \leq C \iint_{\mathcal{O} \times (0, T)} |\psi|^2 \, dx \, dt$$

for all $\psi^T$, with

$$\begin{cases} -\psi_t - \psi_{xx} = \sum_{i=1}^{2} \alpha_i \gamma^i, \quad \gamma^i_t - \gamma^i_{xx} = -\frac{1}{\mu} \psi \mathcal{O}_i \\ \psi|_{t=T} = \psi^T(x), \quad \gamma^i|_{t=0} = 0, \quad \text{etc.} \end{cases}$$

True?
Theorem
Assume: large $\mu$

$\exists \hat{\rho}$ such that, if $\int_{\Omega \times (0, T)} \hat{\rho}^2 |y_i, d|^2 \, dx \, dt < +\infty$, $i = 1, 2$, then:

$\forall y^0 \in L^2(\Omega) \exists$ null controls $f \in L^2(\mathcal{O} \times (0, T))$ & Nash pairs $(v_1(f), v_2(f))$

Idea of the proof:

- Energy estimates for the optimality system for $(y, \phi_1, \phi_2)$
- Energy and Carleman estimates for the adjoint system for $(\psi, \gamma^1, \gamma^2)$

We do need: $\mu$ is large
FIRST, HOW CAN WE COMPUTE A NASH EQUILIBRIUM PAIR? (THE FOLLOWERS)

The goal: $f$ is given. Solve the optimality system

$$\begin{align*}
y_t - \Delta y &= f1_\mathcal{O} - \frac{1}{\mu} \phi_1 1_\mathcal{O}_1 - \frac{1}{\mu} \phi_2 1_\mathcal{O}_2 \\
-\phi_{i,t} - \phi_{i,xx} &= \alpha_i(y - y_{i,a}), \quad i = 1, 2 \\
y_{|t=0} &= y^0(x), \quad \phi_i_{|t=T} = 0, \text{ etc.}
\end{align*}$$

Then take $v_i = \frac{1}{\mu} \phi_i |_{\mathcal{O}_i \times (0, T)}$

For instance: ALG 1 - Fixed point
ALG 1: $(v_1, v_2) \rightarrow y \rightarrow (\phi_1, \phi_2) \rightarrow (v_1, v_2)$
Also: Gradient, Conjugate gradient, etc.

Standard approximations: $P_\ell$-Lagrange FEM’s, Implicit Euler schemes
Numerical analysis and results
Computation of Nash equilibria

A 2D numerical experiment with FreeFem++: http://www.freefem.org/

Figure: The final adapted mesh - Number of vertices: 1460 - Number of triangles: 2781
Figure: The (fixed) leader control $f$ (constant in time)
Numerical analysis and results

Computation of Nash equilibria

Figure: The target $y_{1,d}$ (constant in time)
Numerical analysis and results
Computation of Nash equilibria

Figure: The target $y_{2,d}$ (constant in time)
Numerical analysis and results
Computation of Nash equilibria

Figure: The state $y$ at $t = T$ - Result for $y^0 = 0$, $\mu = 0.15$
Stopping test: $\sum_i \|v_{i,n+1} - v_{i,n}\|/\|v_{i,n+1}\| \leq 10^{-5}$
Figure: The adjoint state $\phi_1$ at $t = 0$
Numerical analysis and results
Computation of Nash equilibria

Figure: The adjoint state $\phi_2$ at $t = 0$
Numerical analysis and results
Computation of Nash equilibria

Iterates versus $\mu$:

Figure: The number of iterates as a function of $\mu$
Stopping test: $\sum_i \|v_{i,n+1} - v_{i,n}\|/\|v_{i,n+1}\| \leq 10^{-5}$
A similar semilinear problem: Compute a Nash equilibrium $(v_1, v_2)$ for

\[
\begin{cases}
y_t - \Delta y + F(y) = f \Omega + v_1 \Omega_1 + v_2 \Omega_2 \\
\text{etc.}
\end{cases}
\]

