Quadratic obstructions to controllability: from ODEs to PDEs

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Take-home message

For small-time local controllability of scalar-input systems, there is nothing good to be expected from a second-order expansion!

old control theory saying

Small-time local controllability

Control system

$$\dot{x} = f(x, u), \tag{1}$$

with $x(t) \in X$, $u(t) \in U$ and $f(0,0) = 0$.

Definition

System (1) is *small-time locally controllable* when

$$\forall T > 0, \forall \eta > 0, \exists \delta > 0, \forall x^*, x^\dagger \in X,$$
$$|x^*| + |x^\dagger| \leqslant \delta \quad \Rightarrow \quad \exists u : [0,T] \to U,$$
$$\|u\| \leqslant \eta, \quad x(0) = x^* \text{ and } x(T) = x^\dagger.$$

Linear test: sufficient Kalman rank condition

Let $X = \mathbb{R}^n$ and

$$\dot{y} = Ay + u(t)b$$

is controllable if and only if

$$\operatorname{rang}\left\{b, Ab, \dots A^{n-1}b\right\} = n. \tag{2}$$

Proposition

Let $f \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$. Assume (A, B), where $A := \partial_x f(0,0)$ et $b := \partial_u f(0,0)$ satisfies (2). Then (1) is STLC.

Scalar-input control-affine systems

Useful first-step

$$\dot{x} = f_0(x) + u f_1(x). \tag{3}$$

Necessary condition (analytic systems)

$$\mathrm{Lie}\left\{f_0, f_1\right\}(0) = \mathbb{R}^n.$$

Lie brackets spaces, for $k \in \mathbb{N}^*$:

$$\begin{aligned}\mathcal{S}_k(0) := \{&\text{iterated brackets of } f_0 \text{ and } f_1,\\ &\text{involving } f_1 \text{ at most } k \text{ times},\\ &\text{evaluated at } 0\}.\end{aligned}$$

For $X, Y \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^n)$, $[X, Y](x) := Y'(x)X(x) - X'(x)Y(x)$.

Known sufficient condition

Proposition [Sussmann, 1983]

Let $f_0, f_1 \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Assume

$$\mathrm{Lie}\{f_0, f_1\}(0) = \mathbb{R}^n,$$
$$\mathcal{S}_{2k+2}(0) = \mathcal{S}_{2k+1}(0), \quad \forall k \in \mathbb{N}.$$

Then (3) is STLC.

Brackets involving f_1 an odd number of times are good, others are bad.

Some necessary conditions

Proposition [Sussmann, 1983 - Stefani, 1986]

Let f_0, f_1 analytic. Assume (3) is STLC Then, for $k \in \mathbb{N}^*$:

$$\mathrm{ad}_{f_1}^{2k}(f_0)(0) \in \mathcal{S}_{2k-1}(0).$$

In particular, for $k = 1$:

$$[f_1, [f_1, f_0]](0) \in \mathcal{S}_1(0).$$

But, what about the condition

$$\mathcal{S}_2(0) = \mathcal{S}_1(0)?$$

A show-stopper

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1, \\ \dot{x}_3 = x_2^2 + x_1^3. \end{cases} \tag{4}$$

Here, $\mathcal{S}_1(0) = \mathbb{R}e_1 + \mathbb{R}e_2$ and $[[f_0, f_1], [f_0, [f_0, f_1]]](0) = -e_3$. So $\mathcal{S}_2(0) \not\subset \mathcal{S}_1(0)$. But Sussmann proved that (4) is STLC.

Questions

- Can we build a quadratic theory for $\dot{x} = f_0(x) + u f_1(x)$?
- Can we move towards a necessary and sufficient condition for STLC?

Our result

With Karine Beauchard, we prove that, for any system

- a quadratic alternative (coercive drift or invariant manifold) holds
- positive results can only come from the cubic (or higher) terms,
- coercive drift prevents a strong version of STLC

Key idea: bring analysis into geometry.

A more precise definition

Let $(E_T, \|\cdot\|_{E_T})$ family of normed vector spaces of functions from $[0, T]$ to $\mathbb{R}$ (e.g., L^∞, C^2, $W^{4,\infty}$).

