NUMERICAL NULL CONTROLLABILITY OF THE HEAT EQUATION THROUGH A VARIATIONAL APPROACH

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Abstract. This work is concerned with the numerical computation of null controls for the heat equation. The goal is to compute a control that drives (an approximation of) the solution from a prescribed initial state at $t = 0$ to zero at $t = T$.

In spite of the diffusion of the heat equation, recent developments indicate that this issue is difficult and still largely open. Most of the existing literature, concerned with controls of minimal $L^2$-norm, make use of dual convex arguments and introduce backward adjoint system. In practice, the null control problem is then reduced to the minimization of a dual conjugate function with respect to the final condition of the adjoint state. As a consequence of the highly regularizing property of the heat kernel, this final condition - which may be seen as the multiplier for the null controllability condition - does not belong to $L^2$, but to a much larger space than can hardly be approximated by finite (discrete) dimensional basis. This phenomenon, unavoidable whatever be the numerical approximation used, strongly deteriorates the efficiency of minimization algorithms.

In this work, we do not use duality arguments and in particular do not introduce any backward heat equation. For the boundary case, the approach consists, first to introduce a class of function satisfying $a priori$ the boundary conditions in space and time - in particular the null controllability condition at time $T$ - and then find among this class one element satisfying the heat equation. This second step is done by minimizing a convex functional, among the admissible corrector functions of the heat equation. The inner case is performed in a similar way.

We present the (variational) approach, discuss the main features of it, and then describe some numerical experiments highlighting the interest of the method.

The method holds in any dimension but, for the sake of simplicity, we provide details in the one-space dimensional case.

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1. Introduction

We are concerned in this work with the null controllability problem for the 1D heat equation for both the boundary and the inner case. We denote by \( T \) any strictly positive real, \( \omega \) any non-empty (small) subset of \((0,1)\) and \( 1_\omega \) the characteristic function of \( \omega \). We introduce the diffusion function \( a \) assumed to be uniformly bounded and strictly positive all over the interval \((0,1)\):

\[
a \in C^1([0,1]), \quad a(x) \geq a_0 > 0 \quad \forall x \in [0,1].
\]

We also introduce in the sequel the notation

\[
q_T = \omega \times (0,T), \quad Q_T = (0,1) \times (0,T), \quad \Sigma_T = \{1\} \times (0,T).
\]

The boundary control problem we consider here can be stated as follows: given any initial data \( u_0 \in L^2(0,1) \), find a control function \( w \in L^2(\Sigma_T) \) such that the unique solution \( u \in C^0([0,T]; H^{-1}(0,1)) \cap L^2(0,T; L^2(0,1)) \) of the homogeneous linear equation

\[
\begin{aligned}
   u_t - (a(x)u_x)_x &= 0 & (x,t) \in Q_T, \\
   u(0,x) &= u_0(x) & x \in (0,1), \\
   u(t,0) &= 0, u(t,1) = w(t) & t \in (0,T)
\end{aligned}
\]

satisfies the null controllability condition

\[
u(\cdot, T) = 0 \quad \text{in} \quad (0,1).
\]

Similarly, the inner (or distributed) control problem may be stated as follows: given any initial data \( u_0 \in L^2(0,1) \), find a control function \( f \in L^2(q_T) \) such that the unique solution \( u \in C^0([0,T]; L^2(0,1)) \cap L^2(0,T; H^1(0,1)) \) of the homogeneous linear equation

\[
\begin{aligned}
   u_t - (a(x)u_x)_x &= f 1_\omega & (x,t) \in Q_T, \\
   u(0,x) &= u_0(x) & x \in (0,1), \\
   u(t,0) &= u(t,1) = 0 & t \in (0,T)
\end{aligned}
\]

satisfies \((1.4)\).

In the one dimensional space case, those controllability problem are known to hold since the seventies: we refer to the earlier contributions \([9, 22]\) for some proof based on spectral arguments. For more recent and general results based on duality arguments and Carleman type estimates, we refer to \([10, 17]\). As is usual in this type of problems, the dual approach allows to reduce the controllability problem to a suitable observability result for the adjoint system. Moreover, in the spirit of the celebrated Hilbert Uniqueness Method introduced by J.-L. Lions, they lead to a practical way to compute controls of a given minimal Sobolev norm.

In order to highlight the underlying difficulties that motivate the search of new methods, let us consider the inner case, which is simpler in many ways with respect to its boundary counterpart. Since there are controls \( f \in L^2(q_T) \) for \((1.5)\), it is natural to look for the one with minimal \( L^2 \)-norm, that is, one seeks to minimize the quadratic functional \( J(v) = \frac{1}{2} \|v\|_{L^2(q_T)}^2 \) over the non-empty set

\[
C(u_0, T) = \{ (u, f) : f \in L^2(q_T), u \text{ solves (1.5)} \text{ and satisfies (1.4)} \}.
\]

Since it is difficult to construct pairs in \( C(u_0, T) \) (and \textit{a fortiori} minimizing sequences !), one may used, following \([3]\), duality arguments to replace the constrained minimization of \( J \) by the unconstrained minimization of its conjugate function \( J^* \) defined as

\[
J^*(\varphi_T) = \frac{1}{2} \int_{q_T} |\varphi|^2 \, dx \, dt + \int_0^1 u_0(x) \varphi(x,0) \, dx
\]
over \( \varphi_T \in \mathcal{H} \), where \( \varphi \) is the adjoint backward state associated with (1.3) such that \( \varphi(\cdot, T) = \varphi_T \). The existence of a positive constant \( C = C(\omega, T) \) (the so-called observability constant) such that
\[
\|\varphi(\cdot, 0)\|_{L^2(\omega, 1)}^2 \leq C(\omega, T)\|\varphi\|_{L^2(\Omega_T)}^2
\]
for all \( \varphi_T \in L^2(0, 1) \) implies that \( J^* \) is coercive on the Hilbert space \( \mathcal{H} \) defined as the completion of \( D(0, 1) \) for the norm \( \|\varphi\|_{L^2(\Omega_T)} \). The control \( f \) of minimal \( L^2(\Omega_T) \)-norm is then given by \( f = \hat{\varphi} L \), where \( \hat{\varphi} \) is associated with the unique minimizer \( \hat{\varphi}_T \) in \( \mathcal{H} \) of \( J^* \). The difficulty, when one wants to approximate such control, is that when one likes to minimize numerically \( J^* \), is that the space \( \mathcal{H} \) is huge, in particular, contains \( H^{-s} \) for every \( s \in \mathbb{N} \), and elements that may not be distributions. Numerical experiments do suggest that the minimizer \( \hat{\varphi}_T \) is very singular (we refer to [3] and also to [20] for more details). Notice that this phenomenon is independent of the choice of \( J \), but is related to the use of dual variables. As we stressed in the abstract, the equality (1.4) can be viewed as an equality in a very small space (due to the strong regularization effect of the heat kernel). Accordingly, the associated multiplier \( \varphi_T \) must belong to a large dual space, much larger than \( L^2(0, 1) \), that cannot be represented numerically. We refer to [7], generalizing [3] for weighted-norms, where the same ill-posedness is shown.

Recently, an alternative way of looking at these problems and avoiding the introduction of dual variables has been introduced in [21]. It is based on the following simple strategy. Instead of working all the time with solutions of the underlying state equation, and looking for one that may comply with the final desired state, one considers a suitable class of functions complying with required initial, boundary, and final conditions, and seeks one of those that is a solution of the state equation. This is in practice accomplished by setting up an error functional defined for all feasible functions, and measuring how far those are from being a solution of the underlying state equation. The task of showing that a problem is controllable amounts to proving that the infimum of the error is a minimum (there is a global minimizer of the error), and that it vanishes. This job requires some interesting analysis as the error functional is not a local, classical integral functional but rather a non-local functional as the ones consider for optimal control for distributed parameter systems ([18]). Once we have a feasible function with zero error, the control is obtained as the trace of this optimal function (or some other function determined in a unique way through it) in the set where we are entitled to act on the system.

One main practical advantage of this variational approach is that the way to get closer to a solution of the problem is by minimizing a functional that cannot get stuck on local minima because the only critical points of the error turn out to be global minimizers with zero error (see next section). Therefore a general strategy for numerical approximation consists in using a typical descent algorithm for this error functional. Exploring this possibility for the problems described above is the main reason for this paper. It is organized as follows. In Section 2, we describe (in a non-technical way) the main ingredients of the variational approach for the heat equation for the boundary case first, following [21]. We then show that [21] may be adapted to address the inner situation. We then move on to provide the details for the numerical approach based on the Polak-Ribiére version of the conjugate gradient algorithm to minimize the error functional. Section 4 presents several experiments and discuss the practical interest of the approach. Section 5 treats a typical non-linear example to stress the flexibility of the approach. The final section provides a simple method to reduce the cost of controls.

