Some results on the controllability of the heat equation

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Outline

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Motivation

General goal

Better understand the control properties of the heat equation.

Null-controllability problem for the heat equation

Let $\Omega \subset \mathbb{R}^d$, $\Gamma \subset \partial\Omega$.
The heat equation with control function $v \in L^2(0, T; L^2(\Gamma))$:

$$\begin{cases} \partial_t u - \Delta_x u = 0, & \text{in } (0, T) \times \Omega, \\ u(t, x) = v(t, x) \mathbf{1}_\Gamma(x), & \text{in } (0, T) \times \partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega. \end{cases}$$

Given $u_0 \in L^2(\Omega)$, find a control function $v \in L^2(0, T; L^2(\Gamma))$ s.t. $u(T, \cdot) = 0$?

$\longrightarrow$ YES [Fursikov Imanuvilov '96, Lebeau Robbiano '95].

Motivation

Despite important numbers of subsequent works based on deep Carleman estimates, there are still some open problems even for the 1-d heat equation with constant coefficient.

Null-controllability in small time $T > 0$

Let $C_{cont}(T)$ be the norm of the control map, which to $u_0 \in L^2(0, L)$ maps the control $v_{u_0} \in L^2(0, T)$ of minimal $L^2(0, T)$-norm such that the solution u of

$$\begin{cases} \partial_t u - \partial_{xx} u = 0 & \text{in } (0, T) \times (0, L), \\ u(t, 0) = 0, u(t, L) = v_{u_0}(t) & \text{in } (0, T), \\ u(0, x) = u_0(x) & \text{in } (0, L). \end{cases}$$

satisfies $u(T) = 0$, i.e.

$$\|v_{u_0}\|_{L^2(0,T)} \leq C_{cont}(T, L) \|u_0\|_{L^2(0,L)} .$$

Known results:

- $C_{cont}(T, L)$ equals the best observability constant, i.e. the best constant s.t.

$$\begin{cases} -\partial_t z - \partial_{xx} z = 0 & \text{in } (0,T) \times (0,L), \\ z(t,0) = z(t,L) = 0 & \text{in } (0,T), \end{cases}$$

$$\|z(0,\cdot)\|_{L^2(0,L)} \leq C_{obs}(T,L) \|\partial_x z(t,L)\|_{L^2(0,T)} .$$

- $\liminf_{T\to 0} T \log(C_{cont}(T,L)) \geq \frac{L^2}{4}$. [Miller '04]
- $\liminf_{T\to 0} T \log(C_{cont}(T,L)) \geq \frac{L^2}{2}$. [Lissy '15]
- $\limsup_{T\to 0} T \log(C_{cont}(T,L)) < \infty$. [Fursikov Imanuvilov '96]
- $\limsup_{T\to 0} T \log(C_{cont}(T,L)) \leq \frac{3^+ L^2}{4}$ [Tenenbaum Tucsnak 07]

⇝ What is the optimal result?

Coron Guerrero's problem (2005): Define $C(\varepsilon, L, T)$ as the cost of the null-control map for the viscous transport eq. :

$$\begin{cases} \partial_t u + \partial_x u - \varepsilon \partial_{xx} u = 0 & \text{in } (0,T)\times(0,L), \\ u(t,0) = v(t), \quad u(t,L) = 0, & \text{in } (0,T), \\ u(0,\cdot) = u_0(\cdot), & \text{in } (0,L), \\ u(T,\cdot) = 0 & \text{in } (0,L), \end{cases}$$

[Coron Guerrero '05]

- If $T < L$, then $\liminf_{\varepsilon \to 0} C(\varepsilon, L, T) = +\infty$.
- If $T > KL$, then $\limsup_{\varepsilon \to 0} C(\varepsilon, L, T) = 0$, for K large enough.

$K = 4.3$ [Coron Guerrero '05],
$K = 4.2$ [Glass '10],
$K = 2\sqrt{3}$ [Lissy '12]. $\longrightarrow$ Optimal K ?

