Minimal controllability time for the heat equation under unilateral state constraint

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The Problem

Consider the 1-D heat equation

$$\begin{aligned} \dot{y}(t,x) &= \partial_x^2 y(t,x) && (t \in \mathbb{R}_+^*,\ x \in (0,1)), \\ \partial_x y(t,0) &= v_0(t) && (t \in \mathbb{R}_+^*), \\ \partial_x y(t,1) &= v_1(t) && (t \in \mathbb{R}_+^*), \end{aligned}$$

with initial condition $\mathrm{y}^0 \geqslant 0$, given,

$$y(0,x) = \mathrm{y}^0(x) \qquad (x \in (0,1)).$$

The aim is to control this system to a constant steady state $\mathrm{y}^1 > 0$

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- for every time $T > 0$ there exists controls v_0 and $v_1 \in L^2(0,T)$ such that $y(T,\cdot) = \mathrm{y}^1$
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Is it possible to find $T > 0$ and controls v_0 and v_1 such that y satisfies $y(T,\cdot) = \mathrm{y}^1$ together with,

$$y(t,x) \geqslant 0 \qquad ((t,x) \in (0,1) \times (0,T) \text{ a.e.})\,?$$

First considerations I

If $\inf_{x\in(0,1)} \mathrm{y}^0(x) > \mathrm{y}^1$, then y^1 cannot be reached in arbitrarily small time T.

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- finally,

$$y(t,\tfrac{1}{2}) > \mathrm{y}^1 \qquad \text{for } t \in \left[0, \frac{1}{\pi^2} \ln \frac{\inf \mathrm{y}^0}{\mathrm{y}^1}\right) .$$

First considerations II

Due to the comparison principle, the constraint

$$y(t,x) \geqslant 0$$

is equivalent to the constraints

$$y(t,0) \geqslant 0 \qquad \text{and} \qquad y(t,1) \geqslant 0\,.$$

First considerations II

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Consequently, we will first consider the control problem

$$\begin{aligned}
\dot y(t,x) &= \partial_x^2 y(t,x) && (t \in \mathbb{R}_+^*\,,\ x \in (0,1))\,,\\
y(t,0) &= u_0(t) && (t \in \mathbb{R}_+^*)\,,\\
y(t,1) &= u_1(t) && (t \in \mathbb{R}_+^*)\,,
\end{aligned}$$

with the control constraints

$$u_0(t) \geqslant 0 \quad \text{and} \quad u_1(t) \geqslant 0 \qquad (t \geqslant 0 \ \text{ a.e.}).$$

The constrained Dirichlet control problem

Consider the 1-D heat equation

$$\dot{y}(t,x) = \partial_x^2 y(t,x) \qquad (t > 0\,,\ x \in (0,1))\,, \tag{1a}$$
$$y(t,0) = u_0(t) \qquad (t > 0)\,, \tag{1b}$$
$$y(t,1) = u_1(t) \qquad (t > 0)\,, \tag{1c}$$

with constant initial condition $\mathrm{y}^0 \in L^2(0,1)$, given,

$$y(0,x) = \mathrm{y}^0(x) \qquad (x \in (0,1))\,.$$

The aim is to control this system to a constant steady state $\mathrm{y}^1 > 0$

$$y(T,x) = \mathrm{y}^1 \qquad (x \in [0,1] \text{ a.e.})\,,$$

with the control constraints

$$u_0(t) \geqslant 0 \quad \text{and} \quad u_1(t) \geqslant 0 \qquad (t > 0 \ \text{ a.e.}).$$

Existence of controls

Proposition

For every $\mathrm{y}^0 \in L^2(0,1)$ *and every* $\mathrm{y}^1 \in \mathbb{R}_+^*$, *there exists a time* $T > 0$ *large enough and controls* $u_0, u_1 \in H^1(0,T)$ *such that*

$$u_0(t) > 0 \quad \text{and} \quad u_1(t) > 0 \qquad (t \in [0,T])$$

and the solution y *of* (1) *satisfies*

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This allows us to define

$$\underline{T}\left(\mathrm{y}^0, \mathrm{y}^1\right) = \inf\left\{T > 0,\ \exists u_0, u_1 \in L^1(0,T) \text{ s.t. } u_0 \geqslant 0,\ u_1 \geqslant 0 \text{ and } y(T,\cdot) = \mathrm{y}^1\right\} \geqslant 0,$$

