APPROXIMATION OF CONTROLS FOR LINEAR WAVE EQUATIONS: A FIRST ORDER MIXED FORMULATION

SANTIAGO MONTANER AND ARNAUD MÜNCH

Abstract. This paper deals with the numerical approximation of null controls for the wave equation posed in a bounded domain of $\mathbb{R}^n$. The goal is to compute approximations of controls that drive the solution from a prescribed initial state to zero at a large enough controllability time. In [Cindea & Münch, *A mixed formulation for the direct approximation of the control of minimal L^2-norm for linear type wave equations*], we have introduced a space-time variational approach ensuring strong convergent approximations with respect to the discretization parameter. The method, which relies on generalized observability inequality, requires H^2-finite element approximation both in time and space. Following a similar approach, we present and analyze a variational method still leading to strong convergent results but using simpler H^1-approximation. The main point is to preliminary restate the second order wave equation into a first order system and then prove an appropriate observability inequality.

1. Introduction

This work is devoted to the numerical study of null-controllability for the wave equation by means of a first order equivalent formulation of the wave equation. In the work [11], a mixed formulation was introduced for the numerical approximation of the control of minimal L^2-norm for the one dimensional wave equation with a potential. More precisely, in [11], under some regularity assumptions on the coefficients, the following kind of control systems were studied:

$$\begin{cases} u_{tt} - (c(x)u_x)_x + d(x,t)u = 0, & (x,t) \in (0,1) \times (0,T), \\ u(0,t) = 0, \quad u(1,t) = v(t), & t \in (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in (0,1). \end{cases}$$

In this work we continue studying mixed formulations for the numerical approximation of controls for wave equations. We shall consider a formulation of the wave equation in terms of a first order hyperbolic system. The main observation is that, if u is a solution to the wave equation

$$u_{tt} - \Delta u = 0, \tag{1.1}$$

and we define the new variables

$$v \triangleq u_t, \quad \mathbf{p} \triangleq \nabla u, \tag{1.2}$$

then the variables $(v, \mathbf{p})$ solve the following first order system

$$\begin{cases} v_t - div\, \mathbf{p} = 0, \\ \mathbf{p}_t - \nabla v = 0. \end{cases} \tag{1.3}$$

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Reciprocally, if we are able to establish well-posedness for (1.3) in some suitable functional spaces, by uniqueness we may recover solutions to (1.1); thus, it will be equivalent to solve (1.1) or (1.3). We will return to this point when discussing the well-posedness of initial-boundary value problems associated to (1.3).

There are two main reasons which motivate the introduction of the variables $(v, \mathbf{p})$. The first reason is that the system (1.3) only involves first order differential operators, a property which will be handy for the finite element approximation of the control of minimal L^2-norm, since it will allow to reduce the regularity required for the finite element spaces. The second reason is that it is a first step before considering other hyperbolic systems where variables analogous to $\mathbf{p}$ are sometime more important than the original variables, such as the system of linear elasticity. This kind of first order —or mixed— formulations of hyperbolic systems have been already considered in the finite element literature; see, for instance [3, 4, 12].

In this work, the first order formulation will appear through the conjugate functional $J^\star$ defined in (1.8) that one has to minimize in order to obtain the control of minimal L^2-norm for the wave equation. In this sense, we will be mainly interested in understanding the *adjoint state* to (1.4) as a solution to a first order system. We will focus on the boundary control.

We recall some facts about the control of minimal L^2-norm for the simple wave equation.