To our knowledge, very few contribution on that topic has appeared since the seminal paper of Carthel-Glowinski-Lions [3] devoted to approximate controllability using duality. This is due to the intrinsic ill-posedness of the problem we have just pointed out. For the null boundary case in one dimensional space, we mention the motion planning method introduced in [16] allowing a semi-explicit expression of controlled solutions in term of Gevrey series. This approach has been adapted and numerically developed recently in [20] to obtain inner controls. The recent works [2] - following [10] - extend [3] with Carleman weighted \( L^2 \)-norm. For a numerical analysis viewpoint, we also indicate readers contributions [2] [15].
2. The variational approach of the null controllability

We are going to describe in this section the basic ingredients of the variational approach in order to apply it to boundary, and inner controllability problems for the 1D heat equation.

2.1. Boundary controllability. Consider first the boundary controllability problem for the heat equation which consists in finding a function \( w \in L^2(\Sigma_T) \), such that the solution of the problem

\[
\begin{cases}
  u_t - (a(x)u_x)_x = 0, & (x, t) \in Q_T, \\
  u(x, 0) = u_0(x), & x \in (0, 1), \\
  u(0, t) = 0, u(1, t) = w(t), & t \in (0, T)
\end{cases}
\]

will comply with \( u(x; T) = 0 \) in \((0, 1)\), so that the state \( u \) with initial distribution given by initial data \( u_0 \) is led to state 0 at time \( T \) under the action of the boundary control \( w \) at the right-end point \( x = 1 \). The data \( u_0 \) is given \textit{a priori}, and the function \( a \) is assumed to be uniformly bounded and strictly positive all over the interval \((0, 1)\).

The main idea of the variational method, as introduced in [21], consists in setting up an error functional which measures the deviation of functions from being a solution of the underlying heat equation, and minimizing such error over the class of feasible functions that comply with initial, boundary, and final conditions. Namely, consider the class of functions

\[ \mathcal{A} = \left\{ u \in H^1(Q_T) : u(x, 0) = u_0(x), u(x, T) = 0, x \in (0, 1), u(0, t) = 0, t \in (0, T) \right\}. \]

assumed non empty. This requirement simply demands some compatibility with the vanishing boundary data for \( x = 0 \), precisely that \( u_0(0) = 0 \) and that \( u_0 \), as the trace of an \( H^1 \) function over \( Q_T \), be slightly more regular than \( L^2(0, 1) \), that is \( u_0 \in H^{1/2}(0, 1) \). According to the regularizing effect of the heat kernel, this assumption may be removed if we assume that the control is zero at time \( t = 0 \). For any \( u \in \mathcal{A} \), we define its corrector \( v \) over \( Q_T \) as the solution of the (elliptic) problem

\[
\begin{cases}
  u_t - v_{tt} - (a(x)(u_x + v_x))_x = 0, & (x, t) \in Q_T, \\
  v_t(x, 0) = v_t(x, T) = 0, & x \in (0, 1), \\
  v(0, t) = v(1, t) = 0, & t \in (0, T).
\end{cases}
\]

(2.1)

Notice that the unique solution of this problem is the minimizer over \( H^1_0(Q_T) = \{ v \in H^1(Q_T), v = 0 \text{ on } (0, 1) \times (0, T) \} \) of the regular quadratic functional

\[
\frac{1}{2} \int_{Q_T} \left( |v_t|^2 + a(x)|v_x|^2 \right) + u_t v + a(x) u_x v_x \, dx \, dt.
\]

The Neumann conditions on the part of the boundary for \( t = 0 \), and \( t = T \), are the natural boundary one. One may also consider Dirichlet conditions. Note how this variational problem determining the corrector \( v \) is a well-defined problem if \( u \in \mathcal{A} \). Even though the corrector function \( v \) was introduced in [21] for each time slice \( t \) to preserve as general a framework as possible, from the point of view of numerical approximation it is advantageous to define such error function globally in the whole time-space domain \( Q_T \) by introducing the additional term \(-v_{tt}\). This has a regularizing effect on the time dependence which is very convenient for numerics.

The error functional is then

\[
E : \mathcal{A} \to \mathbb{R}^+, \quad E(u) = \frac{1}{2} \int_{Q_T} \left( |v_t|^2 + a(x)|v_x|^2 \right) \, dx \, dt,
\]

where \( v \) is the corrector associated with \( u \). It turns out that our problem is controllable if and only if the minimum of the error vanishes. This amounts to proving two facts:

- the infimum of the error \( m \geq 0 \) is attained;
- it vanishes \( m = 0 \).
The first part is achieved by using the direct method, and requires to deal with a quadratic, but possibly degenerate, functional. There is no special difficulty here except for showing that there are bounded minimizing sequences in spite of lack of coercivity. This was also a difficulty in [21] that was resolved by using the linear dependence of \( v \) on \( u \), and the quadratic nature of the error functional, together with an easy lemma to overcome this lack of coercivity. The interesting point to stress is that this lack of coercivity means that not all minimizing sequences will be uniformly bounded, as they would under coercivity, but it suffices to show that there are some bounded minimizing sequences. More specifically, we have the following general fact, which was expressed in a different, and possibly less clear, way in [21].

**Lemma 2.1.** Let \( E : X \to \mathbb{R} \) be a quadratic, convex, non-negative functional defined on a Hilbert space \( X \). Let \( H \) be a closed subspace of \( X \). Then, for arbitrary \( u_0 \in X \), \( E \) always attains its infimum over the manifold \( u_0 + H \).

**Proof.** Without loss of generality, we can assume that \( \inf_{u_0 + H} E = 0 \), for otherwise we replace \( E \) by \( E - m \) if \( m = \inf_{u_0 + H} E > 0 \). Let \( \{ u_j \} \subset H \) be such that \( E(u_0 + u_j) \searrow 0 \), and suppose there is a unitary \( v \in H \) so that \( (u_j, v) \to \infty \). Let \( v^\perp \) be the orthogonal complement of \( \{ v \} \) in \( H \), and let \( \pi \) be the orthogonal projection onto \( v^\perp \). Then \( u_j = \pi u_j + (u_j, v)v \), and

\[
E(u_0 + \pi u_j + (u_j, v)v) = E(u_0 + u_j) \searrow 0.
\]

But since \( E \) is quadratic, and \( (u_j, v) \to \infty \), we should necessarily have

\[
E(v) = 0.
\]

But in this case, because \( E \) cannot be negative,

\[
E(u_0 + \pi u_j) = E(u_0 + u_j) \searrow 0
\]

so that \( u_0 + \pi u_j \) is also minimizing, but has no component in the \( v \)-direction. The arbitrariness of \( v \) implies that there should be bounded minimizing sequences for \( E \), and hence, because of its convexity, it attains its infimum somewhere in \( u_0 + H \). \( \square \)

The application of this lemma in our setting is straightforward, so that the error in (2.2) attains its minimum over \( \mathcal{A} \).

Once we know that the infimum \( m \geq 0 \) is a minimum, we turn to optimality. We introduce the set of admissible variations of \( u \) as follows:

\[
\mathcal{A}_0 = \left\{ U \in H^1(Q_T) : U(x,0) = U(x,T) = 0, x \in (0,1), U(0,t) = 0, t \in (0,T) \right\}
\]

and then define, in a classical way, the variation of \( E \) in the direction \( U \in \mathcal{A}_0 \)

\[
\langle E'(u), U \rangle = \lim_{t \to 0} \frac{E(u + tU) - E(u)}{t}.
\]

We easily obtain that

\[
\langle E'(u), U \rangle = \int_{Q_T} (v_t V_t + a(x)v_x V_x) \, dx \, dt
\]

where \( V \in H^1_{0,x}(Q_T) \) is the corrector function associated with \( U \in \mathcal{A}_0 \), that is, the solution of

\[
\begin{align*}
U_t - V_{tt} - (a(x)(U_x + V_x))_x = 0, & \quad (x,t) \in Q_T, \\
V_t(x,0) = V_t(x,T) = 0, & \quad x \in (0,1), \\
V(0,1) = V(1,t) = 0, & \quad t \in (0,T).
\end{align*}
\]

Multiplying the state equation (2.4) by \( v \), integrating by parts, and taking into account the boundary conditions on \( v \) and \( U \), we transform (2.3) into

\[
\langle E'(u), U \rangle = - \int_{Q_T} (U_t v + a(x)U_x v_x) \, dx \, dt, \quad \forall U \in \mathcal{A}_0.
\]
Now, let us assume that \( u \in \mathcal{A} \) is a minimizer for \( E \), so that \( < E'(u), U > = 0 \) for all \( U \in \mathcal{A}_0 \). This equality implies that \( v \) satisfy the backward heat equation
\[
\begin{aligned}
- v_t - (a(x)v_x)_x &= 0, \quad (x,t) \in Q_T, \\
(1)v_x(t,1) &= 0, \quad t \in (0,T),
\end{aligned}
\]
in addition to the boundary conditions
\[
\begin{aligned}
v_t(x,0) &= v_t(x,T) = 0, \quad x \in (0,1), \\
v(0,t) &= v(1,t) = 0, \quad t \in (0,T).
\end{aligned}
\]
For any positive time \( T > 0 \), this implies, by the unique continuation property, that the corrector \( v \) is the solution of an elliptic linear and well-posed problem in \( H^1(Q_T) \). We refer to [7] where a different variational approach leading to an elliptic problem defined on \( Q_T \) has been introduced and analyzed.