Coron Guerrero's problem is a toy model for all models involving a small viscosity, including conservation laws, or the limit from Navier-Stokes to Euler equations.

Still, a better understanding of these control questions is required.

NB: In 1d, many approaches have been developed besides Carleman estimates:

- Solving a moment problem by constructing a biorthogonal family [Fattorini Russell '71], ...
- The flatness approach [Martin-Rosier-Rouchon '14 '16]
- The backstepping approach [Coron Nguyen '16]
- The transmutation technique [Miller '06, SE Zuazua '12].

$\longrightarrow$ In this talk, I will discuss the reachable set in 1d and the cost of controllability in small times.

Outline

The reachable set

Set $L > 0$ and consider the 1-d heat equation on $(-L, L)$

$$\begin{cases} \partial_t u - \partial_{xx} u = 0 & \text{in } (0,T) \times (-L, L), \\ u(t,-L) = v_-(t) & \text{in } (0,T), \\ u(t,L) = v_+(t) & \text{in } (0,T), \\ u(0,x) = u_0(x) & \text{in } (-L,L). \end{cases}$$

with controls v_-, v_+ at $-L, +L$.

Goal

Describe the reachable set

$$\mathscr{R}_{L,u_0}(T) = \{u(T) \mid u \text{ solving the heat eq. with } v_-,\ v_+ \in L^2(0,T)\}.$$

Classical properties

$$\begin{cases} \partial_t u - \partial_{xx} u = 0 & \text{in } (0,T)\times(-L,L), \\ u(t,-L) = v_-(t) & \text{in } (0,T), \\ u(t,L) = v_+(t) & \text{in } (0,T), \\ u(0,x) = u_0 & \text{in } (-L,L). \end{cases}$$

$$\mathscr{R}_{L,u_0}(T) = \{u(T) \mid u \text{ solving the heat eq. with } v_-,\ v_+ \in L^2(0,T)\}.$$

Null-controllability of the heat equation in arbitrarily small times [Fursikov Imanuvilov '96] implies

- $\mathscr{R}_{L,u_0}(T)$ does not depend on u_0;
- $\mathscr{R}_{L,u_0}(T)$ does not depend on time.

$\longrightarrow$ We simply denote it by $\mathscr{R}_L$.

Known results: Description in terms of the spectrum

[Fattorini Russell '71, SE Zuazua '12]

$$\left\{ u(x) = \sum_{n \geq 1} c_n \sin\left(\frac{n\pi(x+L)}{2L}\right) \text{ s.t. } \sum_n |c_n|^2 n e^{n\pi} < \infty \right\} \subset \mathscr{R}_L.$$

But $\sum_n |c_n|^2 n e^{n\pi} < \infty$ implies:

- Homolorphic extension on the strip $\{|\Im(z)| < L\}$
- $u(x) = \sum_{n\geq 1} c_n \sin\left(\frac{n\pi(x+L)}{2L}\right)$ satisfies $((\partial_{xx})^n u)(-L) = ((\partial_{xx})^n u)(L) = 0$ for all $n \in \mathbb{N}$.

Additional Remark: Sharp description in terms of the spectrum of the operator, see [Fattorini Russell '71].

A recent work

[Martin Rosier Rouchon '16]

Theorem [Martin Rosier Rouchon '16]

1. If $u \in \mathscr{R}_L$, u admits an holomorphic extension to the square $\{x+iy,\ |x|+|y|<L\}$.
2. If u admits an holomorphic expansion on the disk $\{x+iy,\ |x|^2+|y|^2 \leq R^2\}$ with $R = e^{1/(2e)}L$ ($\simeq 1.2L$), then $u \in \mathscr{R}_L$.