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, $T > 0$ and $\mathcal{J} \subseteq \Sigma_T \triangleq \partial\Omega \times (0,T)$, then, for $(u_0, u_1) \in \mathbf{Y} \triangleq L^2(\Omega) \times H^{-1}(\Omega)$ and $v \in L^2(\mathcal{J})$ we consider the control system

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } Q_T \triangleq \Omega \times (0,T), \\ u = v|_{\mathcal{J}}, & \text{on } \Sigma_T, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega. \end{cases} \tag{1.4}$$

Under some regularity assumptions on the boundary $\partial\Omega$ (C^2 will be sufficient for our purposes), there exists a unique solution u to (1.4) with the following regularity

$$u \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega)) \tag{1.5}$$

(see for instance [20]). The null controllability problem for (1.4) at time T is the following: for each $(u_0, u_1) \in \mathbf{Y}$, find $v \in L^2(\Sigma_T)$ supported on $\mathcal{J}$ such that the corresponding solution to (1.4) satisfies

$$u(\cdot, T) = 0,\ u_t(\cdot, T) = 0 \text{ in } \Omega.$$

It is well-known ([18, 20]) that under some *geometrical conditions* on $\mathcal{J}$ the system (1.4) is null-controllable at any *large* time $T > T^\star$ for some $T^\star$ that depends on Ω and $\mathcal{J}$. Moreover, as a consequence of the *Hilbert Uniqueness Method* of J.-L. Lions [20], the null-controllability of (1.4) is equivalent to an observability inequality for the associated adjoint problem. We recall that there is an infinite set of possible choices for the control at time T, thus, the problem of finding the control of minimal L^2-norm arises naturally. In fact, the control of minimal L^2-norm is unique and it can be found as the solution to a minimization problem involving the adjoint state [18, 20].

We define $\mathbf{V} = H_0^1(\Omega) \times L^2(\Omega)$ and denote by $\langle \cdot, \cdot \rangle_{H^{-1}, H_0^1}$ the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$; we recall that

$$\langle u, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega \nabla(-\Delta)^{-1} u(x) \cdot \nabla\varphi(x)\, dx \tag{1.6}$$

for any $u \in H^{-1}(\Omega)$, $\varphi \in H_0^1(\Omega)$, where $(-\Delta)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$ is the inverse of the Dirichlet Laplacian.

It is known that the control $v|_{\mathcal{J}} \in L^2(\Sigma_T)$ of minimal L^2-norm for system (1.4) can be found as $v = \frac{\partial\varphi}{\partial\nu}|_{\mathcal{J}}$, where $\varphi \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ is the solution to the adjoint system

$$\begin{cases} \varphi_{tt} - \Delta\varphi = 0, & \text{in } Q_T, \\ \varphi = 0, & \text{on } \Sigma_T, \\ \varphi(\cdot,T) = \varphi_0,\ \varphi_t(\cdot,T) = \varphi_1, & \text{in } \Omega, \end{cases} \tag{1.7}$$

when $(\varphi_0,\varphi_1) \in \mathbf{V}$ solves the minimization problem

$$\min_{(\varphi_0,\varphi_1)\in\mathbf{V}} J^\star(\varphi_0,\varphi_1) = \frac{1}{2}\int_{\mathcal{J}} |\tfrac{\partial\varphi}{\partial\nu}|^2\, d\sigma\, dt + (u_0,\varphi_t(\cdot,0))_{L^2(\Omega)} - \langle u_1,\varphi(\cdot,0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}. \tag{1.8}$$

Remark 1. *We recall the reader the following existence, uniqueness and boundary regularity result* [20]*: given $(\varphi_0,\varphi_1) \in \mathbf{V}$ and $f \in L^1(0,T;L^2(\Omega))$, there exists a unique solution $\varphi \in C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ to*

$$\begin{cases} \varphi_{tt} - \Delta\varphi = f, & \textit{in } Q_T, \\ \varphi = 0, & \textit{on } \Sigma_T, \\ \varphi(\cdot,0) = \varphi_0,\ \varphi_t(\cdot,0) = \varphi_1, & \textit{in } \Omega. \end{cases} \tag{1.9}$$

