Internal observability for coupled systems of linear partial differential equations

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Scalar equations (1)

Ω smooth bounded domain of $\mathbb{R}^N$ ($N \in \mathbb{N}^*$) or a smooth compact connected Riemannian manifold of dimension N ($N \in \mathbb{N}^*$), with or without boundary.
We consider the following "scalar" evolution equation

$$\begin{cases} \partial_t z &= Pz \text{ in } (0,T) \times \Omega, \\ z(0) &= z^0, \end{cases} \qquad \text{(Scal-Eq)}$$

where P is a linear partial differential operator with domain $D(P) \subset \mathcal{H} = L^2(\Omega, \mathbb{K})$ of arbitrary order with time-independent and (possibly) space-dependent coefficients, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The initial datum z^0 is in $\mathcal{H}$.

Scalar equations (2)

Assume that P is the infinitesimal generator of a strongly continuous semigroup on $\mathcal{H}$. Then, (Scal-Eq-Cont) admits a unique (weak) solution in the space $C^0([0,T],L^2(\Omega))$.
We assume w.l.g. that $0 \notin \rho(P)$, so that for every $k \in \mathbb{N}$, one endows $D(P^k)$ with the norm

$$||z||_{D(P^k)} = ||P^k z||_{L^2(\Omega)}.$$

We also consider an open subset ω of Ω.

Observability inequalities

We assume that system Scal-Eq is observable in the sense that there exists some constant $C > 0$ such that for any $z^0 \in L^2(\Omega)$, one has

$$||z(T)||^2_{L^2(\Omega)} \leqslant C \int_0^T \int_\omega ||z(t,x)||^2 dxdt. \qquad \text{(Obs-Ineg-Scal)}$$

Weak quantative form of a unique continuation property: if $z = 0$ on $(0, T) \times \omega$, then $z(T) = 0$ and then $z \equiv 0$ if we assume that z verifies a backward uniqueness property or is conservative.

Moreover, related to inverse problems: we can reconstruct the final data z^T thanks to the observation $\int_0^T \int_\omega ||z(t,x)||^2 dxdt$.

Link with controllability (1)

We consider the following equation

$$\left\{\begin{array}{ll} \partial_t y & = P^* y + u 1_\omega \text{ in } (0,T)\times\Omega, \\ y(0) & = y^0 \in H, \end{array}\right. \qquad \text{(Scal-Eq-Cont)}$$

where P^* with domain $D(P^*) \subset \mathcal{H} = L^2(\Omega, \mathbb{K})$ is assumed to be the generator of a strongly continuous semigroup. Then, (Scal-Eq-Cont) admits a unique (weak) solution in the space $C^0([0,T],\mathcal{H})$ as soon as $u \in L^2((0,T)\times\Omega)$.

Définition

(Scal-Eq-Cont) is exactly controllable at time T if for every y^0 and y^1 in $\mathcal{H}$, there exists (y,u) solution of (Scal-Eq-Cont) such that $y(0) = y^0$ and $y(T) = y^1$.
(Scal-Eq-Cont) is null controllable at time T if for every $y^0 \in \mathcal{H}$, there exists (y,u) solution of (Scal-Eq-Cont) such that $y(0) = y^0$ and $y(T) = 0$.

Remarks and link between controllability and observability

- exact controllability at time $T \Rightarrow$ null controllability at time T.
- For reversible in time systems, equivalence between the two notions, otherwise in general no equivalence.

Using duality arguments (either computations of adjoints of operators, or the Fenchel-Rockafellar duality theory), one can prove easily:

Theorem

- (Scal-Eq-Cont) *is null-controllable at time* T *if and only if* (Obs-Ineg-Scal) *holds on the solutions of* (Scal-Eq).
- (Scal-Eq-Cont) *is exactly controllable at time* T *if and only if the following inequality observability holds on the solutions of* (Scal-Eq)*:*

$$||z^0||^2_{L^2(\Omega)} \leqslant C \int_0^T \int_\omega ||z(t,x)||^2 dxdt. \qquad \text{(Obs-Ineg-Scal-Ex)}$$

By well-posedness inequality (Obs-Ineg-Scal-Ex) is stronger than (Obs-Ineg-Scal) for irreversible in time systems, and is equivalent to (Obs-Ineg-Scal) for conservative systems.