Item 1 $\simeq$ if u solves $\partial_t u - \partial_{xx} u = v 1_{x \notin (-L,L)}$ in $(0,T) \times \mathbb{R}$,

$$u_T(x) \simeq \int_{\mathbb{R}\setminus(-L,L)} \int_0^T \frac{1}{\sqrt{T-t}} e^{-(x-\tilde{x})^2/4(T-t)} v(t,\tilde{x})\, dt d\tilde{x}.$$

Item 2: The flatness approach [Martin Rosier Rouchon '16]

Our result

Recall

Theorem [Martin Rosier Rouchon 2015]

1. If $u \in \mathscr{R}_L$, u admits an holomorphic extension to the square $\{x + iy,\ |x| + |y| < L\}$.

We proved

Theorem [J. Dardé & S.E. '16]

If u admits an holomorphic extension in the square $\{x + iy,\ |x| + |y| < L_0\}$ for some $L_0 > L$, then $u \in \mathscr{R}_L$.

$\Rightarrow$ Result mainly sharp.

Elements of proof

- A Carleman type estimate ;
- A duality result ;
- Cauchy's formula.

A Carleman type estimate

Theorem

Set $T > L^2/\pi$. Any solution z of

$$\begin{cases} \partial_t z - \partial_{xx} z = 0 & \text{in } (0,\infty)\times(-L,L), \\ z(t,-L) = z(t,L) = 0 & \text{in } (0,\infty), \end{cases}$$

with $z(0,\cdot) \in L^2(-L,L)$ satisfies

$$\int_{-L}^{L} |z(T,x)|^2 e^{\frac{x^2-L^2}{2T}}\, dx + \int_0^\infty \int_{-L}^{L} |z(t,x)|^2 e^{\frac{x^2-L^2}{2t}}\, dt\, dx \leq C \int_0^T t \left(|\partial_x z(t,-L)|^2 + |\partial_x z(t,L)|^2 \right) dt.$$

A time condition: critical points of $t \mapsto e^{-L^2/4t - \pi^2 t/4L^2}$.

Some remarks

"Limiting" Carleman weight $\exp((x^2 - L^2)/4t)$

- Sharp Carleman estimate:

$$z(t,x) = \frac{1}{\sqrt{t}} e^{-(x^2-L^2)/4t} e^{ixL/2t},$$

solves the heat equation.

- The conjugated operator can be written as $A + B$, with

$$A = \partial_t + \frac{x}{t}\partial_x + \frac{1}{2t}, \quad B = -\partial_{xx} - \frac{L^2}{4t^2},$$

so that

$$A^* = -A, \quad B^* = B, \quad [A, B] = -\frac{2}{t} B,$$

Degenerate convexity conditions.
See [Kenig Sjostrand Uhlmann '07].

Other reference:

- The Bergman Space on a Sector and the Heat Equation, [Aikawa, Hayashi, Saito, '90][1].

  The reachable set for the heat equation on the half line $(0, \infty)$ *controlled in* $x = 0$ *with controls in* $L^2(0, T; \, tdt)$ *exactly is the weighted Bergman space in the triangle* $\{x + iy, \, |y| < |x|\}$, *with weight* $\exp(\xi^2/4T)$.

Open question

Can we get a more accurate description of the reachable set in a bounded domain? With an analytic potential?

$\rightsquigarrow$ From the reachable space of the heat equation to Hilbert spaces of holomorphic functions, A. Hartmann, K. Kellay, M. Tucsnak, 2017.

[1] We thank M. Tucsnak for having pointed out this reference.