Proof I
Existence of controls

For the proof, we also refer to Schmidt 1980

Set
$$\tilde{y}(t,x) = y(t,x) - \mathrm{y}^1\,, \quad \tilde{u}_0(t) = u_0(t) - \mathrm{y}^1 \quad \text{and} \quad \tilde{u}_1 = u_1 - \mathrm{y}^1\,,$$
Then, $\tilde{y}$ is solution of (1) with controls $\tilde{u}_0$ and $\tilde{u}_1$ and initial condition
$$\tilde{y}(0,x) = \mathrm{y}^0(x) - \mathrm{y}^1 \qquad (x \in (0,1))\,.$$

Consequently, we aim to prove the existence of a time $T > 0$ and controls $\tilde{u}_0$ and $\tilde{u}_1$ satisfying,
$$\tilde{u}_0(t) > -\mathrm{y}^1 \quad \text{and} \quad \tilde{u}_1(t) > -\mathrm{y}^1 \qquad (t \in (0,T) \ \text{ a.e.})$$
such that
$$\tilde{y}(T,\cdot) = 0\,.$$

For any $T > 0$ the existence of controls $\tilde{u}_0, \tilde{u}_1 \in H^1(0,T)$ such that $\tilde{y}(T,\cdot) = 0$ is ensured by Fattorini-Russel 1971.

Proof II

Existence of controls

In terms of the adjoint system,

$$\begin{aligned} -\dot{z}(t,x) &= \partial_x^2 z(t,x) && (t>0\,,\ x\in(0,1))\,,\\ z(t,0) &= z(t,1) = 0 && (t>0)\,,\\ z(T,x) &= \mathrm{z}^0(x) && (x\in(0,1))\,, \end{aligned}$$

there exists a constant $\tilde{c}(T)>0$ such that,

$$\|z(0,\cdot)\|_{L^2(0,1)}^2 \leqslant \tilde{c}(T)\left(\|\partial_x z(\cdot,0)\|_{H^{-1}(0,T)}^2 + \|\partial_x z(\cdot,1)\|_{H^{-1}(0,T)}^2\right) \qquad (\mathrm{z}^0\in L^2(0,1))\,.$$

This inequality being true in any time interval, we also have

$$\|z(\tfrac{T}{2},\cdot)\|_{L^2(0,1)}^2 \leqslant \tilde{c}(\tfrac{T}{2})\left(\|\partial_x z(\cdot,0)\|_{H^{-1}(0,T)}^2 + \|\partial_x z(\cdot,1)\|_{H^{-1}(0,T)}^2\right)$$

Using the dissipativity properties,

$$\|z(0,\cdot)\|_{L^2(0,1)}^2 \leqslant e^{-C_0\frac{T}{2}}\|z(\tfrac{T}{2},\cdot)\|_{L^2(0,1)}^2\,,$$

Proof III

Existence of controls

we obtain

$$\|z(0,\cdot)\|^2_{L^2(0,1)} \leqslant e^{-C_0 \frac{T}{2}} \tilde{c}(\tfrac{T}{2}) \left(\|\partial_x z(\cdot,0)\|^2_{H^{-1}(0,T)} + \|\partial_x z(\cdot,1)\|^2_{H^{-1}(0,T)} \right).$$

By duality this means that the controls $\tilde{u}_0$ and $\tilde{u}_1$ can be chosen such that

$$\|\tilde{u}_i\|^2_{H^1(0,T)} \leqslant e^{-C_0 \frac{T}{2}} \tilde{c}(\tfrac{T}{2}) \|\mathrm{y}^0 - \mathrm{y}^1\|^2_{L^2(0,1)} \qquad (i \in \{0,1\})$$

Using the embedding $H^1(0,T) \subset L^\infty(0,T)$,

$$\|\tilde{u}_i\|^2_{L^\infty(0,T)} \leqslant C e^{-C_0 \frac{T}{2}} \tilde{c}(\tfrac{T}{2}) \|\mathrm{y}^0 - \mathrm{y}^1\|^2_{L^2(0,1)} \qquad (i \in \{0,1\})$$

Thus, for T large enough,

$$\|\tilde{u}_0\|_{L^\infty(0,T)}, \ \|\tilde{u}_1\|_{L^\infty(0,T)} < \mathrm{y}^1$$

and hence,

$$\tilde{u}_0(t) > -\mathrm{y}^1 \quad \text{and} \quad \tilde{u}_1(t) > -\mathrm{y}^1 \qquad (t \in [0,T] \text{ a.e.}).$$

Minimal control time

Theorem

Let $\mathrm{y}_0 \in L^2(0,1)$ *and* $\mathrm{y}_1 \in \mathbb{R}_+^*$ *with* $\mathrm{y}_0 \neq \mathrm{y}_1$. *Then,*

1. $\underline{T} := \underline{T}\left(\mathrm{y}^0, \mathrm{y}^1\right) > 0$,
2. *there exist nonnegative controls* $\underline{u}_0, \underline{u}_1 \in \mathcal{M}(0, \underline{T})$ *such that the solution* y *with controls* $\underline{u}_0$ *and* $\underline{u}_1$ *satisfies* $y(T, \cdot) = \mathrm{y}^1$.