Notice that, even though there might not be rigorous results to be applied for some particular situation, the decrease of the error to zero is a sure indication that the problem is being controlled.

**Remark 2.2.** There are many ways to define the corrector \( v \). One may, for instance, replace the state equation of (2.1) by the following equation
\[
\begin{aligned}
\begin{array}{l}
u_t = v_{tt} - (a(x)v_x)_x = 0, \quad (x,t) \in Q_T \\
u(0,t) = v(1,t) = 0, \quad t \in (0,T)
\end{array}
\end{aligned}
\]
leading to \( E(u) = \frac{1}{2} \int_{Q_T} (v_t^2 + |v_x|^2) \, dx \, dt \), and the same expression of the first derivative. The choice we made in (2.1) seems the closest to the notion of a corrector for the heat equation.

### 2.2. Inner controllability

Let us now turn to the inner controllability case. This time we assume that the control is acting on a small subset \( \omega \) (for simplicity assumed independent of the time variable) of \( (0,1) \). A first constructive approach applying the previous section could be as follows:

- First, compute the boundary controls acting on \( \partial \omega \times (0,T) \) and driving the solution of the heat equation to rest on the part \( (0,1) \setminus \omega \) at \( T \);
- then, compute the inner control acting on the whole domain \( q_T \) by driving to rest the solution on the part \( \omega \times T \), taking as boundary Dirichlet condition on \( \partial q_T \) the boundary controls of the first step. The inner control acting on \( q_T \) is then a null control for the whole solution at time \( T \).

Let us give a different approach following the ideas of the previous section. We put
\[
\mathcal{A} = \left\{ u \in H^1(Q_T) : u(0,x) = u_0(x), u(x,T) = 0, x \in (0,1), u(0,t) = u(1,t) = 0, t \in (0,T) \right\}
\]
and \( \mathcal{A}_0 = H^1_0(Q_T) \), that is to say, simply \( \mathcal{A}_0 = H^1_0(Q_T) \). In order to ensure that the solution \( u \) satisfies the homogeneous heat equation off \( q_T = \omega \times (0,T) \), we consider the following error functional
\[
E(u) = \frac{1}{2} \int_{Q_T \setminus q_T} \left( |v_t|^2 + a(x)|v_x|^2 \right) \, dx \, dt
\]
where the corrector \( v \) is defined in two pieces:
(1) off $q_T$:

\[
\begin{aligned}
  u_t - v_t - (a(x)(u_x + v_x))_x &= 0, \\
  v &= 0, \\
  v_t(x,0) &= v_t(x,T) = 0,
\end{aligned}
\tag{2.6}
\]

(2) in $q_T$:

\[
\begin{aligned}
  u_t - v_t - (a(x)(u_x + v_x))_x &= 0, \\
  v &= 0, \\
  v_t(x,0) &= v_t(x,T) = 0,
\end{aligned}
\tag{2.7}
\]

Notice that the error associated with a feasible $u$ only depends on the corrector $v$ off $q_T$. However, we would like to be able to update $u$ also inside $q_T$ so as to lead (through the corrector) the error to zero. Then, proceeding as before and using that the corrector $v$ is zero on $\partial q_T$, we obtain that the first variation of $E$ in any direction $U \in \mathcal{A}_0$ is given by

\[
< E'(u), U > = - \int_{Q_T \setminus q_T} (U_t v + a(x)U_x v_x) \, dx \, dt, \quad \forall U \in \mathcal{A}_0.
\]

Notice that the boundary terms on $\partial q_T$, due to the integration by parts in time, vanish thanks to the assumption $v = 0$ on $\partial q_T$ in (2.6). Then, writing that $u$ is critical point of the error, we are left with

\[
\begin{aligned}
  &- (1_{Q_T \setminus q_T} v)_t - (1_{Q_T \setminus q_T} a(x)v_x)_x = 0, \\
  &v = 0, \\
  &v_t(x,0) = v_t(x,T) = 0,
\end{aligned}
\tag{2.8}
\]

Together with the Neumann boundary condition $v_t(\cdot,0) = v_t(\cdot,T) = 0$, (2.8) implies, for any $T > 0$ that $v \equiv 0$ in $Q_T \setminus q_T$, and therefore $E(u) = 0$. Finally, such critical point $u \in \mathcal{A}$ satisfies the following equation

\[
u_t - ((a(x)u_x)_x = (v_t + (a(x)v_x)_x) 1_{q_T}, \quad (x,t) \in Q_T.
\]

Therefore, a control acting on $q_T$ constructed in this way is given by $v_t + (a(x)v_x)_x 1_{q_T}$, $v$ being the corrector associated with the critical point $u$ inside $q_T$. This control belongs at least to $H^{-1}(q_T)$.

**Remark 2.3.** Since the controllability problem is formulated in $Q_T$, we may consider, without further change, the case where the support of the control depends on the time variables, i.e. $q_T = \{(x,t) \in Q_T : g(t) < x < h(t), t \in (0,T)\}$ where $g$ and $h$ are two smooth functions on $[0,T]$ with $0 < g \leq h \leq 1$, $g(t) \neq h(t)$. We refer to [17] for some experiments using a different variational approach.

### 3. Numerical resolution of the minimization problem

As shown in the previous section, the practical search of controls for the heat equation may be reduced to minimization for corrector problems. We describe in this section the minimization procedure to approximate numerically correctors. We give the details for the boundary case, and then point out the main differences for the inner counterpart.

For the boundary case, we have to solve

\[
\begin{aligned}
\text{Minimize} \quad & E(u) = \frac{1}{2} \int_{Q_T} (|v_t|^2 + a(x)|v_x|^2) \, dx \, dt, \\
\text{subject to} \quad & u \in \mathcal{A}
\end{aligned}
\tag{3.1}
\]

with $\mathcal{A} = \{u \in H^1(Q_T); u(x,0) = u_0 \text{ on } (0,1), u(\cdot, T) = 0 \text{ on } (0,1), u(x, t) = 0 \text{ on } \Sigma_T\}$. We endowed the Hilbert space $\mathcal{A}$ with the scalar product

\[
< u, v >_{\mathcal{A}} = \int_{Q_T} (u_t v_t + a(x)u_x v_x + u v) \, dx \, dt, \quad \forall u, v \in \mathcal{A}
\]
and note that $\|u\|_{\mathcal{A}} = \sqrt{\langle u, u \rangle_{\mathcal{A}}}$ for all $u \in \mathcal{A}$. $\mathcal{A}_0$ is endowed with the same scalar product.

3.1. Conjugate gradient algorithm. The Polak-Ribiére version of the conjugate gradient (CG) algorithm to minimize $E$ over $\mathcal{A}$ is as follows (see [13]):

- **Step 0: Initialization** Given any $\varepsilon > 0$ and any $u^0 \in \mathcal{A}$, compute the residual $g^0 \in \mathcal{A}_0$ solution of

  $$(g^0, U)_{\mathcal{A}} = \langle E'(u^0), U \rangle > \forall U \in \mathcal{A}_0.$$ 

  If $\|g^0\|_{\mathcal{A}} \leq \varepsilon$ take $u = u^0$ as an approximation of a minimum of $E$. Otherwise, set $z^0 = g^0$.