Outline

Cost of controllability of the 1-d heat equation

The 1-d heat equation observed at both ends $x = \pm L$:

$$\begin{cases} -\partial_t z - \partial_{xx} z = 0 & \text{in } (0,T) \times (-L,L), \\ z(t,-L) = z(t,L) = 0 & \text{in } (0,T), \end{cases}$$

Goal

Estimate the best constant $C_{obs}(T,L)$ such that $\forall z$,

$$\|z(0,\cdot)\|_{L^2(-L,L)} \leq C_{obs}(T,L) \left(\|\partial_x z(t,L)\|_{L^2(0,T)} + \|\partial_x z(t,-L)\|_{L^2(0,T)} \right).$$

Theorem [J. Dardé & S.E., 2017]

$$\limsup_{T\to 0} T\log(C_{obs}(T,L)) \leq K^+ L^2,$$

with

$$K = \frac{1}{4} + \frac{\Gamma(1/4)^2}{8\sqrt{2}\pi^2}\sum_{n\in\mathbb{N}} \frac{(-1)^n}{(2n+1)}\frac{\Gamma(n+1/4)}{\Gamma(n+7/4)} < 0.7.$$

- $\liminf_{T\to 0} T\log(C_{obs}(T,L)) \geq \frac{L^2}{4}$. [Miller '04]
- $\liminf_{T\to 0} T\log(C_{obs}(T,L)) \geq \frac{L^2}{2}$. [Lissy '15]
- $\limsup_{T\to 0} T\log(C_{obs}(T,L)) < \infty$. [Fursikov Imanuvilov '96]
- $\limsup_{T\to 0} T\log(C_{obs}(T,L)) \leq \frac{3^+L^2}{4}$. [Tenenbaum Tucsnak '07]

$\longrightarrow$ Optimality ??

Ideas of the proof

- Explicit computations based on the previous Carleman type estimate
- An argument of complex analysis.

Explicit computations

Inspired by the previous Carleman estimate:

Theorem [J. Dardé & S.E. 2016]

Set $T > L^2/\pi$. Any solution z of

$$\begin{cases} \partial_t z - \partial_{xx} z = 0 & \text{in } (0,\infty) \times (-L, L), \\ z(t,-L) = z(t,L) = 0 & \text{in } (0,\infty), \end{cases}$$

with $z(0,\cdot) \in L^2(-L,L)$ satisfies

$$\int_{-L}^{L} |z(T,x)|^2 e^{\frac{x^2-L^2}{2T}}\, dx + \int_0^\infty \int_{-L}^{L} |z(t,x)|^2 e^{\frac{x^2-L^2}{2t}}\, dt\, dx \leq C \int_0^T t \left(|\partial_x z(t,-L)|^2 + |\partial_x z(t,L)|^2 \right) dt.$$

$\longrightarrow$ What happens if $T < L^2/2\pi$?

Arbitrary T

Theorem [J. Dardé & S.E. 2016]

Any solution z of

$$\begin{cases} \partial_t z - \partial_{xx} z = 0 & \text{in } (0,\infty)\times(-L,L), \\ z(t,-L) = z(t,L) = 0 & \text{in } (0,\infty), \end{cases}$$

with $z(0,\cdot) \in L^2(-L,L)$ satisfies

$$\int_{-L}^{L} |\nabla (z(T,x) e^{\frac{x^2-L^2}{4T}})|^2 \, dx - \frac{L^2}{4T^2} \int_{-L}^{L} |z(T,x) e^{\frac{x^2-L^2}{4T}}|^2 \, dx \\ \leq C \int_0^T t \left(|\partial_x z(t,-L)|^2 + |\partial_x z(t,L)|^2 \right) \, dt.$$

$\rightsquigarrow$ Should give informations for frequencies $|\xi| > \frac{L}{2T}$.