Observability inequalities in other spaces

Let $\tilde{\mathcal{H}}$ another Hilbert space, that is included in $\mathcal{H}$ (for example a Sobolev space of high order). If the following observability inequality on the solutions of (Scal-Eq) is verified:

$$||z(T)||^2_{L^2(\Omega)} \leqslant C \int_0^T |||z(t,\cdot)1_\omega||^2_{\tilde{\mathcal{H}}} dt. \qquad \text{(Obs-Ineg-Scal-W)}$$

Then by duality we would obtain a controllability result for initial condition in L^2 and with controls in $\tilde{\mathcal{H}}'$ (dual space with pivot space L^2), assuming that (Scal-Eq-Cont) is well-posed in some weaker space (for example $\tilde{\mathcal{H}}'$).

Trying to enhance regularity of the initial condition, one may expect to prove a result of controllability with initial conditions in $\tilde{\mathcal{H}}$ with controls in L^2, which is weaker than before. This operation is not always possible.

Motivations

In real-life applications, many models with systems of equations, with controllability or optimal control issues, and in general underactuated systems (for instance, only one physical quantity can be directly controlled). In general, semi-linear or fully non-linear systems. Amongst others:

- Medicine, treatment of tumors. Some models made of a system of 3 semi-linear equations involving the tumor cell density, the normal cell density and the drug concentration.
- Chemotaxis, e.g. the Keller-Segel equation that describes the change of motion when a population reacts in response to an external chemical stimulus spread in the environment where they live.
- In chemistry, in order to model fast reversible chemical reactions betweens some species.
- ...

Control: local drug/chemical bolus inside the domain.

State of the art

A lot of results by Alabau, Ammar-Khodja, Benabdallah, Boyer, Coron, Dehman, Dupaix, Duprez, Fernandez-Cara, Gonzalez-Burgos, Guerrero, Le Rousseau, Léautaud, Olive, Perez-Garcia, de Teresa, and many others... of different nature and on different classes of equations (parabolic, conservative...), 1D/Multi-D, different couplings (diagonal/non diagonal, constant/time-dependent/space dependent, coercive/non coercive,...), with additional structure (cascade systems) or not, linear/semi-linear...

Some papers of particular interest for our concern:

- Ammar-Khodja et al.'09 DEA (parabolic, time-dependent).
- Ammar-Khodja et al.'09 JEE (parabolic, time-independent).
- Fernandez-Cara et al.'15 COCV (non-diagonalizale diffusion matrix).
- Liard-Lissy'17 MCSS (abstract results for groups of operators).

The finite-dimensional case (1)

Let $n, m \in \mathbb{N}^*$. We consider $A \in \mathcal{M}_n(\mathbb{R})$ and the autonomous differential system

$$\begin{cases} Y' = A^* Y, \\ Y(0) = Y^0 \in \mathbb{R}^N. \end{cases}$$

Consider some $B \in \mathcal{M}_{n,m}(\mathbb{R})$. We want to prove an observability like

$$||Y^0||_2^2 \leqslant \int_0^T ||B^* Y(t)||^2 dt.$$

It holds if and only if

$$\text{Rank}[A|B] = n, \text{ (The Kalman Rank Condition)},$$

where

$$[A|B] := (B|AB|...|A^{n-1}B) \text{ (The Kalman matrix)}.$$

The finite-dimensional case (2)

Consider now the non-autonomous case. We consider some $A \in C^\infty([0,T],\mathcal{M}_n(\mathbb{R}))$ and the non-autonomous system

$$\begin{cases} Y' = A^*(t)Y, \\ Y(0) = Y^0 \in \mathbb{R}^N. \end{cases}$$

Consider some $B \in C^\infty([0,T],\mathcal{M}_{n,m}(\mathbb{R})$. We want to prove an observability like

$$||Y^0||_2^2 \leqslant \int_0^T ||B^*Y(t)||^2 dt.$$

We call

$$B_0 = B \text{ and}$$

$$B_i = AB_{i-1} - B'_{i-1} \text{ for } i \in \mathbb{N}^*.$$

Then observability holds if there exist $\bar{t} \in [0,T]$ such that

$$\text{Span } \{B_k(\bar{t})(\mathbb{R}^m) | k \in \mathbb{N}^*\} = \mathbb{R}^n.$$

It is also necessary for analytic systems.