  For $n \geq 0$, assuming $u^n, g^n, z^n$ being known with $g^n$ and $z^n$ both different from zero, compute $u^{n+1}, g^{n+1}$, and if necessary $z^{n+1}$ as follows:

- **Step 1: Steepest descent** Set $u^{n+1} = u^n - \lambda_n z^n$ where $\lambda_n \in \mathbb{R}$ is the solution of the one-dimensional minimization problem

  $$\minimizes_{\lambda \in \mathbb{R}} E(u^n - \lambda z^n).$$

  Then, compute the residual $g^{n+1} \in \mathcal{A}_0$ from the relation

  $$(g^{n+1}, U)_{\mathcal{A}} = \langle E'(u^{n+1}), U \rangle > \forall U \in \mathcal{A}_0.$$ 

- **Step 2: Convergence testing and construction of the new descent direction.** If $\|g^{n+1}\|_{\mathcal{A}} \leq \varepsilon$ take $u = u^{n+1}$; otherwise compute

  $$\gamma_n = \frac{\langle g^{n+1}, g^{n+1} - g^n \rangle_{\mathcal{A}}}{\langle g^n, g^n \rangle_{\mathcal{A}}}, \quad z^{n+1} = g^{n+1} + \gamma_n z^n.$$

  Then do $n = n + 1$, and return to step 1.

Let us provide more details for two important steps of the algorithm:

- Since $E$ is a quadratic functional with respect to $u$, one may explicitly solve the problem (3.2): we write

  $$E(u^n - \lambda z^n) = E(u^n) - \lambda \int_{Q_T} (v^n Z^n_t + a(x) v^n Z^n_x) \, dx \, dt$$

  $$+ \frac{\lambda^2}{2} \int_{Q_T} (|Z^n|^2 + a(x) |Z^n_x|^2) \, dx \, dt$$

  where $Z^n_x$ is the corrector of $z^n$, solution of

  $$\begin{cases}
  z^n_t - Z^n_{tt} - (a(x)(z^n + Z^n_t))_x = 0, & (x, t) \in Q_T, \\
  Z(\cdot, t) = 0, & \Sigma_t, \quad Z_\Sigma(x, \cdot) = 0, & \Sigma_x
  \end{cases}$$

  so that the optimal parameter is given by

  $$\lambda_n = \frac{\int_{Q_T} (v^n Z^n_t + a(x) v^n Z^n_x) \, dx \, dt}{\int_{Q_T} (|Z^n_t|^2 + a(x) |Z^n_x|^2) \, dx \, dt} = \frac{\int_{Q_T} (z^n u^n + a(x) z^n v^n_x) \, dx \, dt}{\int_{Q_T} (|Z^n_t|^2 + a(x) |Z^n_x|^2) \, dx \, dt}.$$ 

- The computation of the residual $g^n$ is performed as follows. According to the equality

  $$\langle E'(u^n), U \rangle = - \int_{Q_T} (U_t v^n + a(x) U_x v^n_x) \, dx \, dt, \quad \forall U \in \mathcal{A}_0,$$

  $E'(u^n) \in H^{-1}(Q_T)$ may be identified with the linear functional on $\mathcal{A}_0$ defined by

  $$U \rightarrow - \int_{Q_T} (U_t v^n + a(x) U_x v^n_x) \, dx \, dt.$$
It then follows that $g^n$ is the solution of the following linear variational problem: find $g^n \in A_0$ such that
\[
\int_{Q_T} (g^n_t U_t + a(x)g^n_x U_x + g^n U) \, dx \, dt = -\int_{Q_T} (U_t v^n + a(x)U_x v^n_x) \, dx \, dt, \quad \forall U \in A_0,
\]
where $v^n \in H^1_{0,x}(Q_T)$ is the corrector associated with $u^n$. The well-posed elliptic boundary value problem corresponding to this variational formulation is
\[
\begin{cases}
- g^n_{tt} - (a(x)g^n_x)_x + g^n = v^n_t + (a(x)v^n)_x & (x, t) \in Q_T \\
g^n(0, t) = 0, \quad g^n_x(1, t) = 0, & t \in (0, T) \\
g^n(x, 0) = g^n(x, T) = 0, & x \in (0, 1).
\end{cases}
\]
(3.4)

**Remark 3.1.** As we mentioned above, the parameter $\gamma_n$ given by (3.3) corresponds to the Polak-Ribiére version of the conjugate gradient algorithm. In the present quadratic-linear situation, this one should coincide with the Fletcher-Reeves conjugate algorithm for which
\[
\gamma_n = \|g^{n+1}\|^2_A / \|g^n\|^2_A,
\]
since gradients are conjugate to each other ($(g^m, g^n)_A = 0$ for all $m \neq n$). However, we observed that in the parabolic situation (see also [6]) the Polak-Ribiére version (mainly used in nonlinear situations) allows to reduce the loss of the orthogonality, due to the numerical approximation.

The detailed conjugate gradient scheme, written in a variational form, used for the minimization of $E$ is then as follows:

**Step 0: Initialization** $u^0 \in A$ be given, compute the corrector $v^0 \in H^1_{0,x}(Q_T)$ of $u^0$ solution of
\[
\int_{Q_T} (v^n_t \phi_t + a(x)v^n_x \phi_x) \, dx \, dt = -\int_{Q_T} (u^n_t \phi + a(x)u^n_x \phi_x) \, dx \, dt, \quad \forall \phi \in H^1_{0,x}(Q_T),
\]
then compute the gradient $g^0 \in A_0$ solution of
\[
\int_{Q_T} (g^n_t \phi_t + a(x)g^n_x \phi_x + g^n \phi) \, dx \, dt = -\int_{Q_T} (v^n_t \phi + a(x)v^n_x \phi_x) \, dx \, dt, \quad \forall \phi \in A_0,
\]
and set $z^0 = g^0$.

Then, for $n \geq 0$, assuming $u^n, \ n = n, \ z^n, \ v^n$ known, compute $u^{n+1}, n = n+1, z^{n+1}$ and $v^{n+1}$ by:

**Step 1: Steepest descent** Compute the corrector $Z^n \in H^1_{0,x}(Q_T)$ of $z^n$ solution
\[
\int_{Q_T} (Z^n_t \phi_t + a(x)Z^n_x \phi_x) \, dx \, dt = -\int_{Q_T} (z^n_t \phi + a(x)z^n_x \phi_x) \, dx \, dt, \quad \forall \phi \in H^1_{0,x}(Q_T),
\]
and set $u^{n+1} = u^n - \lambda_n z^n \in A$ with
\[
\lambda_n = -\frac{\int_{Q_T} (z^n v^n + a(x)z^n_x v^n_x) \, dx \, dt}{\int_{Q_T} (|z^n|^2 + a(x)|Z^n_x|^2) \, dx \, dt}.
\]
Next, compute the corrector $v^{n+1} \in A_1$ of $u^{n+1}$ solution of
\[
\int_{Q_T} (v_t^{n+1} \phi_t + a(x)v_x^{n+1} \phi_x) \, dx \, dt = -\int_{Q_T} (u^{n+1}_t \phi + a(x)u^{n+1}_x \phi_x) \, dx \, dt, \quad \forall \phi \in H^1_{0,x}(Q_T),
\]
and the gradient $g^{n+1} \in A_0$ solution of
\[
\int_{Q_T} (g^{n+1}_t \phi_t + a(x)g^{n+1}_x \phi_x + g^{n+1} \phi) \, dx \, dt = -\int_{Q_T} (v^{n+1}_t \phi + a(x)v^{n+1}_x \phi_x) \, dx \, dt, \quad \forall \phi \in A_0.
\]

**Step 2: Construction of the new descent direction.** If $\|g^{n+1}\|_A / \|g^0\|_A \leq \varepsilon$, take $u = u^{n+1}$; otherwise compute
\[
\gamma_n = \frac{\langle g^{n+1}, g^{n+1} - g^n \rangle_A}{\langle g^n, g^n \rangle_A}, \quad z^{n+1} = g^{n+1} + \gamma_n z^n.
\]
Then do $n = n + 1$, and return to step 1.

Once the convergence of the algorithm is reached, up to the threshold $\varepsilon$, we take the trace of $u$ on $\Sigma_T$ to define an approximation of the control $w$ of (1.3): $w(t) = u(1, t), t \in (0, T)$. We next compute an approximation of the controlled solution $u$ by solving (1.3): the $L^2$-norm $\|u(1, T)\|_{L^2(0, 1)}$, that may be seen as an a posteriori error, allows to evaluate the efficiency of the approach.