Moreover, $\frac{\partial\varphi}{\partial\nu} \in L^2(\Sigma_T)$ and there exists a constant $C = C(\Omega,T)$ such that

$$\int_{\Sigma_T} |\tfrac{\partial\varphi}{\partial\nu}|^2\, d\sigma\, dt \le C\left(\|(\varphi_0,\varphi_1)\|^2_{\mathbf{V}} + \|f\|^2_{L^1(0,T;L^2(\Omega))}\right). \tag{1.10}$$

A similar result is obtained for backward systems, i.e., when instead of initial conditions at time $t=0$ we impose final conditions at $t=T$. In view of this regularity result, the boundary term which appears in (1.8) *makes sense.*

The coercivity and boundedness by below of $J^\star$ is the consequence of the following estimate: let φ solve (1.7), then there exists k_T such that

$$\|(\varphi(\cdot,0),\varphi_t(\cdot,0))\|^2_{\mathbf{V}} \le k_T \|\tfrac{\partial\varphi}{\partial\nu}\|^2_{L^2(\mathcal{J})}, \tag{1.11}$$

for any $(\varphi_0,\varphi_1) \in \mathbf{V}$.

As it was pointed out in [11] (see also [2]), it is possible to reformulate the minimization problem (1.8) in terms of trajectories of the wave equation. This is possible because the wave equation is reversible in time, thus, there is a bijective relation between the *trajectories* and the *initial data*. For this purpose, in [11] the following *spaces of trajectories* was introduced

$$\widetilde{W} = \left\{\varphi \in L^2(Q_T),\ \varphi = 0 \text{ on } \Sigma_T \text{ such that } \varphi_{tt} - \Delta\varphi = 0 \text{ and } \tfrac{\partial\varphi}{\partial\nu} \in L^2(\Sigma_T)\right\}, \tag{1.12}$$

which allows to restate (1.8) as

$$\min_{\varphi\in\widetilde{W}} \widehat{J}^\star(\varphi) = \frac{1}{2}\int_{\mathcal{J}} |\tfrac{\partial\varphi}{\partial\nu}|^2\, d\sigma\, dt + (u_0,\varphi_t(\cdot,0))_{L^2(\Omega)} - \langle u_1,\varphi(\cdot,0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}. \tag{1.13}$$

At this point, it is possible to reformulate the minimization problem associated to $\widehat{J}^\star$ in terms of the solution to a first order system. Before doing so, we introduce the initial-boundary value problem for the first order system which will be useful for us.

First, we must choose the adequate boundary conditions for $(v,\mathbf{p})$; we observe that, since the solution φ to (1.7) satisfies the boundary condition $\varphi = 0$ on Σ_T, then also $\varphi_t = 0$ on Σ_T, thus, we shall impose the boundary condition $v = 0$ on Σ_T in order to be consistent with (1.2). In fact, this boundary condition will be enough for our purposes and it will not be necessary to impose boundary conditions on $\mathbf{p}$. Regarding the conditions at time $t=T$, the natural choice is to set $v(\cdot,T) = \varphi_1$ and $\mathbf{p}(\cdot,T) = \nabla\varphi_0$, i.e., the initial data for $(v,\mathbf{p})$ should belong to

$$H \triangleq L^2(\Omega) \times \nabla H^1,$$

where

$$\nabla H^1 \triangleq \left\{\nabla\varphi,\, \varphi \in H_0^1(\Omega)\right\}.$$

The space ∇H^1 is a Hilbert space endowed with the inner product

$$(\mathbf{p}, \bar{\mathbf{p}})_{\nabla H^1} = \int_\Omega \mathbf{p}\cdot\bar{\mathbf{p}}\,dx.$$

∇H^1 is a closed subspace of $L^2(\Omega;\mathbb{R}^n) = \left\{(f_1,\dots,f_n),\ f_i \in L^2(\Omega),\ \text{for } i=1,\dots,n\right\}$. We set

$$H(div) = \left\{\mathbf{p} \in \nabla H^1,\ \ div\,\mathbf{p} \in L^2(\Omega)\right\}.$$

We define the Hilbert space $H = L^2(\Omega) \times \nabla H^1$ endowed with the inner product

$$((v,\mathbf{p}),(\bar{v},\bar{\mathbf{p}}))_H = (v,\bar{v})_{L^2(\Omega)} + (\mathbf{p},\bar{\mathbf{p}})_{\nabla H^1}.$$

We also define $V = H_0^1(\Omega) \times H(div)$ and observe that H is the closure of V with respect to the norm $\|\cdot\|_H$.