Goal of the talk

The goal is twofold.

General systems

Assume that (Obs-Ineg-Scal) is true for (Scal-Eq). How to prove with elementary arguments observability inequalities for different system versions of (Scal-Eq), whatever P is?

- With elementary arguments means: mainly basic linear algebra arguments.
- Whatever P is means: can apply for all classes of linear systems of PDEs, as soon as an observability inequality for the scalar equation is known.

Main drawback: will only obtain results of weak observability inequalities (meaning is strong norms) that are notably not adapted to the heat case.

Heat systems

Find a simple NSC in the natural L^2-norm for a general version of systems of heat equations with constant coupling terms and observation.

Constant coupling coefficients and observation

Let $n \in \mathbb{N}^*$ (number of equations) and $m \in \mathbb{N}^*$ (number of observed components), with possibly $m < n$.

- The diagonal case

$$\begin{cases} \partial_t Z &= PZ + A^* Z \quad \text{in} \quad (0,T)\times\Omega, \\ Z(0) &= Z^0, \end{cases} \tag{A-Cst}$$

  with $Z^0 \in \mathcal{H}^n$ and $A^* \in \mathcal{M}_n(\mathbb{K})$.

- Coupling arises in the principal part

$$\begin{cases} \partial_t Z &= DPZ \quad \text{in} \quad (0,\tilde{T})\times\Omega, \\ Z(0) &= Z^0, \end{cases} \tag{D-Cst}$$

  with $Z^0 \in \mathcal{H}^n$ and a diffusion matrix $D = diag(d_1, \ldots, d_n) \in \mathcal{M}_n(\mathbb{K})$ with $d_i > 0$. Observation time: $\tilde{T} := \frac{T}{\min_i d_i}$.

Observation operator: $\mathbb{1}_\omega B^* Z$ ($B \in \mathcal{M}_{n,m}(\mathbb{K})$).

Time-varying coupling and observation

We also consider time-dependent couplings. We assume moreover that observability holds in arbitrary small time in this case.

- The diagonal case

$$\begin{cases} \partial_t Z = PZ + A^*(t)Z & \text{in} \quad (0,T)\times\Omega, \\ Z(0) = Z^0, \end{cases} \tag{A-Time}$$

with $Z^0 \in \mathcal{H}^N$ and $A^* \in C^\infty([0,T],\mathcal{M}_n(\mathbb{K}))$.

- The case where the coupling arises in the principal part

$$\begin{cases} \partial_t Z = D(t)PZ & \text{in} \quad (0,T)\times\Omega, \\ Z(0) = Z^0, \end{cases} \tag{D-Time}$$

with $Z^0 \in \mathcal{H}^N$ and $D \in C^\infty([0,T],\mathcal{M}_n(\mathbb{K}))$ with $D = diag(d_1,\dots,d_n) \in C^\infty([0,T],\mathcal{M}_n(\mathbb{K}))$, where $d_i(t) > 0$ for all $t \in [0,T]$.

The observation operator is given by $\mathbb{1}_\omega B^* Z$, where $B \in C^\infty([0,T],\mathcal{M}_{n,m}(\mathbb{K}))$.