The task: solve the optimality system

\[
\begin{cases}
y_t - \Delta y + F(y) = f \Omega - \frac{1}{\mu} \phi_1 \Omega_1 - \frac{1}{\mu} \phi_2 \Omega_2 \\
-\phi_{i,t} - \Delta \phi_i + F'(y)\phi_i = \alpha_i (y - y_{i,d}) \Omega_{i,d}, \quad i = 1, 2 \\
y|_{t=0} = y^0(x), \quad \phi_i|_{t=T} = 0, \quad \text{etc.}
\end{cases}
\]

Then: \( v_i = \frac{1}{\mu} \phi_i \big|_{\Omega_i \times (0,T)} \)

For globally Lipschitz-continuous \( F \): existence is ensured

For instance: ALG 2 and ALG 3 . . . - Fixed point strategies

ALG 2: \((v_1, v_2) \rightarrow \{y \rightarrow y\} \rightarrow (\phi_1, \phi_2) \rightarrow (v_1, v_2)\)

ALG 3: \((v_1, v_2) \rightarrow y \rightarrow (\phi_1, \phi_2) \rightarrow (v_1, v_2)\)
Numerical experiments with FreeFem - ALG 2 - Iterates versus $\mu$

**Figure:** ALG 2 - The number of iterates as a function of $\mu$ - $y^0 = 0 - F(y) = y(1 + \sin y)$

Stopping test: $\sum_i \| v_{i,n+1} - v_{i,n} \| / \| v_{i,n+1} \| \leq 10^{-5}$
Numerical experiments with FreeFem - ALG 3 - Iterates versus $\mu$

Figure: ALG 3 - The number of iterates as a function of $\mu$ - $y^0 = 0 - F(y) = y(1 + \sin y)$
Stopping test: $\sum_i \|v_{i,n+1} - v_{i,n}\|/\|v_{i,n+1}\| \leq 10^{-5}$
Figure: The state $y$ at $t = T$ - Result for $y^0 = 0 - F(y) = y(1 + \sin y)$
Stopping test: $\sum_i \|v_{i,n+1} - v_{i,n}\|/\|v_{i,n+1}\| \leq 10^{-5}$
Numerical experiments with FreeFem - ALG 3

Figure: The state $y$ at $t = T$ - Result for $y^0 = 0 - F(y) = y(1 + \sin y)$

Stopping test: $\sum_i \|v_{i,n+1} - v_{i,n}\| / \|v_{i,n+1}\| \leq 10^{-5}$
Numerical analysis and results
Computation of Nash equilibria

Figure: ALG 3 - The number of iterates as a function of $\mu$ - $y^0 = 0$

$F(y) = y \log(1 + |y|)^a$, $1 < a < 2$, not sublinear!

Stopping test: $\sum_i \|v_{i,n+1} - v_{i,n}\| / \|v_{i,n+1}\| \leq 10^{-5}$
Numerical analysis and results
Computation of Nash equilibria

Another semilinear system - ALG 3

Figure: The state $y$ at $t = T$ - Result for $y^0 = 0$, $\mu = 2.5$

$F(y) = y \log(1 + |y|)^a$, $1 < a < 2$, not sublinear!

Stopping test: $\sum_i \|v_{i,n+1} - v_{i,n}\|/\|v_{i,n+1}\| \leq 10^{-5}$
Another semilinear system - ALG 3

Figure: The state $y$ at $t = T$. Result for $y_0 = 0, \mu = 2.5$, not sublinear!

Stopping test:
$$\sum_i \|v_i, n + 1 - v_i, n\| / \|v_i, n + 1\| \leq 10^{-5}$$
Another semilinear system - ALG 3

The results for $y^0 = 0, \mu = 2.5$

$F(y) = y \log(1 + |y|)^a$, $1 < a < 2$, not sublinear!