Definition

A systems is E *STLC* when STLC holds with controls small for $\|\cdot\|_{E_T}$.

For any $T > 0$, $\eta > 0$, there exists $\delta > 0$ such that, for any $x^*, x^\dagger \in \mathbb{R}^n$ with $|x^*| + |x^\dagger| \leqslant \delta$, there exists $u \in E_T$ with $\|u\|_{E_T} \leqslant \eta$ such that $x(0) = x^*$ à $x(T) = x^\dagger$.

Holds for Kalman, for $\dot{x}_1 = u$, $\dot{x}_2 = x_1^3$, and so on.

Our alternative - general setting

Theorem [Beauchard, F.M., 2017]

Let $f_0, f_1 \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $f_0(0) = 0$. Let $d := \dim \mathcal{S}_1(0)$. There exists $G \in C^\infty(\mathcal{S}_1(0), \mathcal{S}_1(0)^\perp)$ such that $G(0) = 0$ and $G'(0) = 0$ and a smooth manifold $\mathcal{M}$ of dimension d defined as

$$\mathcal{M} := \{p + G(p), \ \ p \in \mathcal{S}_1(0)\},$$

such that we have the following alternative

Our alternative - case $\mathcal{S}_2(0) = \mathcal{S}_1(0)$

Theorem [Beauchard, F.M., 2017]

For any $\mathcal{T} > 0$, there exists $C, \eta > 0$ such that, for any $T \in (0, \mathcal{T}]$, for any trajectory $(x, u) \in \mathcal{C}^0([0,T], \mathbb{R}^n) \times L^1((0,T), \mathbb{R})$ of (3) with $x(0) = 0$ satisfying $\|u\|_{W^{-1,\infty}} \leqslant \eta$, we have

$$\forall t \in [0,T], \quad \left| \mathbb{P}^\perp x(t) - G\left(\mathbb{P}x(t)\right) \right| \leqslant C \left\|u\right\|_{W^{-1,3}}^3 .$$

Our alternative - case $\mathcal{S}_2(0) \not\subset \mathcal{S}_1(0)$

Theorem [Beauchard, F.M., 2017]

There exists $1 \leqslant k \leqslant d$ such that

$$[\mathrm{ad}_{f_0}^{j-1}(f_1), \mathrm{ad}_{f_0}^{j}(f_1)](0) \in \mathcal{S}_1(0) \quad \text{for } 1 \leqslant j < k,$$
$$[\mathrm{ad}_{f_0}^{k-1}(f_1), \mathrm{ad}_{f_0}^{k}(f_1)](0) \notin \mathcal{S}_1(0).$$

System $\dot{x} = f_0(x) + u f_1(x)$ is not $W^{2k-3,\infty}$ STLC.

Our alternative - case $\mathcal{S}_2(0) \not\subset \mathcal{S}_1(0)$

Theorem [Beauchard, F.M., 2017]

There exists $T^* > 0$ such that, for $\mathcal{T} \in (0, T^*)$, there exists $\eta > 0$ such that, for $T \in (0, \mathcal{T}]$ and any trajectory $(x, u) \in \mathcal{C}^0([0,T], \mathbb{R}^n) \times L^1((0,T), \mathbb{R})$ of system (3) with $x(0) = 0$ and $u \in W_0^{2k-3,\infty}(0,T)$ with $\|u\|_{W^{2k-3,\infty}} \leqslant \eta$, one has

$$\forall t \in [0,T], \quad \left\langle \mathbb{P}^{\perp} x(t) - G\left(\mathbb{P}x(t)\right), d_k \right\rangle \geqslant 0,$$

where

$$d_k := -\mathbb{P}^{\perp}[\mathrm{ad}_{f_0}^{k-1}(f_1), \mathrm{ad}_{f_0}^{k}(f_1)](0).$$

Remarks

- Condition $\mathcal{S}_2(0) = \mathcal{S}_1(0)$ is necessary for SSTLC.
- There are (at most) d quadratic obstructions.
- The k-th one involves a H^{-k} drift.
- All functional spaces are optimal.
- Similar result for $\dot{x} = f(x, u)$.
- Same result for $f_0 \in C^3$ et $f_1 \in C^2$.
- Higher orders?