Some useful matrices and spaces

We introduce

$$K_A := [B|AB|\dots|A^{n-1}B] \in \mathcal{M}_{n,nm}(\mathbb{K}),$$

and

$$K_D := [B|DB|\dots|D^{n-1}B] \in \mathcal{M}_{n,nm}(\mathbb{K}).$$

(Kalman Matrices)
We also introduce the spaces

$$||\varphi||^2_{\mathcal{H}_{n,m}(\omega)} := \sum_{k=0}^{n-1} \int_0^T \int_\omega ||(\partial_t - P)^k \varphi(t,x)||^2 dxdt.$$

and

$$||\varphi||^2_{\mathcal{I}_{n,m}(\omega)} := \sum_{k=0}^{n-1} \int_0^{\tilde{T}} \int_\omega ||\partial_t^{(k)} P^{n-1-k} \varphi(t,x)||^2 dxdt.$$

The result in the constant case

Theorem

Assume that the scalar equation (Scal-Eq) *verifies* (Obs-Ineg-Scal).

- *System* (A-Cst) *is observable in time* T *in norm* $\mathcal{H}_{n,m}(\omega)$, *i.e. there exists* $C > 0$ *s.t. for every* $Z^0 \in D(P^{n-1})$, *the solution* Z *of* (A-Cst) *verifies*
$$||Z(T)||^2_{L^2(\Omega)} \leqslant C||B^*Z||^2_{\mathcal{H}_{n,m}(\omega)} \qquad \text{(Obs-A-Cst)}$$
if and only if
$$rank\ K_A = n. \qquad \text{(Kal-A)}$$
- *System* (D-Cst) *is observable in time* $\tilde{T}$ *in norm* $\mathcal{I}_{n,m}$, *i.e. there exists* $C > 0$ *s.t. for every* $Z^0 \in D(P^{n-1})$, *the solution* Z *of* (D-Cst) *verifies*
$$||Z(\tilde{T})||^2_{D(P^{n-1})} \leqslant C||B^*Z||^2_{\mathcal{I}_{n,m}(\omega)} \qquad \text{(Obs-D-Cst)}$$
if and only if
$$rank\ K_D = n. \qquad \text{(Kal-D)}$$

Some remarks

- Results apply for second (or higher) order in time systems.
- D diagonalizable is crucial. Indeed, the first step of our proof is proving that (D-Cst):

$$||Z(\tilde{T})||^2_{L^2(\Omega)} \leqslant C \int_0^{\tilde{T}} \int_\omega ||Z(t,x)||^2 dxdt.$$

  However, unless D is diagonalizable, one cannot simply deduce this inequality from (Obs-Ineg-Scal) and this kind of inequality may even be hard to obtain. (See Fernandez-Cara et al.'15, COCV).
- The Kalman condition (Kal-D) may be rewritten in a more explicit way: for every $i \in [1, n|]$, there exists $j \in [|1, m|]$ such that $b_{ij \neq 0}$. If $m = 1$, observes each equation.

The result in the time-dependent case (1)

We introduce the notations

$$\overline{B}_0 = B \text{ and}$$
$$\overline{B}_i = A\overline{B}_{i-1} - \overline{B}'_{i-1} \text{ for } i \in \mathbb{N}^*.$$

Theorem

System (A-Time) *is observable in norm* $\mathcal{H}_{n,m}(\omega)$ *in the sense that there exists* $C > 0$ *such that for every* $Z^0 \in D(P^{n-1})$*, the solution* Z *of* (A-Time) *verifies*

$$||z(T)||^2_{L^2(\Omega)} \leqslant C||B^*Z||^2_{\mathcal{H}_{n,m}(\omega)} \qquad \text{(Obs-A-T)}$$

if there exist $\overline{t} \in [0, T]$ *such that*

$$Span\ \{\overline{B}_k(\overline{t})(\mathbb{R}^m) | k \in \mathbb{N}^*\} = \mathbb{R}^n. \qquad \text{(A-Si-Mi)}$$

The result in the time-dependent case (2)

We introduce the notations

$$\tilde{B}_0 = B \text{ and}$$

$$\tilde{B}_i = D\tilde{B}_{i-1} - \tilde{B}'_{i-1} \text{ for } i \in \mathbb{N}^*.$$