Stopping test: $\sum_i \|v_{i,n+1} - v_{i,n}\|/\|v_{i,n+1}\| \leq 10^{-5}$

**FIXED-POINT ITERATE AND ERROR:**

- 0 - 1383.98
- 1 - 640.146
- 2 - 39.2927
- 3 - 3.13866
- 4 - 0.283813
- 5 - 0.0262522
- 6 - 0.0024854
- 7 - 0.000243612
- 8 - 2.5955e-05
- 9 - 2.86158e-06

**FIXED-POINT RATE =** 3.18113
ANOTHER HIERARCHIC STRATEGY: PARETO EQUILIBRIA

\((v_1(f), v_2(f))\) is a Pareto equilibrium if \(\not\exists (w_1, w_2)\) with

\[
J_1(w_1, w_2) \leq J_1(v_1(f), v_2(f)), \quad J_2(w_1, w_2) \leq J_2(v_1(f), v_2(f))
\]

and at least one strict inequality

The goal: \(f\) and \(\lambda \in (0, 1)\) are given. Solve the optimality system

\[
\begin{cases}
  y_t - \Delta y = f^1 \phi + v_1^1 \phi_1 + v_2^2 \phi_2 \\
  -\phi_{i,t} - \phi_{i,xx} = \alpha_i (y - y_i, d), \quad i = 1, 2 \\
  y|_{t=0} = y^0(x), \quad \phi_i|_{t=T} = 0, \text{ etc.}
\end{cases}
\]

Again: ALG 1bis - Fixed point

ALG 1bis: \((v_1, v_2) \rightarrow y \rightarrow (\phi_1, \phi_2) \rightarrow (v_1, v_2)\)

Again: standard approximations
Numerical analysis and results
Computation of Pareto equilibria

Numerical experiments with FreeFem - ALG 1bis - Iterates versus $\lambda$

Figure: ALG 1bis - The number of iterates as a function of $\lambda$ - $y^0 = 0$
Stopping test: $\sum_i \|v_{i,n+1} - v_{i,n}\|/\|v_{i,n+1}\| \leq 10^{-5}$
SOLVING NUMERICALLY THE STACKELBERG-NASH NC PROBLEM?
(COMPUTING THE LEADER AND THE ASSOCIATED FOLLOWERS)

The goal: Find \( f \) such that the solution to

\[
\begin{align*}
\frac{\partial y}{\partial t} - \Delta y &= f^1_0 - \frac{1}{\mu} \phi_1^1 x_1 - \frac{1}{\mu} \phi_2^1 x_2 \\
- \phi_{i,t}^1 - \phi_{i,xx}^1 &= \alpha_i (y - y_{i,d})^1 x_{i,d}, \quad i = 1, 2 \\
y|_{t=0} &= y^0(x), \quad \phi_i|_{t=T} = 0, \text{ etc.}
\end{align*}
\]

satisfies

\[
y(x, T) \equiv 0
\]

The Fursikov-Imanuvilov approach:

\[
\begin{align*}
\text{Minimize} & \quad \iint \rho^2 |y|^2 + \iint_{O \times (0,T)} \rho_0^2 |f|^2 \\
\text{Subject to} & \quad (HN)
\end{align*}
\]

\( \rho \) and \( \rho_0 \) are appropriate, blow up as \( t \to T \)

The advantage: \( y \) necessarily vanishes exactly at \( t = T \) (and so does \( f \))
The resulting task after applying Lagrange's principle:

Solve a 4th-order Lax-Milgram problem

\[
\begin{align*}
\mathbf{a}((\psi, \gamma_1, \gamma_2), (\psi', \gamma'_1, \gamma'_2)) &= \langle \ell, (\psi', \gamma'_1, \gamma'_2) \rangle \\
\forall (\psi', \gamma'_1, \gamma'_2) &\in \mathcal{W}, \quad (\psi, \gamma_1, \gamma_2) \in \mathcal{W}
\end{align*}
\]

\[\exists! \text{ solution for appropriate } \rho \text{ and } \rho_0 \text{ (Carleman inequalities, large } \mu)\]

\[
\int \left[ \rho^{-2} \left\{ (L^* \psi + \sum_i \alpha_i \gamma_i)(L^* \psi' + \sum_i \alpha_i \gamma'_i) + \ldots \right\} + \rho_0^{-2} \mathcal{O} \psi \psi' \right] \\
= \int_\Omega y^0(x) \psi'(x, 0)
\]

\[
\int [z z' + 1 \circ m m'] + \int (z' - \rho^{-1}(L^*(\rho_0 m') + \ldots)) \lambda \\
= \int_\Omega y^0(x) \psi'(x, 0)
\]