In what follows, we will denote

$$M(v,\mathbf{p}) \triangleq (v_t - div\,\mathbf{p}, \mathbf{p}_t - \nabla v). \tag{1.14}$$

In order to state the suitable well-posedness for the first order system, we will consider the closed and densely defined operator $\mathcal{A}$ with domain $D(\mathcal{A}) = V$ defined by

$$\mathcal{A}(v,\mathbf{p}) \triangleq (div\,\mathbf{p}, \nabla v), \quad \forall (v,\mathbf{p}) \in D(\mathcal{A}). \tag{1.15}$$

We show in Section 10 that $\mathcal{A}$ generates a C_0-semigroup of contractions, which we denote $e^{t\mathcal{A}}$ (see Lemma 9).

Definition 1. Let $(v_0,\mathbf{p}_0) \in H$, $(f,\mathbf{F}) \in L^1(0,T;H)$, we say that $(v,\mathbf{p})$ is a *mild solution* to

$$\begin{cases} M(v,\mathbf{p}) = (f,\mathbf{F}), & \text{in } Q_T,\\ v = 0, & \text{on } \Sigma_T,\\ v(\cdot,0) = v_0,\ \mathbf{p}(\cdot,0) = \mathbf{p}_0, & \text{in } \Omega, \end{cases} \tag{1.16}$$

if

$$(v(t),\mathbf{p}(t)) = e^{t\mathcal{A}}(v_0,\mathbf{p}_0) + \int_0^t e^{(t-s)\mathcal{A}}(f(s),\mathbf{F}(s))\,ds \ \ \forall t \in (0,T).$$

As a consequence of well-known facts concerning the theory of semigroups [23] and the definition of *mild solutions*, we have the following well-posedness statement for (1.16).

Lemma 1. *Let $(v_0,\mathbf{p}_0) \in H$, $(f,\mathbf{F}) \in L^1(0,T;H)$, then:*

i) (1.16) *has a unique mild solution $(v,\mathbf{p}) \in C([0,T];H)$ and*

$$\sup_{t\in(0,T)} \|(v(t),\mathbf{p}(t))\|_H \le \|(v_0,\mathbf{p}_0)\|_H + \|(f,\mathbf{F})\|_{L^1(0,T;H)}. \tag{1.17}$$

ii) *If $(f,\mathbf{F}) \in C^1([0,T];H)$ and $(v_0,\mathbf{p}_0) \in V$, then $(v,\mathbf{p}) \in C^1([0,T];H) \cap C([0,T];V)$ and the equation*

$$M(v,\mathbf{p}) = (f,\mathbf{F})$$

holds in H.

Remark 2. *Since $-\mathcal{A}$ also generates a C_0-semigroup, if $(g,\mathbf{G}) \in L^1(0,T;H)$ and $(w_T,\mathbf{q}_T) \in H$, then the backward problem*

$$\begin{cases} M(w,\mathbf{q}) = (g,\mathbf{G}), & \text{in } Q_T,\\ w = 0, & \text{on } \Sigma_T,\\ w(\cdot,T) = w_0,\ \mathbf{q}(\cdot,T) = \mathbf{q}_0, & \text{in } \Omega, \end{cases} \tag{1.18}$$

is also well-posed in the sense of mild solutions.