Theorem

System (D-Time) *is observable in norm* $\mathcal{I}_{n,m}(\omega)$ *in the sense that there exists* $C > 0$ *such that for every* $Z^0 \in D(P^{n-1})$*, the solution* Z *of* (D-Time) *verifies*

$$||z(T)||^2_{L^2(\Omega)} \leqslant C||B^*Z||^2_{\mathcal{I}_{n,m}(\omega)} \qquad \text{(Obs-D-T)}$$

if there exist $\tilde{t} \in [0, T]$ *such that*

$$Span\ \{\tilde{B}_k(\tilde{t})(\mathbb{R}^m) | k \in \mathbb{N}^*\} = \mathbb{R}^n. \qquad \text{(D-Si-Mi)}$$

Example: the Schrödinger equation

$(A,B) \in \mathcal{M}_n(\mathbb{K}) \times \mathcal{M}_{n,m}(\mathbb{C})$ constant matrices. We consider $-\Delta$ with domain $\mathcal{D}(-\Delta) = H^2(\Omega) \times H^1_0(\Omega)$. We call $H^{2n}_{(0)} = D((-\Delta)^n)$.

$$\begin{cases} \partial_t Y = \Delta Y + A^* Y & \text{in} \quad (0,T)\times\Omega, \\ Y(0,\cdot) = Y^0 \in H^{2n-2}_{(0)}. \end{cases} \qquad \text{(Wave-Sys)}$$

Observation operator: $\mathbb{1}_\omega B^* Z$ ($B \in \mathcal{M}_{n,m}(\mathbb{K})$). May require geometric conditions on ω like the Geometric Control Condition (Lebeau'92 (JMPA)). Case already studied in Liard-Lissy'17 (MCSS), were we obtained (by duality)

$$||Z(T)||^2_{(H^{2n-2}_{(0)})'} \leqslant C||B^*Z||^2_{L^2((0,T)\times\omega))}.$$

Here applying the previous result, we have

$$||Z(T)||^2_{L^2(\Omega)} \leqslant C||B^*Z||^2_{\mathcal{H}_{n,m}(\omega)} \leqslant C||B^*Z||^2_{L^2((0,T),H^{2n-2}_{(0)})}.$$

It is in general not so easy to "gain regularity on each side" without losing anything, see Dehman-Lebeau'09 (SICON), so a little bit weaker.
In this case, we would expect to have a result in natural space L^2.

Proof: observation of each component

Consider (A-Cst) and assume that (Kal-A) is verified. First prove

$$||Z(T)||^2_{L^2(\Omega)} \leqslant C \int_0^T \int_\omega ||Z(t,x)||^2 dxdt. \qquad \text{(Obs-All)}$$

Quite easy: if Z verifies (A-Cst), then $\tilde{Z} := \exp(-tA^*)Z$ verifies

$$\begin{cases} \partial_t \tilde{Z} = P\tilde{Z} & \text{in} \quad (0,T) \times L^2(\Omega)^n, \\ \tilde{Z}(0) = Z^0. \end{cases}$$

Hence, applying inequality (Obs-Ineg-Scal) on each line we obtain that

$$||\tilde{Z}(T)||^2_{L^2(\Omega)} \leqslant C \int_0^T \int_\omega ||\tilde{Z}(t,x)||^2 dxdt.$$

Inequality (Obs-All) is then easily deduced by remarking that

$$C_1||\tilde{Z}(t,x)^2|| \leqslant ||Z(t,x)^2|| \leqslant C_2||\tilde{Z}(t,x)^2||.$$

Proof: Kalman condition means equivalence of norms

Now, let us consider $||B^*Z||^2_{\mathcal{H}_{n,m}(\omega)}$. Using the definition of this space and equation (A-Cst), we deduce that

$$||B^*Z||^2_{\mathcal{H}_{n,m}(\omega)} = \sum_{k=0}^{n-1} \int_0^T \int_\omega ||B^*A^{*k}Z(t,x)||^2 dxdt.$$

Since (Kal-A) is verified, the following map

$$z = (z_1, \ldots z_n) \in \mathbb{R}^n \mapsto \sum_{k=0}^{n-1} ||B^*A^{*k}z||^2$$

defines a norm on $\mathbb{R}^n$, equivalent to the euclidean norm $z \mapsto ||z||^2$. Hence, we obtain

$$||B^*Z||^2_{\mathcal{H}_{n,m}(\omega)} \geqslant C \int_0^T \int_\omega ||Z(t,x)||^2 dxdt,$$

which enables us to deduce (Obs-A-Cst) thanks to (Obs-All).