$\rightsquigarrow$ We work on

$$w(t,x) = z(t,x) \exp\left(\frac{x^2 - L^2}{4t}\right).$$

It satisfies the equation

$$\begin{cases} \partial_t w + \frac{x}{t}\partial_x w - \partial_{xx} w - \frac{L^2}{4t^2} w + \frac{1}{2t} w = 0, & \text{in } (0,\infty) \times (-L,L), \\ w(t,-L) = w(t,L) = 0, & \text{in } (0,\infty), \\ w(t=0,x) = 0, & \text{in } (-L,L). \end{cases}$$

• We extend w by 0 outside $(-L,L)$,

$$\begin{cases} \partial_t w + \frac{x}{t}\partial_x w - \partial_{xx} w - \frac{L^2}{4t^2} w + \frac{1}{2t} w = \pm\partial_x w(t,\pm L)\delta_{\pm L}, \\ \qquad \text{in } (0,\infty) \times \mathbb{R}, \\ w(t=0,x) = 0, \qquad \text{in } \mathbb{R}. \end{cases}$$

w satisfies:

$$\begin{cases} \partial_t w + \dfrac{x}{t}\partial_x w - \partial_{xx} w - \dfrac{L^2}{4t^2} w + \dfrac{1}{2t} w = \pm \partial_x w(t, \pm L)\delta_{\pm L}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,\infty)\times\mathbb{R}, \\ w(t=0,x) = 0, \qquad\qquad\qquad\qquad \text{in } \mathbb{R}. \end{cases}$$

$\rightsquigarrow$ Fourier transform $\hat{w}(t,\xi) = \mathscr{F}_{x\to\xi} w(t)$ satisfies:

$$\begin{cases} \partial_t \hat{w} - \dfrac{\xi}{t}\partial_\xi \hat{w} + \xi^2 \hat{w} - \dfrac{L^2}{4t^2}\hat{w} - \dfrac{1}{2t}\hat{w} = \pm \partial_x w(t, \pm L) e^{\pm i\xi L}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,\infty)\times\mathbb{R}, \\ \hat{w}(t=0,x) = 0, \qquad\qquad\qquad\qquad \text{in } \mathbb{R}. \end{cases}$$

$\rightsquigarrow$ This is a transport equation !

In particular, it can be solved explicitly...

$$\partial_t \hat{w} - \frac{\xi}{t}\partial_\xi \hat{w} + \xi^2 \hat{w} - \frac{L^2}{4t^2}\hat{w} - \frac{1}{2t}\hat{w} = \pm \partial_x w(t, \pm L) e^{\pm i \xi L}.$$

Along the characteristics

$$\frac{d\xi}{dt} = -\frac{\xi}{t}, \quad \xi(T) = \xi_0,$$

i.e. $\xi(t) = \xi_0 T/t$

$$\frac{d}{dt}\left(\frac{\hat{w}(t, \xi_0 T/t)}{\sqrt{t}}\right) + \left(\xi_0^2 T^2 - \frac{L^2}{4}\right)\frac{1}{t^2}\left(\frac{\hat{w}(t, \xi_0 T/t)}{\sqrt{t}}\right) = \pm \frac{1}{\sqrt{t}}\partial_x w(t, \pm L) e^{\pm i \xi_0 TL/t}.$$

Therefore,

$$\hat{w}(T,\xi_0) = \pm \int_0^T \sqrt{\frac{T}{t}} \partial_x w(t,\pm L) e^{\pm i\xi_0 TL/t} e^{(\xi_0^2 T^2 - \frac{L^2}{4})(\frac{1}{T} - \frac{1}{t})} \, dt,$$

provided the integrals converge as $t \to 0$, i.e. for

$$\xi_0 \in \Omega = \left\{ a + ib, \text{ with } |a| \geq |b| + \frac{L}{2T} \right\},$$

$$(\Re(\mp i\xi_0 TL + (\xi_0^2 T^2 - L^2/4)) \geq 0).$$

Besides, for $\xi_0 \in \Omega$, we get

$$|\hat{w}(T,\xi_0)| \leq C e^{L|\Im(\xi_0)|} \left\| \partial_x w(t,\pm L) \right\|_{L^2(0,T)} .$$

We have

- There exists $C > 0$ such that

$$\forall \xi_0 \in \Omega = \{a + ib, \text{ with } |a| \geq |b| + \frac{L}{2T}\},$$

$$|\hat{w}(T, \xi_0)| \leq C e^{L|\Im(\xi_0)|} \|\partial_x w(t, \pm L)\|_{L^2(0,T)}.$$

- Supp $w(T) \subset [-L, L]$, therefore
  - $\hat{w}(T, \cdot)$ is an entire function.
  - $\forall \xi_0 \in \mathbb{C}$, $|\hat{w}(T, \xi_0)| \leq e^{L|\Im(\xi_0)|} \|w(T, \cdot)\|_{L^1(-L,L)}$.