Similarly to the boundary regularity result stated in Remark 1, we have a boundary regularity result for solutions to (1.18). In this situation we will prove in Lemma 3 that if $(w, \mathbf{q}) \in C([0,T]; H)$ is a mild solution to (1.18), then $\mathbf{q}\cdot\nu \in L^2(\Sigma_T)$ and for some constant $C = C(\Omega, T)$ the inequality

$$\|\mathbf{q}\cdot\nu\|^2_{L^2(\Sigma_T)} \le C\left(\|(w_T, \mathbf{q}_T)\|^2_H + \|(g, \mathbf{G})\|^2_{L^1(0,T;H)}\right),$$

will hold true.

We are now in position to restate the minimization problem (1.13) in terms of a solution to a first order system of the form (1.18). Since we want to identify w with φ_t and $\mathbf{q}$ with $\nabla\varphi$, the natural way to proceed is to consider a minimization problem analogous to (1.13) in which the quantities $\nabla\varphi$ and φ_t are replaced by $\mathbf{p}$ and v respectively.

If we denote by $U((g, \mathbf{G}), (w_0, \mathbf{q}_0)) \triangleq (w, \mathbf{q})$ the mild backward solution to (1.18) associated to a particular data $(g, \mathbf{G}) \in L^2(0,T;H)$ and $(w_0, \mathbf{q}_0) \in H$, then we define the following *spaces of trajectories*

$$W = \Big\{U((g, \mathbf{G}), (w_0, \mathbf{q}_0)) \in C([0,T]; H),\ (g, \mathbf{G}) \in L^2(0,T;H),\ (w_0, \mathbf{q}_0) \in H\Big\},$$

$$W_0 = \Big\{U((0,0), (w_0, \mathbf{q}_0)) \in C([0,T]; H),\ (w_0, \mathbf{q}_0) \in H\Big\}.$$

With this definition, the adequate minimization problem in terms of the first order system is

$$\min_{(w,\mathbf{q})\in W_0} \widehat{J}^\star(w, \mathbf{q}) = \frac{1}{2}\int_{\mathcal{J}} |\mathbf{q}\cdot\nu|^2\, d\sigma\, dt + (u_0, w(0))_{L^2(\Omega)} - (\nabla(-\Delta)^{-1}u_1, \mathbf{q}(0))_{L^2(\Omega)}. \tag{1.19}$$

As in the case of the wave equation, the coercivity and lower bound of (1.19) can be obtained as a consequence of an observability inequality which takes the form

$$\|(w_0, \mathbf{q}_0)\|_H \le C\|\mathbf{q}\cdot\nu\|_{L^2(\mathcal{J})} \tag{1.20}$$

for any solution to (1.18) under suitable conditions on $\mathcal{J}$ and T.

The main objective of this work is to adapt the theoretical and numerical analysis done in [11], replacing the extremal problem (1.13) in φ, solution of a second order equation by the equivalent extremal problem (1.19) in $(w, \mathbf{q})$, solution of a first order system. Following [11], (1.19) is addressed by means of a mixed formulation which involves $(n+1)$ Lagrange multipliers. The paper is organized as follows. In Section 2, we prove the generalized observability inequality (2.15) for mild solution of $(w, \mathbf{q})$ of (1.18). This provides an appropriate functional setting and leads to the well-posedeness of the mixed formulation (3.1), which is the optimality system for (1.19). The dual problem, involving only the Lagrange multiplier variables is introduced and analyzed in Section 4. In Section 5, we exploit some properties of the multipliers to derive a stabilized mixed formulation *à la Barbosa-Hughes* [1] whose solution coincides with the initial mixed formulation. This allows to circumvent the Babuška-Brezzi inf-sup condition. Section 6 introduces a conformal H^1-approximation of the mixed formulation leading to error estimates in the one dimensional case in Section 7. Section 8 is devoted to numerical experiments for a discontinuous initial condition. Eventually, Section 9 concludes with some perspectives.