Proof: the converse sense

The fact that (Obs-A-Cst) implies (Kal-A) is classical and can be handled for example by using the strategy of Ammar-Khodja et al.'09 (DEA).

- First prove the result for $m = 1$ by transforming (A-Cst) in the non-controllable Brunovsky canonical form, so that the system is in cascade form. The observation is made on the last components that are decoupled from the rest of the system, so that observability cannot hold.
- We treat the general case $m > 1$ by transforming (A-Cst) into a block triangular system where each diagonal block is in the Brunovsky canonical form.

Proof: observation of each component

Assume that (Kal-D) is verified. First prove

$$||Z(\tilde{T})||^2_{L^2(\Omega)} \leqslant C \int_0^{\tilde{T}} \int_\omega ||Z(t,x)||^2 dxdt. \qquad \text{(Obs-All)}$$

We consider the $i-th$ line of system (D-Cst) on $(0, \tilde{T})$, i.e.

$$\begin{cases} \partial_t z_i & = d_i P Z_i \quad \text{in} \quad (0, \tilde{T}) \times L^2(\Omega), \\ z_i(0) & = z_i^0. \end{cases}$$

Change of unknowns $\tilde{z}_i(t,x) := z_i(t/d_i, x)$. $\tilde{z}$ is now defined on $(0, T_i)$, where $T_i = d_i \tilde{T}$, and $\tilde{z}$ verifies (Scal-Eq).
One has $T_i \geqslant T$, hence using (Obs-Ineg-Scal) and the WP (Scal-Eq) we obtain

$$||\tilde{z}_i(T_i)||^2_{L^2(\Omega)} \leqslant C \int_0^{T_i} \int_\omega ||\tilde{z}_i(t,x)||^2 dxdt,$$

Changing variables and adding on i we obtain what we want.

Proof: Kalman condition means equivalence of norms

Now, we consider $||B^*Z||^2_{\mathcal{I}_{n,m}(\omega)}$. Using the definition of this space and equation (A-Cst), we deduce that

$$||B^*Z||^2_{\mathcal{I}_{n,m}(\omega)} = \sum_{k=0}^{n-1} \int_0^T \int_\omega ||B^* D^{*k} P^{n-1} Z(t,x)||^2 dxdt.$$

Since (Kal-D) is verified, the following map

$$z = (z_1, \dots z_n) \in \mathbb{R}^n \mapsto \sum_{k=0}^{n-1} ||B^* D^{*k} z||^2$$

is a norm on $\mathbb{R}^n$, equivalent to the euclidian one $z \mapsto ||z||^2$. Hence, we obtainthat

$$||B^*Z||^2_{\mathcal{I}_{n,m}(\omega)} \geqslant C \int_0^T \int_\omega ||P^{n-1} Z(t,x)||^2 dxdt.$$

Applying (Obs-All) to $P^{n-1}Z$, instead of Z, we know that

$$||Z(T)||^2_{D(P^{n-1})} \leqslant C ||B^*Z||^2_{\mathcal{I}_{n,m}(\omega)},$$

from which we deduce (Obs-D-Cst).

Almost everything works the same

- Observation on each component by making a change of unknown (first case, using the resolvent) or variable.
- Look at the observation term. Crucial point:

$$x = (x_1, \dots x_n) \in \mathbb{R}^n \mapsto \sum_{k=0}^{n-1} ||\overline{B}_i(t)^* x||^2$$

is a norm on $\mathbb{R}^n$, equivalent to the euclidean one $x \mapsto ||x||^2$, which means that there exists some constant $C(t) > 0$ such that

$$\sum_{k=0}^{n-1} ||\overline{B}_i^*(t)x||^2 \geqslant C(t)||x||^2.$$

Restricting (t_0, t_1) if necessary, we may always assume that $C(t) > C$ for $t \in [t_0, t_1]$ since $C(t)$ can be chosen as the smallest singular value of $[|\overline{B}_0|\overline{B}_1 \dots |\overline{B}_n|]^*$, which is continuous with respect to t, so that we the argument is the same.