The solution y, of the Dirichlet control problem with controls in the set of Radon measures, is defined by transposition.

Remark

$\underline{T}\left(\mathrm{y}^0, \mathrm{y}^1\right) > 0$ *even if* $\mathrm{y}^0 < \mathrm{y}^1$.

Proof of $\underline{T} > 0$ I

Define $y_n(t) = \int_0^1 y(t,x)\sin(n\pi x)\,\mathrm{d}x$, where y is solution of (1). We have

$$\dot{y}_n(t) = \int_0^1 \partial_x^2 y(t,x)\sin(n\pi x)\,\mathrm{d}x = -n\pi\int_0^1 \partial_x y(t,x)\cos(n\pi x)\,\mathrm{d}x$$
$$= n\pi\left(u_0(t) - (-1)^n u_1(t)\right) - (n\pi)^2 y_n(t)$$

with $y_n(0) = \int_0^1 \mathrm{y}^0(x)\sin(n\pi x)\,\mathrm{d}x := \mathrm{y}_n^0$. Thus,

$$y_n(T) = e^{-(n\pi)^2 T}\mathrm{y}_n^0 + n\pi\int_0^T e^{-(n\pi)^2(T-t)}\left(u_0(t) - (-1)^n u_1(t)\right)\,\mathrm{d}t\,.$$

If $y(T,x) \equiv \mathrm{y}_1$, we have $y_n(T) = \int_0^1 \mathrm{y}_1 \sin(n\pi x)\,\mathrm{d}x = \dfrac{1-(-1)^n}{n\pi}\mathrm{y}_1$.

Consequently,

$$\frac{1-(-1)^n}{n\pi}\mathrm{y}^1 - e^{-(n\pi)^2 T}\mathrm{y}_n^0 = n\pi\int_0^T e^{-(n\pi)^2(T-t)}\left(u_0(t) - (-1)^n u_1(t)\right)\,\mathrm{d}t\,.$$

Proof of $\underline{T} > 0$ II

For $n = 2p$,

$$\int_0^T e^{(2p\pi)^2 t} \left(u_0(t) - u_1(t)\right) \mathrm{d}t = \frac{\mathrm{y}^0_{2p}}{2p\pi}\,,$$

For $n = 2p+1$,

$$\frac{2\,\mathrm{y}^1}{(2p+1)\pi} - e^{-(2p+1)^2\pi^2 T}\,\mathrm{y}^0_{2p+1} = (2p+1)\pi \int_0^T e^{-(2p+1)^2\pi^2(T-t)} \left(u_0(t) + u_1(t)\right) \mathrm{d}t\,.$$

But,

$$e^{-(2p+1)^2\pi^2 T} \leqslant e^{-(2p+1)^2\pi^2(T-t)} \leqslant 1 \qquad (t \in [0,T])\,.$$

u_0 and u_1 being nonnegative,

$$\begin{aligned} e^{-(2p+1)^2\pi^2 T} \int_0^T \left(u_0(t) + u_1(t)\right) \mathrm{d}t \leqslant \int_0^T e^{-(2p+1)^2\pi^2(T-t)} \left(u_0(t) + u_1(t)\right) \mathrm{d}t \\ \leqslant \int_0^T \left(u_0(t) + u_1(t)\right) \mathrm{d}t\,, \end{aligned}$$

Proof of $\underline{T} > 0$ III

We have obtained,

$$\frac{2\,\mathrm{y}^1}{(2p+1)^2\pi^2} - e^{-(2p+1)^2\pi^2 T}\,\frac{\mathrm{y}^0_{2p+1}}{(2p+1)\pi} \leqslant \int_0^T (u_0(t)+u_1(t))\,\mathrm{d}t$$
$$\leqslant e^{(2p+1)^2\pi^2 T}\,\frac{2\,\mathrm{y}^1}{(2p+1)^2\pi^2} - \frac{\mathrm{y}^0_{2p+1}}{(2p+1)\pi}\,.$$

If for every $T>0$ there exists nonnegative controls u_0^T and u_1^T steering y_0 to y_1 in time T, then

$$\lim_{T\to 0}\int_0^T \left(u_0^T(t)+u_1^T(t)\right)\,\mathrm{d}t = \frac{2\,\mathrm{y}^1}{(2p+1)^2\pi^2} - \frac{\mathrm{y}^0_{2p+1}}{(2p+1)\pi} := \gamma\in\mathbb{R} \qquad (p\in\mathbb{N})\,.$$