Reformulation: a 2nd-order mixed problem after integration by parts

\[
\begin{align*}
\alpha ((z, m), (z', m')) + \beta ((z', m'), \lambda) &= \langle \tilde{\ell}, (z', m') \rangle \\
\beta ((z, m), \lambda') &= 0 \\
\forall (z', m', \lambda') &\in Z \times M \times \Lambda, \quad (z, m, \lambda) \in Z \times M \times \Lambda
\end{align*}
\]
Numerical analysis and results
Numerical solution of the Stackelberg-Nash null control problem

\[
\begin{align*}
\alpha ((z, m), (z', m')) + \beta ((z', m'), \lambda) &= \langle \bar{\ell}, (z', m') \rangle \\
\beta ((z, m), \lambda') &= 0 \\
\forall (z', m', \lambda') \in Z \times M \times \Lambda, \quad (z, m, \lambda) \in Z \times M \times \Lambda
\end{align*}
\]

**Approximation:** mixed $P_1 - P_2$-Lagrange FEM’s
Techniques already applied for NC of (nonlinear) heat, wave, Stokes, Navier-Stokes, ... 
[EFC-München, Cindea-EFC-München, EFC-München-Souza, ...]
A 1D numerical experiment with FreeFem

Figure: The domain and the mesh - $\Omega = (0, 1)$, $\mathcal{O}_1 = (0, 0.2)$, $\mathcal{O}_2 = (0.8, 1)$ - $T = 0.5$
- Number of vertices $(x_i, t_i)$: 3521 - Number of triangles: 6820
Numerical analysis and results
Numerical solution of the Stackelberg-Nash null control problem

Figure: The state $y - y^0 \equiv 10 \sin x - \mu = 1 - y_1,d = y_2,d = 0$
Numerical analysis and results
Numerical solution of the Stackelberg-Nash null control problem

Figure: The leader $f$
Figure: The leader $f$
EXTENSIONS

- **Boundary followers, distributed leader**: OK under similar conditions

\[
\begin{align*}
\dot{y} - y_{xx} &= f^1 \text{d}, \quad (x, t) \in (0, 1) \times (0, T) \\
y(0, t) &= v_1(t), \quad y(1, t) = v_2(t), \quad t \in (0, T) \\
y(x, 0) &= y^0(x), \quad x \in (0, 1)
\end{align*}
\]

**Costs:**
\[
\alpha_i \int_0^T \int_{\mathcal{Q}} |y - y_{i,d}|^2 + \mu \int_0^T |v_i|^2 \, dt, \quad i = 1, 2
\]

- **Distributed followers, boundary leader**: OK again

\[
\begin{align*}
\dot{y} - y_{xx} &= v_1^1 \text{d} + v_2^2 \text{d}, \quad (x, t) \in (0, 1) \times (0, T) \\
y(0, t) &= f, \quad y(1, t) = 0, \quad t \in (0, T) \\
y(x, 0) &= y^0(x), \quad x \in (0, 1)
\end{align*}
\]

- **However**: boundary followers + boundary leader is unknown!

We would need: 
\[
\|\psi\|_{t=0}^2 + \sum_{i=1}^2 \int_Q \hat{\rho}^{-2} |\gamma^i|^2 \leq C \int_0^T \rho_*^{-2} |\psi_x(0, t)|^2 \, dt
\]

\[
\begin{align*}
-\dot{\psi} - \psi_{xx} &= \sum_{i=1}^2 \alpha_i \gamma^i \text{d}, \quad \gamma^i - \gamma^i_{xx} = -\frac{1}{\mu} \psi^1 \text{d} \\
\psi|_{t=T} &= \psi^T(x), \quad \gamma^i|_{t=0} = 0, \quad \text{etc.}
\end{align*}
\]
EXTENSIONS (Cont.)