As in [11], we use a conformal space-time finite element methods for the discretization of our infinite dimensional extremal problem (1.19), written in a variational form. This "optimize-then-discretize" approach guarantees the strong convergence of the approximation with respect to the discretization parameter. For other space-time finite element methods in similar contexts, we refer to [19] and the references therein. This contrasts with the well-known numerical pathologies (due to discrete high frequencies) appearing when the "discretize-then-optimize" approach is employed (exhibited in [16] and enhanced in [13]) and which requires very specific treatments ([14, 15, 21]).

2. Boundary observability inequality

We will employ a Rellich-Necas type identity. Let $\beta : \Omega \to \mathbb{R}^n$ and $\mathbf{p} : \Omega \to \mathbb{R}^n$ be two $C^1(\Omega; \mathbb{R}^n)$ vector fields. It is easy to check that the following identity holds

$$\begin{aligned} 2div\,((\beta\cdot\mathbf{p})\mathbf{p}) - div\,(\beta|\mathbf{p}|^2) &= 2(\beta\cdot\mathbf{p})div\,\mathbf{p} + 2(J\beta\,\mathbf{p})\cdot\mathbf{p} - div\,\beta|\mathbf{p}|^2 \\ &\quad + 2\left((J\mathbf{p}\,\beta)\cdot\mathbf{p} - (J\mathbf{p}\,\mathbf{p})\cdot\beta\right), \end{aligned} \tag{2.1}$$

where $J\mathbf{u}$ is the Jacobian matrix $(J\mathbf{u})_{ij} = \partial_j u^i$ of a vector field $\mathbf{u} = (u^1, \ldots, u^n)$.

Lemma 2. *If $\mathbf{p} \in \nabla H^1$ is a smooth vector field, then identity* (2.1) *becomes*

$$2div\,((\beta\cdot\mathbf{p})\mathbf{p}) - div\,(\beta|\mathbf{p}|^2) = 2(\beta\cdot\mathbf{p})div\,\mathbf{p} + 2(J\beta\,\mathbf{p})\cdot\mathbf{p} - div\,\beta|\mathbf{p}|^2. \tag{2.2}$$

Proof. If $\mathbf{p} \in V$ is smooth, then there exists a smooth function φ such that $\mathbf{p} = \nabla\varphi$. Therefore, the Hessian matrix $D^2\varphi = J\mathbf{p}$ is symmetric and we have

$$(J\mathbf{p}\,\beta)\cdot\mathbf{p} - (J\mathbf{p}\,\mathbf{p})\cdot\beta = (D^2\varphi\,\beta)\cdot\mathbf{p} - (D^2\varphi\,\mathbf{p})\cdot\beta = (D^2\varphi\,\beta)\cdot\mathbf{p} - (D^2\varphi\,\beta)\cdot\mathbf{p} = 0.$$

□

We recall that, if $\mathbf{p} \in H(div)$, then its normal trace $\mathbf{p}\cdot\nu$ is well defined as a distribution in $H^{-\frac{1}{2}}(\partial\Omega)$ and Green's formula is satisfied:

$$\int_\Omega v\,div\,\mathbf{p}\,dx + \int_\Omega \mathbf{p}\cdot\nabla v\,dx = \langle \mathbf{p}\cdot\nu, v\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}}, \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$. Next Lemma improves the regularity of $\mathbf{p}\cdot\nu$ when $(v, \mathbf{p})$ is a solution to (1.3) belonging to $C^1(0, T; H) \cap C(0, T; V)$ and allows us to define the normal trace $\mathbf{p}\cdot\nu$ when $(v, \mathbf{p})$ a mild solution of (1.16).