We are interested in

$$\begin{cases} \partial_t Z = D^* \Delta Z + A^* Z & \text{in} \quad (0,T)\times\Omega, \\ Z(0) = Z^0, \end{cases} \tag{1}$$

with $Z^0 \in L^2(\Omega)^n$, $A^* \in \mathcal{M}_n(\mathbb{R})$ and $D^* \in \mathcal{M}_n(\mathbb{R})$ verifying

$$\langle D^* X, X\rangle \geqslant C||X||^2, \forall X \in \mathbb{R}^n. \tag{Ellip}$$

Sufficient to ensure the well-posedness.
Observation: done on

$$\sum_{i=1}^{m} B_i^* Z 1_{\omega_i},$$

where B_i is the i-th column of $B \in \mathcal{M}_{n,m}(\mathbb{K})$, and ω_i $(i \in [|1,n|])$ are some open subsets of Ω (the ω_i's may be disjoint).
Let $\{\lambda_k\}_{k\geq 1}$ be the eigenvalues of $-\Delta$ with Dirichlet boundary conditions and $e_k \in H_0^1(\Omega)$ be the corresponding normalized eigenfunctions. We also introduce the one-parameter $(\lambda > 0)$ family of matrices

$$K(\lambda) := [B|(-\lambda D + A)B| \dots \ |(-\lambda D + A)^{n-1}B].$$

Main result

Theorem

System (1) is observable on $(0, T)$ in the sense that for every $Z^0 \in L^2(\Omega)^n$, the solution Z of (1) verifies

$$||Z(T)||^2 \leqslant C \int_0^T \int_\omega \sum_{i=1}^m ||B_i^* Z(t,x) 1_{\omega_i}(x)||^2 dxdt$$

$$\text{if and only if } \operatorname{rank} K(\lambda_p) = n, \forall p \geqslant 1. \qquad \text{(Kalm-Spec)}$$

Same condition as in Ammar-Khodja et al.'09 JEE , but was restricted in this article to diagonalizable D and to $\omega_i = \omega$ (same observation subset).

Comments

Comments

- No extra assumption on the Jordan blocks of D contrary to Fernandez-Cara et al.'15 COCV.
- Internal or boundary controls with different control domains in Olive'12 MCSS, for $D = Id$.
- Strategy works for each elliptic operator for which a Lebeau-Robbiano inequality is known.
- Does not work for time-dependent or space-dependent coupling terms. Does not work for conservative systems.

1. Introduction
   - Observability inequalities in the scalar case
   - The case of systems
2. The proof
   - The first case
   - The second case
   - The time-dependent case
3. A particular case: systems of heat equations
   - Introduction and main result
   - The proof
4. Perspectives

Proof (1): decompositions

We only prove the difficult part, i.e. the reverse implication. We assume that (Kalm-Spec) holds. We use the Lebeau-Robbiano strategy. Decompose Z^0 as

$$Z^0(x) = \sum_{k=1} Z_k^0 e_k(x), \; Z_k^0 \in \mathbb{R}^n.$$

Then solution Z of (1) can be written as

$$Z(t,x) = \sum_{k=1}^{\infty} Z_k(t) e_k(x),$$

where Z_k is the unique solution of the ordinary differential system

$$\begin{cases} Z_k' &= (-\lambda_k D^* + A^*) Z_k, \\ Z_k(0) &= Z_k^0. \end{cases} \qquad \text{(ODE-k)}$$

Proof (2): Lebeau-Robbiano strategy

Lebeau-Robbiano'95: for any non-empty subset ω_i of Ω, there exists $C_i > 0$ such that for any $J > 0$ and any finite linear combination of the e_k $(k \leqslant J)$ given by $e(x) := \sum_{k \leqslant J} a_k e_k(x)$, we have

$$\sum_{k \leqslant J} |a_k|^2 = \int_{\Omega} \left(\sum_{k \leqslant J} a_k e_k(x) \right)^2 dx \leqslant C_i e^{C_i \sqrt{\lambda_J}} \int_{\omega_i} \left(\sum_{k \leqslant J} a_k e_k(x) \right)^2 dx. \tag{Spec-Ineq}$$