The minimization of the functional $E$ related to the inner case (see (2.5)) is very similar. The main difference is that the corrector have to be solved independently in and off $q_T$ (see (2.6) and (2)). The additional condition is $v = 0$ on $\partial_\omega \times (0, T)$. It is important to note that these correctors are linked through the descent direction $g^p$, solution of the problem posed in all of the domain $Q_T$:

\begin{equation}
\int_{Q_T} (g^p_i U_t + a(x) g^0_i U_x + g^0 U) \, dx \, dt = - \int_{Q_T} (U_t v^0 + a(x) U_x v^0) \, dx \, dt, \quad \forall U \in \mathcal{A}_0 = H^1_0(Q_T).
\end{equation}

3.2. Numerical approximation. For “large” integers $N_x$ and $N$, we set $\Delta x = 1/N_x$, $\Delta t = T/N$, and $h = (\Delta x, \Delta t)$. Let us denote by $\mathcal{P}_{\Delta x}$ the uniform partition of $[0, 1]$ associated with $\Delta x$, and let us denote by $\mathcal{Q}_h$ the uniform quadrangulation of $Q_T$ associated with $h$. In particular,

$\mathcal{Q}_T = \bigcup_{K \in \mathcal{Q}_h} K$.

The following (conformal) finite element approximation of $H^1(Q_T)$ is introduced:

$X_h = \{ \varphi_h \in C^0([0, 1] \times [0, T]) : \varphi_h|_K \in \{ P_m, x \otimes P_1, 1 \}(K) \text{ \forall } K \in \mathcal{Q}_h \}$.

Here, $P_{m, x}$ denotes the space of polynomial functions of order $m$ in the variable $x$. Accordingly, the functions in $X_h$ reduce on each quadrangle $K \in \mathcal{Q}_h$ to a polynomial of the form $A + Bx + Ct + Dxt$ involving 4 degrees of freedom. Obviously, the space $X_h$ is a conformal approximation of $L^2(Q_T)$. We will also consider the space

$X_{0h} = \{ \varphi_h \in X_h : \varphi_h(0, t) = \varphi_h(1, t) = 0 \text{ \forall } t \in (0, T) \}$, $X_{uh} = \{ \varphi_h \in X_h : \varphi_h(0, t) = 0 \text{ \forall } t \in (0, T), \varphi_h(x, 0) = u_0(x), \varphi_h(x, T) = 0 \text{ \forall } x \in (0, 1) \}$.

$X_{0h}$ and $X_{uh}$ are finite-dimensional subspaces of $A$ and $H^1_{0, x}(Q_T)$, respectively (and also of $L^2(0, T; H^1(0, 1))$). The functions $\varphi_h \in X_{0h}$ are uniquely determined by their values at the nodes $(x_j, t_j)$ of $\mathcal{Q}_h$ such that $0 < x_j < 1$.

Therefore, for any $h$, we consider the following problem, which is an approximation of (3.1):

\begin{equation}
\begin{cases}
\text{Minimize} & E_h(u_h) = \frac{1}{2} \int_{Q_T} (|v_{h,t}|^2 + a(x)|v_{h,x}|^2) \, dx \, dt, \\
\text{subject to} & u_h \in X_{uh}.
\end{cases}
\end{equation}

According to the conjugate gradient algorithm, this minimization problem is reduced to the resolution of well-posed elliptic problems defined on $Q_T$ in order to compute corrector functions $v_h \in X_{0h}$.

Once the optimal function $u_h$, minimizer of $E$ over $X_h$ is obtained, the control $w_h$ is defined by $w_h = u_h$ on $\Sigma_T$. In order to check the quality of the control $w_h$, piecewise linear along $\Sigma_T$, one may compute such solution in $X_{uh}$, with the solution $\overline{u}_h$ of (1.3) starting from $u_0$ at time $t = 0$ and such that $\overline{u}_h = u_{0, \Delta x}$. $\overline{u}_h$ is computed using, for the time discretization, the two-step implicit Gear scheme of order two in time (see for instance [14]). We set

$\Phi_{\Delta x} = \{ z \in C^0([0, 1]) : z|_k \in \mathcal{P}_{1, x}(k) \text{ \forall } k \in \mathcal{P}_{\Delta x} \}$,

a finite dimensional subspace of $L^2(0, 1)$. Functions in $\Phi_{\Delta x}$ are uniquely determined by their values at the nodes of $\mathcal{P}_{\Delta x}$.

The Gear scheme, which is of order two, is then combined with a $\mathcal{P}_1$-finite element discretization in space as follows:

1. We first set $u_h|_{t=0} = u_{0, \Delta x}$.

(2) Secondly, \( u_h|_{t=t_1} \) is the solution of the linear problem in \( \Psi \in \Phi_{\Delta x} \)
\[
\begin{cases}
  \int_0^1 \frac{1}{\Delta t} (\Psi - \overline{u}_h|_{t=0}) z \, dx + \frac{1}{2} \int_0^1 a(x)(\Psi + \overline{u}_h|_{t=0})_x z_x \, dx = 0 \\
  \forall z \in \Phi_{\Delta x}.
\end{cases}
\]

(3) For given \( n = 2, \ldots, N_\tau - 1, \) \( \Psi^* = \overline{u}_h|_{t=t_{n-1}} \) and \( \overline{\Psi} = \overline{u}_h|_{t=t_n}, \overline{u}_h|_{t=t_{n+1}} \) is the solution of the linear problem in \( \Psi \in \Phi_{\Delta x} \)
\[
\begin{cases}
  \int_0^1 \frac{1}{2\Delta t} (3\Psi - 4\overline{\Psi} + \Psi^*) z \, dx + \int_0^1 a(x)\Psi_x z_x \, dx = 0 \\
  \forall z \in \Phi_{\Delta x}.
\end{cases}
\]

The \( L^2 \)-norm \( \| u_h(\cdot, T) - \overline{u}_h(\cdot, T) \|_{L^2(0,1)} = \| \overline{u}_h(\cdot, T) \|_{L^2(0,1)} \) allows to analyze \emph{a posteriori} how the constraint (1.4) is satisfied. Recall that \( \overline{u}_h \), obtained by an integration in time, solves the heat equation.

This same numerical approximation is used for the inner case.

4. Numerical experiments

We now present some numerical experiments, and analyze the behavior of the computed controls with respect to the data, and \( h \). We assume for simplicity that \( \Delta x = \Delta t \), that is we consider only uniform meshes \( \mathcal{Q}_h \).

4.1. Experiment 1: Boundary Case. We assume that the function \( u_0 \) to be controlled is the first mode of the Laplacian, that is
\[
u_0(x) = \sin(\pi x), \quad x \in (0, 1).
\]
Moreover, we assume that the diffusion function \( a \) is constant equal to \( a(x) = a_0 = 1/4 \) in \((0, 1)\), and take a controllability time equal to \( T = 1/2 \). These two values lead to a rather \emph{stiff} case in the context of null boundary controllability for the heat equation.

We take \( \varepsilon = 10^{-5} \) as the value for the stopping criterion of the conjugate gradient algorithm. The algorithm is initialized with \( u^0 \in \mathcal{A} \) defined by \( u^0(x,t) = u_0(x)(1 - t/T)^2, (x,t) \in Q_T \).

Table 1 gives various norms of the solution \( u_h \in \mathcal{A} \) with respect to \( h \), and clearly suggests the convergence of the approximation. Figure 1 depicts the evolution of \( E(u^0) \) and the residual \( \| g_0^h \|_\mathcal{A} \) (in log-log-scale) with respect to the iteration of the conjugate gradient corresponding to \( \Delta x = \Delta t = 1/100 \). The algorithm requires 2013 iterations to fulfill \( \| g_0^h \|_\mathcal{A} \leq \varepsilon \).

As is typical when the heat equation is involved, the slope of the residual decreases significantly after the first iteration. This phenomenon is also possibly due to the lack of coercivity of \( E \). We check however that the functional \( E(u^0) \) decreases with respect to the iteration and reaches a small value, here of the order \( O(10^{-6}) \).