Lemma 3. *Let Ω be a bounded C^2 regular domain and $(f, \mathbf{F}) \in C^1(0, T; H)$. Let $(v, \mathbf{p}) \in C^1(0, T; H) \cap C(0, T; V)$ be a solution to*

$$\begin{cases} M(v, \mathbf{p}) = (f, \mathbf{F}), & \text{in } Q_T, \\ v = 0, & \text{on } \Sigma_T, \\ v(\cdot, 0) = 0, \;\; \mathbf{p}(\cdot, 0) = 0, & \text{in } \Omega. \end{cases} \tag{2.4}$$

Then $\mathbf{p}\cdot\nu \in L^2(\Sigma_T)$ and there exists a constant $C = C(\Omega)$ such that

$$\|\mathbf{p}\cdot\nu\|_{L^2(\Sigma_T)} \le C\sqrt{1+T}\|(f, \mathbf{F})\|_{L^1((0,T);H)}. \tag{2.5}$$

Consequently, there is a unique bounded map

$$\Lambda : L^1(0, T; H) \to L^2(\Sigma_T)$$

such that $\Lambda(f, \mathbf{F}) = \mathbf{p}\cdot\nu$ when $(v, \mathbf{p}) \in C^1(0, T; H) \cap C(0, T; V)$.

Accordingly, we will denote $\mathbf{p}\cdot\nu = \Lambda(f, \mathbf{F})$ when $(v, \mathbf{p})$ is a mild solution to (2.4) *with $(f, \mathbf{F}) \in L^1(0, T; H)$ and, in that case,* (2.5) *also holds.*

Proof. We prove (2.5) assuming that f is smooth and $\mathbf{F} = \nabla\varphi$ with $\varphi \in C_c^\infty(\Omega \times (0, T))$, then a density argument implies (2.5) for arbitrary $(f, \mathbf{F}) \in C^1(0, T; H)$. If $\mathbf{p}$ satisfies

$$\mathbf{p}_t - \nabla v = \mathbf{F}, \quad \mathbf{p}(x, 0) = 0,$$

then

$$\mathbf{p}(x, t) = \int_0^t \nabla v(x, \tau)\,d\tau + \int_0^t \mathbf{F}(x, \tau)\,d\tau \quad \text{ in } Q_T. \tag{2.6}$$

$\mathbf{F}$ is compactly supported in $\Omega \times (0,T)$, hence (2.6) implies that $\mathbf{p} = \int_0^t \nabla v(x,\tau)\,d\tau$ when $x \in \partial\Omega$. Since $v = 0$ on Σ_T, $\nabla v = (\nabla v \cdot \nu)\nu$ on Σ_T, therefore, $\mathbf{p}(x,t)$ must be a vector field perpendicular to Σ_T when $(x,t) \in \Sigma_T$; this implies that $\mathbf{p} = (\mathbf{p}\cdot\nu)\nu$ on Σ_T.

The regularity of the boundary implies that there exists a smooth vector field β satisfying $\beta = \nu$ on $\partial\Omega$. With this choice of β we integrate identity (2.2) on Q_T to obtain

$$\int_{\Sigma_T} |\mathbf{p}\cdot\nu|^2\,dx\,dt \le C\left(\int_{Q_T} |\mathbf{p}|^2\,dx\,dt\right) + 2\left|\int_{Q_T} (\beta\cdot\mathbf{p})div\,\mathbf{p}\,dx\,dt\right|. \tag{2.7}$$

Employing the equations solved by $(v,\mathbf{p})$ and integrating by parts we find

$$\begin{aligned}
\int_{Q_T} (\beta\cdot\mathbf{p})div\,\mathbf{p}\,dx\,dt &= \int_{Q_T} (\beta\cdot\mathbf{p})v_t\,dx\,dt - \int_{Q_T}(\beta\cdot\mathbf{p})f\,dx\,dt \\
&= -\int_{Q_T} (\beta\cdot\nabla v)v\,dx\,dt - \int_{Q_T} (\beta\cdot\mathbf{F})v\,dx\,dt - \int_{Q_T}(\beta\cdot\mathbf{p})f\,dx\,dt \\
&\quad + \int_\Omega \beta\cdot\mathbf{p}(x,T)v(x,T)\,dx\,dt \\
&= \frac{1}{2}\int_{Q_T} div\,\beta|v|^2\,dx\,dt - \int_{Q_T} (\beta\cdot\mathbf{F})v\,dx\,dt - \int_{Q_T}(\beta\cdot\mathbf{p})f\,dx\,dt \\
&\quad + \int_\Omega \beta\cdot\mathbf{p}(x,T)v(x,T)\,dx\,dt.
\end{aligned}$$