- More followers, coefficients, non-scalar parabolic systems, other functionals, boundary controls, higher dimensions, etc.
- **Semilinear** systems: OK for Lipschitz-continuous $F$

\[
\begin{align*}
  y_t - y_{xx} &= F(x, t; y) + f_1 \circ \sum_{i=1}^{m} v_i 1_{\partial_i} \\
  y(0, t) &= y(1, t) = 0, \quad t \in (0, T), \text{ etc.}
\end{align*}
\]

- **ECT:** OK
- **Local constraints:** OK
  For instance, $v_i \in L^2(\mathcal{O}_i \times (0, T))$, $v_i(x, t) \in L_i \text{ (closed)}$
**AN INTERESTING QUESTION:**

All this holds for large $\mu$ - What about small $\mu$?

Recall: $J_i(v_1, v_2) := \frac{\alpha_i}{2} \int_{\mathcal{O}_i \times (0, T)} |y - y_{i,d}|^2 + \frac{\mu}{2} \int_{\mathcal{O}_i \times (0, T)} |v_i|^2, \quad i = 1, 2$

\[
\begin{align*}
    y_t - y_{xx} &= f^1 \phi - \frac{1}{\mu} (\phi_1 1_{\mathcal{O}_1} + \phi_2 1_{\mathcal{O}_2}) \\
    -\phi_{i,t} - \phi_{i,xx} &= \alpha_i (y - y_{i,d}) 1_{\mathcal{O}_i,d} \quad \Leftrightarrow \quad (\text{Id.} - \frac{1}{\mu} \Lambda)(v_1, v_2) = (v_{1,0}, v_{2,0}) \\
    v_i &\in L^2(\mathcal{O}_i \times (0, T))
\end{align*}
\]

for some compact, self-adjoint $\Lambda$

Fredholm's alternative + Hilbert-Schmidt

$\Rightarrow \exists \mu_1 > \mu_2 > \ldots$ (independent of $f$), with $\mu_n \to 0^+$ such that

$\exists$ Nash equilibrium for all $\mu \neq \mu_n$ for all $n$

Do we have NC for these $\mu$?
FINAL COMMENTS:

- Other hierarchical strategies? Stackelberg-Pareto controllability?
  \[ \lambda J'_1(v_1, v_2) + (1 - \lambda) J'_2(v_1, v_2) = 0, \quad \lambda \in (0, 1) \]

For each \( f \), we get a family of equilibria \( (v^1_\lambda(f), v^2_\lambda(f)) \), with \( \lambda \in (0, 1) \)

\[
\begin{align*}
  y_t - y_{xx} &= f^1 \phi_1 \mu_{1,1} + \frac{1}{1-\lambda} \phi_2 \mu_{1,2} \\
  -\phi_t - \phi_{xx} &= \alpha_1 \lambda (y - y_{1,a}) \mu_{1,1,a} + \alpha_2 (1 - \lambda)(y - y_{2,a}) \mu_{1,2,a} \\
  \ldots
\end{align*}
\]

\[
\begin{align*}
  -\psi_t - \psi_{xx} &= \alpha_1 \lambda \gamma_1 \mu_{1,1,a} + \alpha_2 (1 - \lambda) \gamma_2 \mu_{1,2,a} \\
  \gamma'_t - \gamma'_{xx} &= -\frac{1}{\mu} \left( \frac{1}{1-\lambda} \psi \mu_{1,1} + \frac{1}{1-\lambda} \psi \mu_{1,2} \right) \\
  \ldots
\end{align*}
\]

∃ some kind of “common” null controls?
∃ average null controls, i.e. \( f \) such that \( \left( \int_0^1 y d\lambda \right)(T) = 0? \)

- Navier-Stokes? OPEN, as well as the standard NC problem
  Work in progress: local results (for small \( y_0 \))
  [with Araruna, Guerrero and Santos]
REFERENCES:

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AN ANNOUNCEMENT:
Doc-Course on Partial Differential Equations: Analysis, Numerics and Control
http://www.imus.us.es/DOC-COURSE18/en/presentation

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