<table>
<thead>
<tr>
<th>( \Delta x = \Delta t )</th>
<th>1/25</th>
<th>1/50</th>
<th>1/100</th>
<th>1/200</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon ) CG iteration</td>
<td>846</td>
<td>2132</td>
<td>2014</td>
<td>2834</td>
</tr>
<tr>
<td>( | u_h |_{L^2(Q_T)} )</td>
<td>4.78 \times 10^{-1}</td>
<td>5.96 \times 10^{-1}</td>
<td>4.81 \times 10^{-1}</td>
<td>4.87 \times 10^{-1}</td>
</tr>
<tr>
<td>( | u_h |_{H^1(Q_T)} )</td>
<td>6.024</td>
<td>6.658</td>
<td>5.920</td>
<td>6.021</td>
</tr>
<tr>
<td>( | u_h |_{L^2(\Sigma_T)} )</td>
<td>1.369</td>
<td>1.487</td>
<td>1.392</td>
<td>1.418</td>
</tr>
<tr>
<td>( | \overline{\Psi}(\cdot, T) |_{L^2(0,1)} )</td>
<td>1.95 \times 10^{-2}</td>
<td>9.65 \times 10^{-3}</td>
<td>8.39 \times 10^{-3}</td>
<td>6.04 \times 10^{-3}</td>
</tr>
<tr>
<td>( | u_h - \overline{u}<em>h |</em>{L^2(Q_T)} )</td>
<td>1.45 \times 10^{-2}</td>
<td>6.31 \times 10^{-3}</td>
<td>2.01 \times 10^{-3}</td>
<td>9.34 \times 10^{-4}</td>
</tr>
<tr>
<td>( E(u_h) )</td>
<td>4.88 \times 10^{-6}</td>
<td>8.37 \times 10^{-7}</td>
<td>1.22 \times 10^{-6}</td>
<td>8.29 \times 10^{-7}</td>
</tr>
</tbody>
</table>

Table 1. \( u_0(x) = \sin(\pi x), \quad T = 1/2, \quad a_0 = 1/4, \quad \Delta x = \Delta t = 1/100 - \varepsilon = 10^{-5} \)

Numerical results with respect to \( h = (\Delta x, \Delta t) \).

Figure 2 depicts the corresponding solution \( u_h \in \mathcal{A} \), and corrector \( v_h \in H_{0,1}^1(Q_T) \). The trace of \( u_h \) on \( \Sigma_T \) is given in Figure 3-left. The control obtained is quite irregular near \( t = T \). This is reminiscent of what it is obtained in [16] by computing exactly the controlled heat solution in
Figure 1. $u_0(x) = \sin(\pi x)\ T = 1/2\ a_0 = 1/4\ \Delta x = \Delta t = 1/100 - \log_{10}$ of $E(h(u^n_h))$ (dashed line) and $\log_{10}(\|g_h^n\|_A)$ (full line) vs. the iteration $n$ of the conjugate gradient algorithm.

the one dimensional space by means of the motion planning method (we also refer to [20] for an adaptation to the inner case using the so-called transmutation method). Finally, the solution $\pi_h$ of (1.3) with $w(t) = u_h(1, t)$ is plotted at time $T$ on Figure 3 right. We compute that the $L^2$-norm of $\pi_h(1, T)$, what we called the a posteriori error, is $\|\pi_h(\cdot, T)\|_{L^2(0,1)} \approx 8.39 \times 10^{-3}$. This is an acceptable value that can be improved by reducing $\varepsilon$ and $h$. Notice that the stiffness matrices involved in the resolution of the elliptic problems in step 1 are standard and well-conditioned. Notice also that a small gap between $u$ and $\pi$ (in particular at time $T$) is a priori unavoidable, since they are approximated, and computed in a different way.

Figure 2. $u_0(x) = \sin(\pi x)\ T = 1/2\ a_0 = 1/4\ \Delta x = \Delta t = 1/100$ - Solution in $u_h \in A_h$ (Left) and corresponding corrector $v_h$ (Right) along $Q_T$.

It is also interesting to note that this method allows to obtain non trivial controlled solution of the heat equation with zero initial data, that is in $A_0$. Figure 4 depicts one such solution obtained with the initial function $u^0(x, t) = \sin(\pi x)t^2(1-t/T)^2$. For $\varepsilon = 10^{-6}$, the algorithm converges after 1 242 iterations, and we get $E_h(u_h^{1242}) \approx 6.63 \times 10^{-9}$ and $\|\pi_h(\cdot, T)\|_{L^2(0,1)} \approx 2.89 \times 10^{-5}$. 

The solution $\pi_h$ of (1.3) with $w(t) = u_h(1, t)$ is plotted at time $T$ on Figure 3 right. We compute that the $L^2$-norm of $\pi_h(1, T)$, what we called the a posteriori error, is $\|\pi_h(\cdot, T)\|_{L^2(0,1)} \approx 8.39 \times 10^{-3}$. This is an acceptable value that can be improved by reducing $\varepsilon$ and $h$. Notice that the stiffness matrices involved in the resolution of the elliptic problems in step 1 are standard and well-conditioned. Notice also that a small gap between $u$ and $\pi$ (in particular at time $T$) is a priori unavoidable, since they are approximated, and computed in a different way.

The solution $\pi_h$ of (1.3) with $w(t) = u_h(1, t)$ is plotted at time $T$ on Figure 3 right. We compute that the $L^2$-norm of $\pi_h(1, T)$, what we called the a posteriori error, is $\|\pi_h(\cdot, T)\|_{L^2(0,1)} \approx 8.39 \times 10^{-3}$. This is an acceptable value that can be improved by reducing $\varepsilon$ and $h$. Notice that the stiffness matrices involved in the resolution of the elliptic problems in step 1 are standard and well-conditioned. Notice also that a small gap between $u$ and $\pi$ (in particular at time $T$) is a priori unavoidable, since they are approximated, and computed in a different way.

The solution $\pi_h$ of (1.3) with $w(t) = u_h(1, t)$ is plotted at time $T$ on Figure 3 right. We compute that the $L^2$-norm of $\pi_h(1, T)$, what we called the a posteriori error, is $\|\pi_h(\cdot, T)\|_{L^2(0,1)} \approx 8.39 \times 10^{-3}$. This is an acceptable value that can be improved by reducing $\varepsilon$ and $h$. Notice that the stiffness matrices involved in the resolution of the elliptic problems in step 1 are standard and well-conditioned. Notice also that a small gap between $u$ and $\pi$ (in particular at time $T$) is a priori unavoidable, since they are approximated, and computed in a different way.

The solution $\pi_h$ of (1.3) with $w(t) = u_h(1, t)$ is plotted at time $T$ on Figure 3 right. We compute that the $L^2$-norm of $\pi_h(1, T)$, what we called the a posteriori error, is $\|\pi_h(\cdot, T)\|_{L^2(0,1)} \approx 8.39 \times 10^{-3}$. This is an acceptable value that can be improved by reducing $\varepsilon$ and $h$. Notice that the stiffness matrices involved in the resolution of the elliptic problems in step 1 are standard and well-conditioned. Notice also that a small gap between $u$ and $\pi$ (in particular at time $T$) is a priori unavoidable, since they are approximated, and computed in a different way.
Accordingly, this means that any linear combination of such nontrivial solution in $A_0$ with the previous ones in $A$ remains a controlled solution of the heat equation. We will get back to this notion in Section 6. The non uniqueness of our minimization problem may also be checked by considering different initial function $u_0 \in A$. 

The experiments also confirm that the situation is more favorable, notably with respect to the speed of convergence of the algorithm, when the control acts on both sides, that is on $x = 0$ and $x = 1$. Figure 5 shows the controlled solution with initial data $u_0(x) = \sin(\pi x) + \sin(2\pi x) + \sin(3\pi x)$ in that situation. For a same value of $\varepsilon$, the $L^2$-norm of the corrector as well as the $a posteriori$ error are lower than in the previous situation: $E_h(u_h^{\varepsilon=1.242}) \approx 6.63 \times 10^{-9}$ and $\|\pi_h(\cdot, T)\|_{L^2(0,1)} \approx 2.89 \times 10^{-5}$ after 855 iterations. 

We also emphasize that we may consider the more realistic situation where null Neumann boundary limit holds on the free part, here $x = 0$. It suffices to start with $u^0 \in A_N = \{ u \in$. 

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**Figure 3.** $u_0(x) = \sin(\pi x)$, $T = 1/2$, $a_0 = 1/4$, $\Delta x = \Delta t = 1/100$ - Full line: Trace $u_h(x = 1, t)$ vs. $t \in (0, T)$ (Left) and a posteriori solution $\pi_h(x, T)$ vs. $x \in (0, 1)$ (Right); Dashed line: same quantities obtained with an additional compact support function in time.