$\longrightarrow$ We can use Phragmén Lindelöf type theorems.

$$\rho(a,b) = \log\left(\frac{|\hat{w}(T, a+ib)|}{C\,\|\partial_x w(t,\pm L)\|_{L^2(0,T)}}\right)$$

- is a subharmonic function on $\Omega^c = \{(a,b), \text{ with } |a| < |b| + L/2T\}$.
- satisfies
$$\rho(a,b) \leq L|b| \text{ on } \partial\Omega^c,$$
- $|\rho(a,b)| \leq L|b| + C$ for $(a,b) \in \Omega^c$.

Consequently, for all $\varepsilon > 0$, $\rho \leq (1+\varepsilon)h$ in Ω^c, where

$$\begin{cases} \Delta h = 0 & \text{in } \Omega^c, \\ h(a,b) = L|b| & \text{on } \partial\Omega^c, \\ \limsup_{|b|\to\infty} \inf_{a \text{ s.t.}(a,b)\in\Omega^c} \dfrac{h(a,b)}{|b|} \geq L. & \end{cases}$$

Rescaling $h(a,b) = \frac{L^2}{2T}\tilde{h}(\tilde{a},\tilde{b})$, where

$$\begin{cases} \Delta \tilde{h} = 0 & \text{in } \mathcal{O}, \\ \tilde{h}(a,b) = |b| & \text{on } \partial\mathcal{O}, \\ \limsup_{|b|\to\infty} \inf_{a \text{ s.t.}(a,b)\in\mathcal{O}} \dfrac{\tilde{h}(a,b)}{|b|} \geq 1, \end{cases}$$

where $\mathcal{O} = \{(a,b), \text{ with } |a| < |b| + 1\}$.

We define $k(a,b) = \tilde{h}(a,b) - |b|$:

$$\begin{cases} \Delta k = \delta_{[-1,1]} & \text{in } \mathcal{O}, \\ k(a,b) = 0 & \text{on } \partial\mathcal{O}, \end{cases}$$

k can be explicitly computed by conformal mappings.

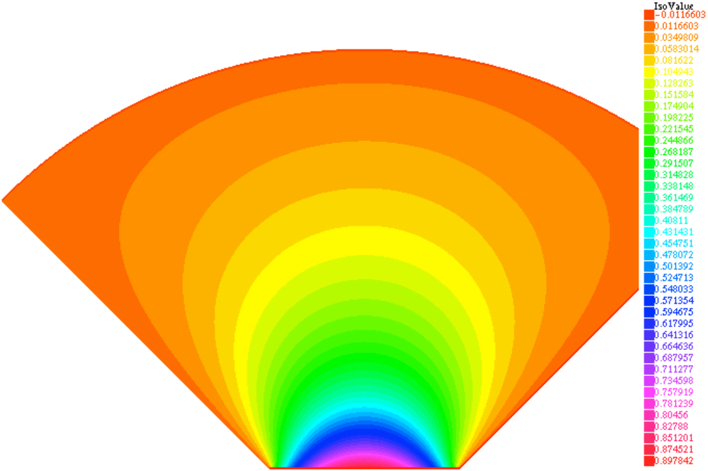

Figure : Plot of the function k.

$$\sup_{[-1,1]} k(\cdot,0) = k(0,0) = \frac{\Gamma(1/4)^2}{4\sqrt{2}\pi^2} \sum_{n\in\mathbb{N}} \frac{(-1)^n}{(2n+1)} \frac{\Gamma(n+1/4)}{\Gamma(n+7/4)} < 0.9.$$