Hence,

$$\mathrm{y}^0_{2p+1} = \frac{2\mathrm{y}^1}{(2p+1)\pi} - (2p+1)\pi\gamma \qquad (p\in\mathbb{N})\,.$$

$\mathrm{y}^0\in L^2(0,1)$, ensures that $\sum_{n=0}^{\infty}\left|\mathrm{y}^0_n\right|^2<\infty$ and hence $\gamma=0$, $\mathrm{y}^0_{2p+1}=\dfrac{2\mathrm{y}^1}{(2p+1)\pi}$ and

$$\lim_{T\to 0}\int_0^T \left(u_0^T(t)+u_1^T(t)\right)\,\mathrm{d}t = 0\,.$$

Proof of $\underline{T} > 0$ IV

Since $u_0^T \geqslant 0$ and $u_1^T \geqslant 0$, we can also conclude

$$\lim_{T\to 0} \int_0^T u_0^T(t)\,\mathrm{d}t = \lim_{T\to 0} \int_0^T u_1(t)\,\mathrm{d}t = 0\,.$$

consequently passing to the limit $T \to 0$ in

$$\int_0^T e^{(2p\pi)^2 t} \left(u_0^T(t) - u_1^T(t)\right)\,\mathrm{d}t = \frac{\mathrm{y}^0_{2p}}{2p\pi}\,,$$

we obtain

$$\mathrm{y}^0_{2p} = 0 \qquad (p \in \mathbb{N}^*)\,.$$

All in all, since the family $\left\{\sqrt{2}\sin(n\pi\,\cdot)\right\}_{n\in\mathbb{N}^*}$ is an orthonormal basis of $L^2(0,1)$, we conclude that y^0 can be steered to y^1 in arbitrarily small time with nonnegative controls if and only if

$$\mathrm{y}^0(x) = \mathrm{y}^1 \qquad (x \in (0,1))\,.$$

Proof of Controllability in the minimal time $\underline{T}$ I

Define $(\varepsilon_k)_{k\in\mathbb{N}}$ a sequence of positive numbers converging to 0.
For every $k \in \mathbb{N}$, there exist nonnegative controls $u_0^k, u_1^k \in L^1(0, \underline{T} + \varepsilon_k)$, so that the solution y satisfies $y(\underline{T} + \varepsilon_k, \cdot) = \mathrm{y}^1$.
Define $\bar{\varepsilon} = \sup_{k\in\mathbb{N}} \varepsilon_k$.

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For every $k \in \mathbb{N}$, there exist nonnegative controls $u_0^k, u_1^k \in L^1(0, \underline{T} + \varepsilon_k)$, so that the solution y satisfies $y(\underline{T} + \varepsilon_k, \cdot) = \mathrm{y}^1$.
Define $\bar{\varepsilon} = \sup_{k\in\mathbb{N}} \varepsilon_k$.

According to

$$\frac{2\,\mathrm{y}^1}{(2p+1)\pi} - e^{-(2p+1)^2\pi^2 T}\,\mathrm{y}^0_{2p+1} = (2p+1)\pi \int_0^T e^{-(2p+1)^2\pi^2(T-t)} \left(u_0^k(t) + u_1^k(t)\right) \mathrm{d}t\,,$$

we obtain,

$$\|u_0^k\|_{L^1(0,\underline{T}+\bar{\varepsilon})} + \|u_1^k\|_{L^1(0,\underline{T}+\bar{\varepsilon})} = \int_0^{\underline{T}+\varepsilon_k} \left(u_0^k(t) + u_1^k(t)\right) \mathrm{d}t$$

$$\leqslant \inf_{p\in\mathbb{N}} \left(e^{(2p+1)^2\pi^2(\underline{T}+\varepsilon_k)} \frac{2\,\mathrm{y}^1}{(2p+1)^2\pi^2} - \frac{\mathrm{y}^0_{2p+1}}{(2p+1)\pi} \right)$$

$$\leqslant \frac{2e^{\pi^2(\underline{T}+\bar{\varepsilon})}\,|\mathrm{y}^1|}{\pi^2} + \frac{|\mathrm{y}^0_1|}{\pi} \leqslant \infty\,.$$

Proof of Controllability in the minimal time $\underline{T}$ II

In conclusion,

- The sequences $(u_0^k)_k$ and $(u_1^k)_k$ are bounded in $L^1(0, \underline{T} + \bar{\varepsilon})$;
- $(u_0^k)_k$ and $(u_1^k)_k$ have their support contained in $[0, \underline{T} + \varepsilon_k]$, with $\varepsilon_k \to 0$;
- Thus, they are (up to a subsequence) weakly convergent in the sense of measures to some nonnegative controls $\underline{u}_i$ in $\mathcal{M}([0, \underline{T}])$;
- These limits ensure the control requirements in the minimal control time $\underline{T}$.