An impossible movement at quadratic order

Let $L = 2\pi$. Consider the Korteweg-de-Vries system

$$\begin{cases} \psi_t + \psi_x + \psi_{xxx} + \psi\psi_x = 0, \\ \psi(t,0) = \psi(t,L) = 0, \\ \psi_x(t,L) = u(t). \end{cases} \tag{5}$$

At linear order, direction $\phi(x) := 1 - \cos(x)$ is lost. Indeed $\phi_x + \phi_{xxx} = 0$ and $\phi(0) = \phi(L) = \phi_x(L) = 0$.

Proposition [Rosier, 1997]

For any $T > 0$, and $y^*, y^\dagger \in L^2(0,L)$ such that $\langle y^*, \phi \rangle = \langle y^\dagger, \phi \rangle = 0$, there exists a solution of the linearized system of (5) such that $y(0) = y^*$ and $y(T) = y^\dagger$.

An impossible movement at quadratic order

Proposition [Coron, Crépeau, 2003]

Let $T > 0$. Consider a trajectory of

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t,0) = y(t,L) = 0, \end{cases}$$

such that $y(0,\cdot) = y(T,\cdot) = 0$ and a trajectory of

$$\begin{cases} z_t + z_x + z_{xxx} = -yy_x, \\ z(t,0) = z(t,L) = 0, \end{cases}$$

such that $z(0,\cdot) = 0$. Then $\langle z(T,\cdot), \phi \rangle = 0$.

However, the movement is possible at cubic order.

A drift of the first type

Consider the Schrödinger system

$$\begin{cases} i\psi_t = -\psi_{xx} - u(t)\mu(x)\psi, \\ \psi(t,0) = 0, \\ \psi(t,1) = 0. \end{cases} \tag{6}$$

Let $\varphi_j(x) := \sqrt{2}\sin(j\pi x)$ for $j \geqslant 1$ and study local controllability around the ground state $\psi_1(t,x) := e^{-i\pi^2 t}\varphi_1(x)$.

A drift of the first type

Proposition [Beauchard, Laurent, 2010]

Let $\mu \in H^3((0,1),\mathbb{R})$. Assume

$$\forall j \in \mathbb{N}^*, \quad \frac{c}{j^3} \leqslant \left|\langle \mu\varphi_1, \varphi_j \rangle\right|.$$

Then (6) is L^2-STLC.

Controllability comes from the linearized system. If there exists $k \in \mathbb{N}^*$ such that $\langle \mu\varphi_1, \varphi_k \rangle = 0$, then direction φ_k is lost at linear order.

A drift of the first type

Proposition [Beauchard, Morancey, 2014]

Let $\mu \in H^3((0,1),\mathbb{R})$. Assume there exists $k \in \mathbb{N}^*$ such that

$$\langle \mu\varphi_1, \varphi_k \rangle = 0,$$
$$\langle (\mu')^2\varphi_1, \varphi_k \rangle \neq 0.$$

Then (6) is not H^{-1} STLC.

Proof relies on a drift quantified by the H^{-1} size of the control. Let $A_0 := -\partial_{xx}$ and $A_1 := \mu$. Compute

$$[A_1, [A_0, A_1]] = 2(\mu')^2$$

Scalar-input Burgers system

$$\begin{cases} \psi_t - \psi_{xx} + \psi\psi_x = u(t)\mathbb{1}_{[0,1]} & (0,T)\times(0,1), \\ \psi(t,0) = 0 & (0,T), \\ \psi(t,1) = 0 & (0,T), \\ \psi(0,x) = \psi_0(x) & (0,1). \end{cases} \tag{7}$$

Theorem [F.M., 2015]

System (7) is not L^2 STLC. There is a drift of size $\|u\|^2_{H^{-5/4}}$.