Therefore, for all $\varepsilon > 0$ and for all $\xi_0 \in [-L/2T, L/2T]$,

$$|\hat{w}(T,\xi_0)| \leq C_\varepsilon \left\|\partial_x w(t,\pm L)\right\|_{L^2(0,T)} \exp\left((1+\varepsilon)\frac{L^2}{2T}k(0,0)\right).$$

We also have, by the Carleman estimate,

$$\int_{|\xi_0|>L/2T} |\hat{w}(T,\xi_0)|^2 \, d\xi_0 \leq C \int_0^T |\partial_x w(t,\pm L)|^2 \, dt.$$

$$\rightsquigarrow \quad \|w(T)\|_{L^2(-L,L)} \leq C \exp\left((1+\varepsilon)\frac{L^2}{2T}k(0,0)\right) \|\partial_x w(t,\pm L)\|_{L^2(0,T)}$$

Hence the result as $w(T,x) = z(T,x)\exp((x^2 - L^2)/4T)$.

We have thus obtained

$$\left\| z(T,x)\exp((x^2-L^2)/4T) \right\|_{L^2(-L,L)} \leq C\exp\left((1+\varepsilon)\frac{L^2}{2T}k(0,0) \right) \|\partial_x z(t,\pm L)\|_{L^2(0,T)},$$

i.e.

$$\left\| z(T,x)\exp(x^2/4T) \right\|_{L^2(-L,L)} \leq C\exp\left(\left(\frac{1}{4} + \frac{(1+\varepsilon)k(0,0)}{2} \right) \frac{L^2}{T} \right) \|\partial_x z(t,\pm L)\|_{L^2(0,T)}.$$

Outline

Clean estimate on the cost of controllability of the 1d heat equation.

- Applies to controllability from one side in 1 space dimension by symmetry arguments. (*Coeff* $=$ 1)
- Applies to distributed controllability in 1 space dimension by cut-off arguments. (*Coeff* $=$ 1)
- Directly applies in higher space dimension when the control acts on the whole boundary. (*Coeff* $=$ 1)
- We can use the transmutation method [Miller '06] to get estimates on the cost of controllability of the heat equation in cases in which the corresponding wave equation is controllable (*Geometric Control Conditions*).

Coron Guerrero's problem (2005): Define $C(\varepsilon, L, T)$ as the cost of the null-control map for the viscous transport eq. :

$$\begin{cases} \partial_t u + \partial_x u - \varepsilon \partial_{xx} u = 0 & \text{in } (0,T)\times(0,L), \\ u(t,0) = v(t), \quad u(t,L) = 0, & \text{in } (0,T), \\ u(0,\cdot) = u_0(\cdot), & \text{in } (0,L), \\ u(T,\cdot) = 0 & \text{in } (0,L), \end{cases}$$

[Coron Guerrero '05]

- If $T < L$, then $\liminf_{\varepsilon \to 0} C(\varepsilon, L, T) = +\infty$.
- If $T > KL$, then $\limsup_{\varepsilon \to 0} C(\varepsilon, L, T) = 0$, for K large enough.

$K = 4.3$ [Coron Guerrero '05], $K = 4.2$ [Glass '10],
$K = 2\sqrt{3}$ ($\simeq 3.46$) [Lissy '12]. $K = 4\sqrt{K_0}$ ($\simeq 3.34$) [J.D. S.E.]

$\longrightarrow$ Optimal K ?

Thank you for your attention!

Ref1: On the reachable set for the one-dimensional heat equation.
Jérémi Dardé, Sylvain Ervedoza, 2016.

Ref2: On the cost of observability in small times for the one-dimensional heat equation.
Jérémi Dardé, Sylvain Ervedoza, 2017, in preparation.