**Figure 4.** $u_0 = 0$, $T = 1/2$, $a_0 = 1/4$, $\Delta x = \Delta t = 1/100$ - Control acting on $0 - \varepsilon = 10^{-6}$. Solution in $u_h \in A_h$ (Left) and corresponding corrector $v_h$ (Right) along $Q_T$. $E_h(u_h^{\varepsilon=1.242}) \approx 6.63 \times 10^{-9}$ and $\|\pi_h(\cdot, T)\|_{L^2(0,1)} \approx 2.89 \times 10^{-5}$.
Figure 5. $u_0(x) = \sin(\pi x) + \sin(2\pi x) + \sin(3\pi x)$, $T = 1/2$, $a_0 = 1/4$, $\Delta x = \Delta t = 1/100$ - Control acting on $\{0, 1\}$ - $\varepsilon = 10^{-5}$ - Solution in $u_h \in A_h$ (Left) and corresponding corrector $v_h$ (Right) along $Q_T$.

$H^1(Q_T), u(\cdot, T) = 0, u(\cdot, 0) = u_0, u_x(0, t) = 0$, and impose that both the descent direction and the corrector have null derivatives at $x = 0$. Figure 5 shows the function $u_h \in A_N$ associated with $u_0(x) = \sin(\pi x)^2$. With $\varepsilon = 10^{-5}$, the convergence is reached after 3 431 iterations, and we get $\|\pi_h(\cdot, T)\|_{L^2(0, 1)} \approx 1.31 \times 10^{-2}$. The convergence is slower in that case, because null Neumann boundary condition - contrary to null Dirichlet one - does not emphasize the dissipation of the solution.

Finally, let us comment on a simple way to smooth out the control near $t = T$, and therefore avoid the oscillations we mentioned at the beginning of this section (see Figure 6-Left). It suffices to replace at each iteration $n$ the descent direction $g^n$ by $c(t) g^n$ with any smooth positive function $c$ such that $c(T) = c'(T) = 0$. Figure 5 gives (in dashed line) the quantities $u_h(1, \cdot)$ on $(0, T)$ and $\pi(\cdot, T)$ on $(0, 1)$ obtained with $c(t) = \sin(\pi t/T)^2$ (in that case, notice that the solution is also smoothed at $t = 0$). This modification has the effect to reduce the a posteriori error $\|\pi_h\|_{L^2(0, 1)}$ but to increase the number of iterations. Notice also that the $L^2$-norm of the trace is larger (see Table 5).
We observe similar results with Dirichlet boundary condition on \((0, 1) \times \{0, T\}\) in (2.1). The CG algorithm converges faster and leads to a control with smaller \(L^2\)-norm. The \textit{a posteriori} error \(\|\tilde{u}_h\|_{L^2(\{0, T\})}\) is however larger.

4.2. Experiment 2: Inner Case. Let us consider the following data \(\omega = (0.3, 0.6)\), \(T = 1/2\) and \(a(x) = a_0 = 1/4\) used notably in ([7] [8] [20]). The initial data to be controlled is again \(u_0(x) = \sin(\pi x)\).

Table 1 collects some numerical values obtained with the CG algorithm and \(\varepsilon = 10^{-6}\), \(c(t) = \sin(\pi t/T)^2\). The situation is more favorable than the boundary case in the sense that the number of iterations to reach a relative residual of order 10^{-6} (instead of 10^{-5} in Table 1) is significantly reduced. As a consequence, the \textit{a posteriori} error \(\|\tilde{u}_h\|_{L^2(\{0, T\})}\) is smaller, of the order \(O(10^{-5})\). We also notice that the function \(v_{h,t} + \langle a(x)v_{h,x}\rangle\) is actually bounded with respect to \(h\) for the \(L^2(Q_T)\)-norm, and converges as \(h \rightarrow 0\). This is the effect of the compactly support function \(c(t)\), and an additional smoothing procedure on the descent direction \(g^n\): \(g^n\), solution of (3.5), is replaced by \(\tilde{g}^n\), unique solution in \(H^1_0(Q_T)\) of
\[
(I - (\Delta x)^2 \partial_{xx} - (\Delta t)^2 \partial_{tt})\tilde{g}^n = g^n \text{ on } Q_T, \quad \tilde{g}^n = 0 \text{ on } \partial Q_T.
\]

The controlled solution \(u_h \in \mathcal{A}\) as well as the corresponding corrector \(v_h\) are depicted on Figure 7 for \((\Delta t, \Delta x) = (1/100, 1/100)\). The control function \(v_{h,t} + \langle a(x)v_{h,x}\rangle\), very small out of \(\omega\) is given in Figure 8 left. This control, obtained with the initial guess \(u_0^0(x) = \sin(\pi x)(1-t/T)^2\), is quite different from controls obtained by duality arguments in [20]. Mainly concentrated on the boundary of \(\omega\), his \(L^2\)-norm is larger : for \(h = (1/100, 1/100)\), we obtain \(\|v_{h,t} + \langle a(x)v_{h,x}\rangle\|_{L^2(Q_T)} \approx 2.839\), about twice the HUM-control obtained in [20].

<table>
<thead>
<tr>
<th>(\Delta x = \Delta t)</th>
<th>1/25</th>
<th>1/50</th>
<th>1/100</th>
<th>1/200</th>
</tr>
</thead>
<tbody>
<tr>
<td>CG iteration</td>
<td>135</td>
<td>192</td>
<td>231</td>
<td>361</td>
</tr>
<tr>
<td>(|u_h|_{L^2(Q_T)})</td>
<td>2.53 \times 10^{-1}</td>
<td>2.58 \times 10^{-1}</td>
<td>2.57 \times 10^{-1}</td>
<td>2.61 \times 10^{-1}</td>
</tr>
<tr>
<td>(|u_{tt}|_{H^1(Q_T)})</td>
<td>1.301</td>
<td>1.336</td>
<td>1.337</td>
<td>1.352</td>
</tr>
<tr>
<td>(|v_{h,t} + \langle a(x)v_{h,x}\rangle|_{L^2(Q_T)})</td>
<td>1.675</td>
<td>2.641</td>
<td>2.839</td>
<td>2.981</td>
</tr>
<tr>
<td>(|\tilde{v}<em>{h}(\cdot, T)|</em>{L^2({0, T})})</td>
<td>7.23 \times 10^{-5}</td>
<td>5.43 \times 10^{-5}</td>
<td>4.30 \times 10^{-5}</td>
<td>2.91 \times 10^{-5}</td>
</tr>
<tr>
<td>(|u_h - \tilde{u}<em>h|</em>{L^2(Q_T)})</td>
<td>3.21 \times 10^{-5}</td>
<td>7.31 \times 10^{-5}</td>
<td>5.10 \times 10^{-5}</td>
<td>1.58 \times 10^{-5}</td>
</tr>
<tr>
<td>(E(u_h))</td>
<td>4.12 \times 10^{-7}</td>
<td>3.34 \times 10^{-7}</td>
<td>4.16 \times 10^{-7}</td>
<td>2.36 \times 10^{-7}</td>
</tr>
</tbody>
</table>

Table 3. \(u_0(x) = \sin(\pi x)\), \(T = 1/2\), \(a_0 = 1/4\), \(\Delta x = \Delta t = 1/100 - \omega = (0.3, 0.6) - \varepsilon = 10^{-6}\). Numerical results with respect to \(h = (\Delta x, \Delta t)\) with a compact support function in time.

5. Remarks on a non-linear situation

As a good way to emphasize the flexibility of the variational approach to adapt itself to various different settings, we are going to indicate the changes needed for a typical non-linear situation.
where a low order non-linear perturbation is considered. Namely, we will look at the problem of finding a control $w$, so that the solution of the problem

\[
\begin{cases}
    u_t - (a(x)u_x)_x + f(u) = 0, & (x,t) \in Q_T, \\
    u(x,0) = u_0(x), & x \in (0,1), \\
    u(0,t) = 0, u(1,t) = w(t), & t \in (0,T)
\end{cases}
\]

(5.1)

will comply with $u(x,T) = 0$ for all $x \in (0,1)$. System (5.1) is known to be controllable, uniformly with respect to the data $u_0$ and $T$, if the nonlinear function $f(s)$ grows slower than $s \log^{3/2}(1 + |s|)$ as $|s| \to +\infty$ (we refer to [3]).
The procedure for such nonlinear system is similar. We define the corrector associated with $u$, through the problem

$$\begin{aligned}
& u_t - v_t - (a(x)(u_x + v_x)) + f(u) = 0, \quad (x, t) \in Q_T, \\
& v_t(x, 0) = v_t(x, T) = 0, \quad x \in (0, 1), \\
& v(0, t) = v(1, t) = 0, \quad t \in (0, T)
\end{aligned}$$

while the error functional $E$ is still defined by (2.2). However, due to the nonlinear dependence of the corrector with respect to $v$, we cannot ensure, at least using the arguments we employed for the linear situation, that the infimum of $E$ over $A$ is reached. We then go on in a formal way assuming the well-posedness of the minimization problem. We obtain that the first derivative of $E$ is given by

$$< E'(u), U > = - \int_{Q_T} (U_t v + U_x v_x + (f'(u) \cdot U) v) \, dx \, dt, \quad \forall U \in A_0,$$

leading to the characterization of the corrector $v$ associated with any optimal $u$ (assumed to exist in $A$)

$$\begin{aligned}
& v_t + (a(x)v_x)_x + f'(u)v = 0, \quad (x, t) \in Q_T, \\
& v_t(x, 0) = v_t(x, T) = 0, \quad x \in (0, 1), \\
& v(0, t) = v(1, t) = a(1)v_x(1, t) = 0, \quad t \in (0, T).
\end{aligned}$$

Once again, the solution of this system vanishes in $Q_T$ so that the minimizer of $E$ is a solution of the nonlinear heat equation (5.1).