□

Lower bounds on $\underline{T}$

When y^0 is a constant initial condition, $\underline{T} := \underline{T}(\mathrm{y}^0, \mathrm{y}^1)$ satisfies

1. if $\mathrm{y}^1 < \mathrm{y}^0$,

$$\underline{T} > \frac{1}{\pi^2} \log \frac{\mathrm{y}^0}{\mathrm{y}^1} \qquad \text{and} \qquad \sup_{p \in \mathbb{N}^*} \frac{1}{(2p+1)^2} \left(\frac{\mathrm{y}^1}{\mathrm{y}^0} - e^{-(2p+1)^2 \pi^2 \underline{T}} \right) \leqslant \frac{\mathrm{y}^1}{\mathrm{y}^0} e^{\pi^2 \underline{T}} - 1 \,.$$

For $\mathrm{y}^0 \equiv 5$ and $\mathrm{y}^1 \equiv 1$, we obtain (numerically): $\underline{T} \geqslant 0.165297$;

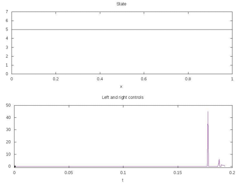
2. if $\mathrm{y}^1 > \mathrm{y}^0$,

$$\frac{\mathrm{y}^1}{\mathrm{y}^0} - e^{-\pi^2 \underline{T}} \leqslant \inf_{p \in \mathbb{N}^*} \frac{1}{(2p+1)^2} \left(\frac{\mathrm{y}^1}{\mathrm{y}^0} e^{(2p+1)^2 \pi^2 \underline{T}} - 1 \right) \,.$$

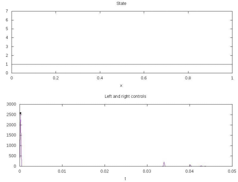
For $\mathrm{y}^0 \equiv 1$ and $\mathrm{y}^1 \equiv 5$, we obtain (numerically): $\underline{T} \geqslant 0.023076$;

Numerical examples

- From $\mathrm{y}^0 \equiv 5$ to $\mathrm{y}^1 \equiv 1$, $\underline{T}\left(\mathrm{y}^0, \mathrm{y}^1\right) \simeq 0.1931$.



- From $\mathrm{y}^0 \equiv 1$ to $\mathrm{y}^1 \equiv 5$, $\underline{T}\left(\mathrm{y}^0, \mathrm{y}^1\right) \simeq 0.0438$.



Heat equation with nonnegative state constraint I

Consider the 1-D heat equation

$$\begin{aligned} \dot y(t,x) &= \partial_x^2 y(t,x) + \mathbf{1}_\omega(x) w(t,x) && (t>0\,,\ x\in(0,1))\,,\\ \partial_x y(t,0) &= v_0(t) && (t>0)\,,\\ \partial_x y(t,1) &= v_1(t) && (t>0)\,, \end{aligned}$$

with given initial condition $\mathrm{y}^0 \geqslant 0$,

$$y(0,\cdot) = \mathrm{y}^0 \in L^2(0,1)\,.$$

The aim is to control this system to a constant steady state $\mathrm{y}^1 > 0$

$$y(T,x) = \mathrm{y}^1 \qquad (x\in(0,1)\ \text{a.e.})\,,$$

with the state constraint,

$$y(t,x) \geqslant 0 \qquad (t\geqslant 0\,,\ x\in(0,1)\ \ \text{a.e.}).$$

We assume $\omega \subset (0,1)$ is such that there exists an interval $(a,b) \subset (0,1)\setminus\omega$.

Heat equation with nonnegative state constraint II

For $v_0, v_1 \in L^2(0,T)$ and $w \in L^2((0,T)\times\omega)$, define

$$u_a = y(\cdot, a) \in L^2(0,T) \quad \text{and} \quad u_b = y(\cdot, b) \in L^2(0,T)\,.$$

Furthermore, $y|_{(a,b)}$ is solution of

$$\begin{aligned} \dot y(t,x) &= \partial_x^2 y(t,x) && (t>0\,,\ x\in(a,b))\,,\\ y(t,a) &= u_a(t) && (t>0)\,,\\ y(t,b) &= u_b(t) && (t>0)\,,\end{aligned}$$

Consequently, if v_0, v_1 and w are controls in time $T>0$ such that

$$y(t,x) \geqslant 0 \qquad \text{and} \qquad y(T,x) = \mathrm{y}^1\,,$$

then we have

$$u_a(t) \geqslant 0 \quad \text{and} \quad u_b(t) \geqslant 0 \qquad (t\in[0,T] \ \text{ a.e.})$$

and hence T cannot be arbitrarily small unless $\mathrm{y}^0|_{(0,1)\setminus\omega} = \mathrm{y}^1|_{(0,1)\setminus\omega}$.