Small-time against small-viscosity

Let $T = \varepsilon \ll 1$. Let $\bar{\psi}(\tau, x) := \varepsilon\psi(\varepsilon\tau, x)$ and $\bar{u}(\tau) := \varepsilon^2 u(\varepsilon\tau)$.

$$\begin{cases} \bar{\psi}_\tau - \varepsilon\bar{\psi}_{xx} + \bar{\psi}\bar{\psi}_x & = \bar{u}(t) & (0,1)\times(0,1), \\ \bar{\psi}(\tau,0) & = 0 & (0,1), \\ \bar{\psi}(\tau,1) & = 0 & (0,1), \\ \bar{\psi}(0,x) & = 0 & (0,1). \end{cases} \tag{8}$$

No initial data. Everything is small.

Quadratic approximation

If $\bar{u} = \mathcal{O}(\eta)$, with $\eta \ll 1$, write $\bar{\psi}(t,x) = \eta y(t,x) + \eta^2 z(t,x) + \eta^3 r(t,x)$, where

$$\begin{cases} y_t - \varepsilon y_{xx} & = \bar{u}(t) & (0,1)\times(0,1), \\ y(t,0) = y(t,1) & = 0 & (0,1), \\ y(0,x) & = 0 & (0,1) \end{cases} \tag{9}$$

and

$$\begin{cases} z_t - \varepsilon z_{xx} & = -y y_x & (0,1)\times(0,1), \\ z(t,0) = z(t,1) & = 0 & (0,1), \\ z(0,x) & = 0 & (0,1). \end{cases} \tag{10}$$

Core idea

Look for a sign argument

$$\forall \bar{u}, \quad \int_0^1 z(1,x)\rho(x)\mathrm{d}x \geq 0, \tag{11}$$

where ρ must be chosen wisely. Write

$$\langle z(1,\cdot), \rho \rangle = \int_0^1 \int_0^1 K^\varepsilon(s_1,s_2)\bar{u}(s_1)\bar{u}(s_2)\mathrm{d}s_1\mathrm{d}s_2. \tag{12}$$

Computation of the kernel

$$\begin{cases} G_t - \varepsilon G_{xx} &= 0 \qquad (0,1)\times(0,1), \\ G(t,0) = G(t,1) &= 0 \qquad (0,1), \\ G(0,x) &= 1 \qquad (0,1) \end{cases} \tag{13}$$

and

$$\begin{cases} \Phi_t - \varepsilon \Phi_{xx} &= 0 \qquad (0,1)\times(0,1), \\ \Phi(t,0) = \Phi(t,1) &= 0 \qquad (0,1), \\ \Phi(0,x) &= \rho \qquad (0,1) \end{cases} \tag{14}$$

and

$$K^{\varepsilon}(s_1,s_2) = \frac{1}{2}\int_{s_1\vee s_2}^{1}\int_0^1 \Phi(1-s,x)G(s-s_1,x)G(s-s_2,x)\mathrm{d}x\mathrm{d}s.$$

Asymptotic kernel

$$\int_0^1 \Phi_x(1-t,x)G(t-s_1,x)G(t-s_2,x)\mathrm{d}x$$

$$= \frac{1}{2}\int_0^1 \Phi_x(1-t,x)\left(G(t-s_1,x)G(t-s_2,x)-1\right)\mathrm{d}x \qquad \int \Phi_x = 0$$

$$= \int_0^{\frac{1}{2}} \Phi_x(1-t,x)\left(G(t-s_1,x)G(t-s_2,x)-1\right)\mathrm{d}x \qquad \text{parity}$$

$$\approx \int_0^{\frac{1}{2}} \rho_x(x)\left(\operatorname{erf}\left(\frac{x}{\sqrt{4\varepsilon(t-s_1)}}\right)\operatorname{erf}\left(\frac{x}{\sqrt{4\varepsilon(t-s_2)}}\right)-1\right)\mathrm{d}x \qquad \text{almost}$$

$$\approx 2\sqrt{\varepsilon}\int_0^{\frac{1}{4\sqrt{\varepsilon}}} \rho_x\left(2\sqrt{\varepsilon}x\right)\left(\operatorname{erf}\left(\frac{x}{\sqrt{(t-s_1)}}\right)\operatorname{erf}\left(\frac{x}{\sqrt{(t-s_2)}}\right)-1\right)\mathrm{d}x$$

$$\sim -2\sqrt{\varepsilon}\rho_x(0)\int_0^{+\infty}\left(1-\operatorname{erf}\left(\frac{x}{\sqrt{(t-s_1)}}\right)\operatorname{erf}\left(\frac{x}{\sqrt{(t-s_2)}}\right)\right)\mathrm{d}x.$$