As we mentioned earlier, even if we are not able to show the well-posedness of the minimization problem, the decrease of the error to zero is a sure indication that the problem is being, at least approximately, controlled. Let us simply mention that, in the conjugate gradient algorithm, the function $g^a$ in the steepest descent step is the solution of the linear formulation

$$\int_{Q_T} (g^a \phi_t + a(x)g^a_x \phi_x) \, dx \, dt = - \int_{Q_T} (v^a \phi_t + a(x)v^a_x + f'(u^a)v^a) \, dx \, dt, \quad \forall \phi \in A_0.$$

Figure 9 and 10 are concerned with the case $f(s) = -a s \log(1 + |s|)$, $a = 5$. $f$ belongs to $C^1(\mathbb{R})$ and $f'(s) = \log(1 + |s|) + |s|/(1 + |s|)$. The other data are kept unchanged. This nonlinear term prevents the diffusion in time of the heat solution, that is the $L^2$-norm $\|u(\cdot, t)\|_{L^2(0, 1)}$ increases with respect to $t$: in the uncontrolled situation, we get $\|u(\cdot, T)\|_{L^2(0, 1)} \approx 1.25$ at $t = 2$; the error we get is $\|u(\cdot, T)\|_{L^2(0, 1)} \approx 2.05 \times 10^{-4}$ for the linear case. As for the linear situation, our approach allows to drive the solution in a closed neighborhood of zero: the $a$ posteriori error we get is $\|u(\cdot, T)\|_{L^2(0, 1)} \approx 1.92 \times 10^{-2}$. Compared to the linear situation, we observe also that the solution $u_h \in A$ has a similar shape, the difference being the magnitude (see Figure 10-left). Notice that we have used the compact support function $c(t) = \sin^2(\pi t/T)$ so that $u_h$ is smooth near $T$. The nonlinearity increases slightly the number of the iterations, here 2788, to reach the same threshold $\varepsilon = 10^{-5}$. We also plot the difference $u_h - \overline{u}_h$ in $Q_T$ (see Figure 10-right) so as to measure the influence of the corrector $v_h$. For larger values of $a$, the algorithm does not converge anymore. Similar phenomena are observed for smaller or larger values of $T$, for instance $T = 0.25$, and $T = 2$. We also obtain convergence results for the case $f(s) = -5s \log(1 + |s|)$, more critical than the previous situation since $f$ is non-positive. The number of iterations is greater (6883) as well as the $L^2(\Sigma_T)$-norm of the control. Similar remarks hold for “more” nonlinear function such as $f(s) = a|s|^p$, $p \in \mathbb{N}$ or $f(s) = a \exp(s)$ provided that $a$ or $\|u_0\|_{L^2(0, 1)}$ be small enough.

6. Reducing the norm of the control

By minimizing the error functional $E$ defined by (2.2), we do not control any norm, in particular the $L^2$-norm, of the solution on $\Sigma_T$. From a practical viewpoint, it is interesting to minimize such norm. A possibility is to take advantage of the fact that the method allows to obtain non trivial controlled solutions in $A$ with null initial condition $u_0$, that is, solutions in $A_0$. 


Figure 9. $u_0(x) = \sin(\pi x) - f(s) = -5s \log(1 + |s|)$ - $T = 1/2$, $a_0 = 1/4$, $\Delta x = \Delta t = 1/100$ - log$_{10}(E_h(u_h^N))$ (dashed line) and log$_{10}(|g_h^n|_A)$ (full line) vs. the iteration $n$ of the CG algorithm.

Figure 10. $u_0(x) = \sin(\pi x) - f(s) = -5s \log(1 + |s|)$, $T = 1/2$, $a_0 = 1/4$, $\Delta x = \Delta t = 1/100$ - Control acting on $x = 0$ - $\varepsilon = 10^{-5}$. Solution in $u_h \in A_h$ (Left) and gap $u_h - \overline{u}_h$ (Right) along $Q_T$. $E_h(u_h^{N=2788}) \approx 3.33 \times 10^{-6}$, $\|g_h^{N=2788}\|_A \approx 9.89 \times 10^{-6}$ and $\|\overline{u}_h(.,T)\|_{L^2(0,1)} \approx 1.92 \times 10^{-2}$.

Suppose a family $\{u_k\}_{k \in [1,N]}$ of $N$ elements in $A_0$ is given. Then, for any $\alpha_n \in \mathbb{R}$, $n = 1 \cdots N$, and any $u \in A$,

$$u^N(x,t) = u(x,t) + \sum_{k=1}^{N} \alpha_k u_k(x,t), \quad (x,t) \in Q_T$$

still belongs, in the linear situation of Section 2, to $A$. The minimization of $\|u^N(1,t)\|_{L^2(0,1)}$ is then reduced to a quadratic minimization on $\{\alpha_k\}_{k=1,N}$. The method we propose to construct the family $\{u_k\}_{k \in [1,N]}$ is as follows: we first compute $N$ elements $v_k$, $k = 1, \cdots, N$ in $A_0$ using the conjugate gradient algorithm with initial guesses $u_0^0(x,t) = x \sin(k\pi t/T)^2$: then, we orthogonalize
these elements using the Gram-Schmidt procedure with the scalar product associated with $\mathcal{A}$:

$$u_k = v_k - \sum_{n=1}^{k-1} \langle v_k, u_n \rangle_A u_n.$$  

Figure 11 shows the trace of $u^N \in \mathcal{A}$ along $\Sigma_T$ obtained with $N = 10$ as well as the trace of $u \in \mathcal{A}$ corresponding to $u^0(x) = \sin(\pi x)(1-t/T)^2$ (see Figure 2). We obtain $\|u^N_h\|_{L^2(\Sigma_T)} \approx 0.981$ significantly lower than $\|u_h\|_{L^2(\Sigma_T)} \approx 1.392$. Larger values of $N$, which require a finer mesh in time so as to capture the oscillating functions $\sin(m\pi t/T)$, do not allow a significant additional reduction of the $L^2(\Sigma_T)$-norm.

This constructive approach which allows to jump from a local minimum of $E$ to another one does not apply for the nonlinear situation of Section 5. On the other hand, the more flexible approach which consists to minimize at the same time the error functional $E$ and the $L^2$-norm of the trace with respect to $u$, does not lead to satisfactory results, as it depends too much on the initial guess $u^0$. In that respect, a possible strategy could be to initialize the CG algorithm with an approximate control obtained from the dual approach (see [3]).

7. Concluding remarks

The variational approach we discussed here to construct numerical controls is very different in nature from the usual one [3, 8, 20] making use of dual variable to deal with the constraint $u(\cdot, T) = 0$. In the context of parabolic equations, this difference is significant because the variational approach avoids the approximation of singular functional spaces and therefore ill-posed problems. Here, the problem is elliptic and leads to standard and well-posed formulations. A quantitative comparison with the dual approach for the boundary situation remains however to be done. The method extends to any target - trajectory for the heat equation -, to higher dimensions, and to any system for which a unique continuation property is known.

It is also remarkable to note that this variational approach allows to solve inverse problems. Let us mention, in particular, the highly ill-posed backward heat problem which consists to determine the solution of the heat equation at time $t = 0$ from the solution $u_T$ at any positive time $T$. It suffices to define the functional spaces $\mathcal{A}$ and $\mathcal{A}_0$ respectively as follows : $\mathcal{A} = \{ u \in H^1(Q_T), u(0,t) = u(1,t) = 0, u(x,T) = u_T(x), (x,t) \in Q_T \}$ and $\mathcal{A}_0 = \{ u \in H^1(Q_T), u(0,t) = u(1,t) = 0, u(x,T) = 0, (x,t) \in Q_T \}$. We plan to analyze this situation in a near future.
References