Numerical example I

Consider the 1-D heat equation with Neumann controls

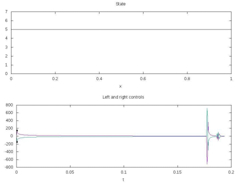
$$\begin{aligned} \dot{y}(t,x) &= \partial_x^2 y(t,x) && (t>0\,,\ x\in(0,1))\,,\\ \partial_x y(t,0) &= v_0(t) && (t>0)\,,\\ \partial_x y(t,1) &= v_1(t) && (t>0)\,, \end{aligned}$$

with the state constraint,

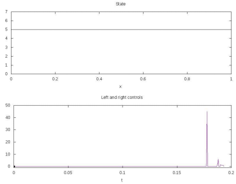
$$y(t,x) \geqslant 0 \qquad (t\geqslant 0\,,\ x\in(0,1)\ \text{ a.e.}).$$

Numerical example II

- From $y^0 \equiv 5$ to $y^1 \equiv 1$, $\underline{T}(y^0, y^1) \simeq 0.1938$.



Remind that with Dirichlet controls, we had,



The constrained Dirichlet control problem in a ball

Set $D = B(0,1) \subset \mathbb{R}^d$. We consider the control system

$$\begin{aligned} \dot{y}(t,x) &= \Delta y(t,x) & (t > 0\,,\ x \in D)\,, \\ y(t,x) &= u(t,x) & (t > 0\,,\ x \in \partial D)\,, \end{aligned}$$

with initial condition in $L^2(D)$,

$$y(0,x) = \mathrm{y}^0(x) \qquad (x \in D)\,.$$

The aim is to steer y to a constant target $\mathrm{y}^1 \in \mathbb{R}_+^*$ with nonnegative controls $u \in L^2(0,T;L^2(\partial D))$, i.e.

$$u(t,x) \geqslant 0 \qquad (t > 0\,,\ x \in \partial D \ \text{ a.e.})\,.$$

Minimal control time

Proposition

Set $\mathrm{y}^0 \in L^2(D)$ *and* $\mathrm{y}^1 \in \mathbb{R}_+^*$ *with* $\mathrm{y}^0 \neq \mathrm{y}^1$.
Then there exists $T > 0$ *and a strictly positive control* $u \in L^2(0, T, L^2(\partial D))$, *such that* y *satisfies* $y(T, \cdot) = \mathrm{y}^1$.

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Set $\mathrm{y}^0 \in L^2(D)$ *and* $\mathrm{y}^1 \in \mathbb{R}_+^*$ *with* $\mathrm{y}^0 \neq \mathrm{y}^1$.
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Thus, we can define,

$$\underline{T}\left(\mathrm{y}^0, \mathrm{y}^1\right) = \inf\left\{T > 0\,,\ \exists u \in L^1((0,T)\times\partial D) \text{ s.t. } u \geqslant 0 \text{ and } y(T,\cdot) = \mathrm{y}^1\right\} \geqslant 0\,.$$

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Proposition

Set $y^0 \in L^2(D)$ and $y^1 \in \mathbb{R}_+^*$ with $y^0 \neq y^1$.
Then there exists $T > 0$ and a strictly positive control $u \in L^2(0, T, L^2(\partial D))$, such that y satisfies $y(T, \cdot) = y^1$.

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Theorem

Given $y^0 \in L^2(D)$ and $y^1 \in \mathbb{R}_+^*$, with $y^0 \neq y^1$.
Then we have

1. $\underline{T}(y^0, y^1) > 0$;
2. *In time $T = \underline{T}(y^0, y^1)$, there exist a control $\underline{u} \in \mathcal{M}([0, \underline{T}] \times \partial D)$, $\underline{u} \geqslant 0$, steering y^0 to y^1 in time $\underline{T}$.*

Proof of $\underline{T} > 0$ I

Proof (for $y^0 \in \mathbb{R}$).
Define $(\lambda_n)_{n\in\mathbb{N}^*}$ and $(p_n)_{n\in\mathbb{N}^*}$ solutions of the Sturm-Liouville problems:

$$p_n''(r) + \frac{d-1}{r} p_n'(r) = -\lambda_n p_n(r) \qquad (r \in (0,1)),$$
$$p_n(1) = p_n'(0) = 0,$$

in order to fix p_n, we enforce:

$$p_n(0) = 1 \qquad \text{and define} \qquad \alpha_n = p_n'(1) \neq 0 .$$

We have $\lambda_n > 0$ and λ_n two by two distinct (Pöschel-Trubowitz 1987).
Let us then define:

$$\varphi_n(x) = p_n(|x|) \qquad (x \in D),$$

so that we have,

$$\Delta\varphi_n(x) = -\lambda_n \varphi_n(x) \qquad (x \in D),$$
$$\varphi_n(x) = 0 \qquad (x \in \partial D)$$

and $\nabla\varphi_n(x) \cdot \mathrm{n}(x) = \alpha_n,$ for every $x \in \partial D$.