Asymptotic kernel

$$\int_0^{+\infty} \left(1 - \mathrm{erf}(\alpha x)\mathrm{erf}(\beta x)\right) \mathrm{d}x = \frac{1}{\alpha\beta}\sqrt{\frac{\alpha^2+\beta^2}{\pi}}. \tag{15}$$

Hence

$$\begin{aligned} K^\varepsilon(s_1,s_2) &= \int_{s_1\vee s_2}^1 A^\varepsilon(t,s_1,s_2)\mathrm{d}t \propto \sqrt{\varepsilon}\int_{s_1\vee s_2}^1 \sqrt{(t-s_1)+(t-s_2)}\mathrm{d}t \\ &\propto \sqrt{\varepsilon}\cdot\left[(2t-s_1-s_2)^{\frac{3}{2}}\right]_{s_1\vee s_2}^1 \propto \sqrt{\varepsilon}K^0(s_1,s_2), \end{aligned}$$

where

$$K^0(s_1,s_2) = (2-s_1-s_2)^{3/2} - |s_1-s_2|^{3/2}.$$

Coercivity of the asymptotic kernel

Letting $u_1(t) := \int_0^t u$

$$\begin{aligned}
&\langle K^0 u, u\rangle \\
&= \int_0^1 \int_0^1 \left((2 - s - s')^{3/2} - |s - s'|^{3/2} \right) u(s) u(s') \mathrm{d}s \mathrm{d}s' \\
&= \frac{3}{4} \int_0^1 \int_0^1 \left((2 - s - s')^{-1/2} + |s - s'|^{-1/2} \right) u_1(s) u_1(s') \mathrm{d}s \mathrm{d}s' \\
&= \text{positive term} + \text{coercive term in } \|u_1\|^2_{H^{-1/4}} .
\end{aligned}$$

Estimating higher-order residues

Use weakly singular integral operators theory.

Proposition [Torres 1991 - Youssfi, 1996]

Let $L : [0,1]^2 \to \mathbb{R}$ with $\kappa > 0$ and $1/2 < \delta \leqslant 1$ such that

$$\begin{aligned} |L(x,y)| &\leqslant \kappa |x-y|^{-1/2}, \\ |L(x,y) - L(x',y)| &\leqslant \kappa |x-x'|^{\delta} |x-y|^{-1/2-\delta}, \\ |L(x,y) - L(x,y')| &\leqslant \kappa |y-y'|^{\delta} |x-y|^{-1/2-\delta}. \end{aligned}$$

Then, there exists $C(\delta)$ such that

$$|\langle L\phi, \phi \rangle| \leqslant \kappa C(\delta) \, \|\phi\|^2_{H^{-1/4}} \, .$$

Estimating higher-order residues

It is easy to get cubic estimates, but hard to obtain the correct norms to have estimates such as

$$|\langle r(T,\cdot),\rho\rangle| \leqslant C\,\|u\|_{L^2}\,\|u\|_{H^{-5/4}}^2\,.$$

So the drift dominates the residues for small L^2 controls.

Comments and perspectives

- We did too much work. Focus on $0 - 0$ controls for the linear system.
- We don't obtain the drift direction itself as an object.
- We could wish to apply the method to
  - Korteweg de Vries (for some critical lengths),
  - Schrödinger (for some lost directions).
- Can we observe H^{-s} drifts for any $s \geqslant 0$?
- Does there exist systems where the quadratic order

Take-home message

For small-time local controllability of scalar-input systems, there is nothing good to be expected from a second-order expansion!

old control theory saying

- If the linearized system is already controllable, then stop there.
- If not, quadratic terms will either
  - lead to a quadratic coercive drift (you are doomed),
  - or slightly bend the reachable space (there is still hope).

This is proved in finite dimension (Karine Beauchard, F.M.) and still open for PDEs (but already supported by at least 5 papers).