Proof (2)

Writing (Spec-Ineq) for each component of $B^* Z_k$ and adding on n we obtain that there exists $C > 0$ such that

$$\sum_{k \leqslant J} ||B^* Z_k(t)||^2 \leqslant C e^{C\sqrt{\lambda_J}} \sum_{j=1}^{J} \sum_{i=1}^{n} \int_{\omega_i} ||B_i^* Z_k(t) e_k(x)||^2 dx.$$

Integrating between 0 and T, we obtain

$$\int_0^T \sum_{k \leqslant J} ||B^* Z_k(t)||^2 dt \leqslant C e^{C\sqrt{\lambda_J}} \sum_{j=1}^{J} \sum_{i=1}^{n} \int_0^T \int_{\omega_i} ||B_i^* Z_k(t) e_k(x)||^2 dx dt.$$

We go back to (ODE-k). Thanks to the assumption (Kalm-Spec), we deduce that for every $k \in [|1, J|]$, system (ODE-k) is observable and we have the existence of some constant $C(\lambda_k) > 0$ such that

$$||Z_k(T)||^2 \leqslant C(\lambda_k) \int_0^T ||B^* Z_k(t)||^2 dt. \qquad \text{(Ineg-Zk)}$$

Proof (3)

Moreover, one can prove that there exists $p_1, p_2 \in \mathbb{N}$ (depending on n but independent of k) such that (Ineg-Zk) holds with

$$C(\lambda_k) \leqslant C\left(1+\frac{1}{T^{p_1}}\right)\lambda_k^{p_2}.$$

This is a consequence of the computations made in Seidman'88.
We just have to regroup all the previous inequalities and we obtain

$$\sum_{k\leqslant J} ||Z_k(T)||^2 \leqslant \sum_{k\leqslant J} C(\lambda_k)e^{C\sqrt{\lambda_J}}\int_0^T\int_\omega \sum_{j=1}^{J} ||Z_k(t)e_k(x)||^2 dxdt$$
$$\leqslant \tilde{C}\left(1+\frac{1}{T^{p_1}}\right)e^{\tilde{C}\sqrt{\lambda_J}}\int_0^T\int_\omega \sum_{j=1}^{J} Z_k(t)e_k(x)||^2 dxdt$$

for some new constant $\tilde{C} \geqslant C$. It is classical that we can obtain the desired observability inequality by coupling this equality with a dissipation estimate (cf. e.g. Miller'10 DCDS-B).

Summary

Perspectives (1)

- In the case of systems (A-Cst), (D-Cst), (A-Time) and (D-Time), can we obtain observability inequalities in the natural L^2-norm? If not, what are the optimal norms we can put?
- In the case of systems (D-Cst) and (D-Time), can we obtain the same Kalman or Silverman-meadows condition for non-diagonalizable coupling matrices D^*?
- In the case where P is self-adjoint but no spectral inequality is known, or in the case of parabolic systems of order two with time and space-dependent coefficients, can we obtain by other means necessary and sufficient conditions similar to (Kalm-Spec) for general systems of the form $\partial_t Z = D^* P Z + A^* Z$?
- Find necessary and sufficient conditions for general systems of the form $\partial_t Z = D^* P Z + A^* Z$ for unitary groups of operators like Schrödinger or wave equations.

Perspectives (2)

- In the case of systems (A-Cst), (D-Cst), (A-Time) and (D-Time), can we obtain observability inequalities with different (and possibly disjoint) observation subsets ω_i?
- Can we obtain the same characterizations if we couple different dynamics (for example systems of mixed heat and wave equations)? (cf. Zuazua'16, SCL)
- Can obtain the some results for an infinite number of coupled equations (i.e. $n = \infty$)?

The end

Thank you for your attention.