Proof of $\underline{T} > 0$ II

Let $T > 0$ and $u^T \in L^1((0,T) \times \partial D)$ be a nonnegative control such that y (with initial condition $\mathrm{y}^0 \in \mathbb{R}_+^*$) satisfies $y(T,\cdot) = \mathrm{y}^1$.

For every $n \in \mathbb{N}^*$, we define $y_n(t) = \displaystyle\int_D y(t,x)\varphi_n(x)\,\mathrm{d}x$. Integrating by parts, we obtain,

$$\begin{aligned}\dot{y}_n(t) &= \int_D \Delta y(t,x)\varphi_n(x)\,\mathrm{d}x \\ &= -\int_{\partial D} y(t,x)\nabla\varphi_n(x)\cdot \mathrm{n}(x)\,\mathrm{d}\Gamma_x + \int_D y(t,x)\Delta\varphi_n(x)\,\mathrm{d}x \\ &= -\lambda_n y_n(t) - \alpha_n \int_{\partial D} u^T(t,x)\,\mathrm{d}\Gamma_x\end{aligned}$$

and hence, $$y_n(T) = e^{-\lambda_n T} y_n(0) - \alpha_n \int_0^T e^{-\lambda_n(T-t)} \int_{\partial D} u^T(t,x)\,\mathrm{d}\Gamma_x\,\mathrm{d}t.$$

Setting $\mathrm{y}_n^i := \mathrm{y}^i \displaystyle\int_D \varphi_n(x)\,\mathrm{d}x = -\omega_{d-1}\frac{\alpha_n}{\lambda_n}\mathrm{y}^i$ for $i \in \{0,1\}$, we obtain

$$\frac{\omega_{d-1}}{\lambda_n}\left(\mathrm{y}^1 - e^{-\lambda_n T}\mathrm{y}^0\right) = \int_0^T e^{-\lambda_n(T-t)} \int_{\partial D} u^T(t,x)\,\mathrm{d}\Gamma_x\,\mathrm{d}t\,.$$

Proof of $\underline{T} > 0$ III

Since $u^T \geqslant 0$ and $\lambda_n > 0$, we obtain:

$$e^{-\lambda_n T} \int_0^T \int_{\partial D} u^T(t,x)\, \mathrm{d}\Gamma_x\, \mathrm{d}t \leqslant \frac{\omega_{d-1}}{\lambda_n} \left(\mathrm{y}^1 - e^{-\lambda_n T} \mathrm{y}^0\right) \leqslant \int_0^T \int_{\partial D} u^T(t,x)\, \mathrm{d}\Gamma_x\, \mathrm{d}t\,,$$

that is to say,

$$\frac{\omega_{d-1}}{\lambda_n} \left(\mathrm{y}^1 - e^{-\lambda_n T} \mathrm{y}^0\right) \leqslant \int_0^T \int_{\partial D} u^T(t,x)\, \mathrm{d}\Gamma_x\, \mathrm{d}t \leqslant \frac{\omega_{d-1}}{\lambda_n} \left(e^{\lambda_n T} \mathrm{y}^1 - \mathrm{y}^0\right).$$

Thus, if, for every $T > 0$, such a nonnegative control u^T exists, we have

$$\lim_{T \to 0} \int_0^T \int_{\partial D} u^T(t,x)\, \mathrm{d}\Gamma_x\, \mathrm{d}t = \frac{\omega_{d-1}}{\lambda_n} \left(\mathrm{y}^1 - \mathrm{y}^0\right) := \gamma \in \mathbb{R} \qquad (n \in \mathbb{N}^*)\,.$$

This is impossible since the values of λ_n are two by two distinct and $\mathrm{y}^0 \neq \mathrm{y}^1$.

Proof of controllability in time $\underline{T}$

For every $T > \underline{T}$, we have,

$$\|u^T\|_{L^1((0,T)\times\partial D)} = \int_0^T \int_{\partial D} u^T(t,x)\,\mathrm{d}\Gamma_x\,\mathrm{d}t \leqslant \frac{\omega_{d-1}}{\lambda_n}\left(e^{\lambda_n T}\mathrm{y}^1 - \mathrm{y}^0\right).$$

This, together with the vague convergence to measures ensures that (up to a subsequence)

u^T converges to some nonnegative control $\underline{u} \in \mathcal{M}([0,\underline{T}] \times \partial D)$.

□

Consequences for the n-D heat equation with nonnegative state constraints I

Consider the control problem:

$$\begin{aligned} \dot{y}(t,x) &= \mathrm{div}\,(A\nabla y(t,x)) + \mathbf{1}_\omega(x)w(t,x) && (t>0\,,\ x\in\Omega)\,,\\ \nabla y(t,x)\cdot n(x) &= v(t,x) && (t>0\,,\ x\in\partial\Omega)\,, \end{aligned}$$

with the constant and nonnegative initial condition,

$$y(0,\cdot)=\mathrm{y}^0\in L^2(\Omega),\qquad \text{s.t.}\quad \mathrm{y}^0\geqslant 0\,.$$

where Ω is an open bounded and regular set of $\mathbb{R}^d$, $A\in\mathbb{R}^{d\times d}$ is a positive matrix, and $\omega\subset\Omega$.

Given $\mathrm{y}^1\in\mathbb{R}_+^*$, the aim is to find controls v and w such that

$$y(T,\cdot)=\mathrm{y}^1\quad\text{ and }\quad y(t,x)\geqslant 0\,.$$

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$$y(T,\cdot) = \mathrm{y}^1 \quad \text{and} \quad y(t,x) \geqslant 0\,.$$

Setting $A = P^\top P$, $\tilde{x} = Px$ and $\tilde{y}(t, P^\top x) = y(t,x)$, it is enough to control the system

$$\begin{aligned} \dot{\tilde{y}}(t,\tilde{x}) &= \Delta\tilde{y}(t,\tilde{x}) + \mathbf{1}_{P\omega}(\tilde{x})\tilde{w}(t,\tilde{x}) && (t>0\,,\ \tilde{x}\in P\Omega)\,,\\ \nabla\tilde{y}(t,\tilde{x})\cdot\tilde{n}(\tilde{x}) &= \tilde{v}(t,\tilde{x}) && (t>0\,,\ \tilde{x}\in P\partial\Omega)\,,\end{aligned}$$

$$\tilde{y}(0,\tilde{x}) = \tilde{\mathrm{y}}^0(\tilde{x}) \geqslant 0 \qquad (\tilde{x}\in P\Omega)\,,$$

with the constraints $\quad \tilde{y}(T,\tilde{x}) = \mathrm{y}^1 \quad$ and $\quad \tilde{y}(t,\tilde{x}) \geqslant 0.$

Consequences for the n-D heat equation with nonnegative state constraints II

We are back to a system of the form:

$$\begin{aligned} \dot{y}(t,x) &= \Delta y(t,x) + \mathbf{1}_\omega(x) w(t,x) && (t>0\,,\ x\in\Omega)\,,\\ \nabla y(t,x)\cdot n(x) &= v(t,x) && (t>0\,,\ x\in\partial\Omega)\,,\\ y(0,x) &= \mathrm{y}^0(x) \geqslant 0 && (x\in\Omega)\,, \end{aligned}$$

with the constraints,

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with the constraints,

$$y(T,\cdot) = \mathrm{y}^1 \quad \text{and} \quad y(t,x) \geqslant 0\,.$$

Assume there exists $x_0 \in \Omega$ and $\varepsilon > 0$ such that $B(x_0,\varepsilon) \subset \Omega \setminus \omega$.

Set $T > 0$ and assume the exists such controls v and w with $v \in L^2((0,T)\times\partial\Omega)$ and $w \in L^2((0,T)\times\omega)$, due to regularity results (see Lions-Magenes 1968), we have $u_0 \in L^2((0,T)\times\partial B(x_0,\varepsilon))$, with

$$u_0(t,\cdot) = y(t,\cdot)|_{\partial B(x_0,\varepsilon)}\,.$$

Further more, $y \geqslant 0$ ensures that $u_0 \geqslant 0$. Consequently,

T cannot be arbitrarily small.

Our proofs are based on spectral decomposition and this can be used to prove similar results for:

- Controllability to any kind of nonnegative steady state;
- 1-D parabolic equation, $\dot{y} = \partial_x (a(x)\partial_x y) - p(x)\partial_x y$ with internal and/or boundary control;
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Some additional open questions

- Structure and uniqueness of the nonnegative Dirichlet controls in the minimal time $\underline{T}$?
- How the time optimal control is related to the adjoint state?
  Numerical examples: $1 \to 5$ $5 \to 1$

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THANK YOU, FOR YOUR ATTENTION!