Consequently, the Cauchy-Schwarz inequality implies

$$\left|\int_{Q_T} (\beta\cdot\mathbf{p})div\,\mathbf{p}\,dx\,dt\right| \le C\left(\sup_{t\in(0,T)}\int_\Omega \|(v(t),\mathbf{p}(t))\|_H^2\,dt + \|(f,\mathbf{F})\|^2_{L^1(0,T;H)}\right). \tag{2.8}$$

The inequality (2.8) together with (2.7) yields

$$\|\mathbf{p}\cdot\nu\|^2_{L^2(\Sigma_T)} \le C\left((1+T)\sup_{t\in(0,T)}\|(v(t),\mathbf{p}(t))\|_H^2 + \|(f,\mathbf{F})\|^2_{L^1((0,T);H)}\right). \tag{2.9}$$

Finally, we apply (1.17). □

Let $x_0 \in \mathbb{R}^n$ and denote $\Sigma^+ = \{x \in \Sigma, (x-x_0)\cdot\nu > 0\}$, $\Sigma_T^+ = \Sigma^+ \times (0,T)$.

Lemma 4. *Let $(v,\mathbf{p})$ be a mild solution to (1.16) with $(f,\mathbf{F}) = 0$. There is $T^\star = T^\star(\Omega) > 0$ such that for any $T > T^\star$ there is a constant $C(\Omega,T) > 0$ such that:*

$$\|(v_0,\mathbf{p}_0)\|_H \le C(\Omega,T)\|\mathbf{p}\cdot\nu\|_{L^2(\Sigma_T^+)}. \tag{2.10}$$

Proof. It is well known [18] that, if $u \in C^1(0,T;L^2(\Omega)) \cap C(0,T;H_0^1(\Omega))$ is a weak solution to

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } Q_T, \\ u = 0, & \text{on } \Sigma_T, \\ u(\cdot,0) = u_0 \in H_0^1(\Omega),\ u_t(\cdot,0) = u_1 \in L^2(\Omega), \end{cases} \tag{2.11}$$

then there exists $T^\star > 0$ such that for any $T > T^\star$ the following boundary observability estimate holds:

$$\|(u_0,u_1)\|_{H_0^1(\Omega)\times L^2(\Omega)} \le C(\Omega,T)\|\nabla u\cdot\nu\|_{L^2(\Sigma_T^+)}. \tag{2.12}$$

We are going to use (2.12) to obtain a boundary observability estimate for (1.16) when $(f,\mathbf{F}) = 0$. If $(v_0,\mathbf{p}_0) \in H$, then there exists $\varphi_0 \in H_0^1(\Omega)$ such that $\mathbf{p}_0 = \nabla\varphi_0$. We set $\hat{u}_0 = \varphi_0$, $\hat{u}_1 = v_0$, we let $\hat{u}$ be the solution to (2.11) with $(u_0,u_1) = (\hat{u}_0,\hat{u}_1)$ and define

$(\hat{v}, \hat{\mathbf{p}}) = (\hat{u}_t, \nabla \hat{u})$. If we apply the observability inequality (2.12) to $\hat{u}$ we obtain that for $T > T^\star$ the following estimate holds:

$$\|(v_0, \mathbf{p}_0)\|_H \leq C(\Omega, T) \|\hat{\mathbf{p}} \cdot \nu\|_{L^2(\Sigma_T^+)}. \tag{2.13}$$

It is easy to check that $(\hat{v}, \hat{\mathbf{p}})$ is a mild solution to (1.16) with $(f, \mathbf{F}) = 0$; in fact it is the unique solution, so $(v, \mathbf{p}) = (\hat{v}, \hat{\mathbf{p}})$ and the proof is finished. □

Lemma 5. *Assume that $(v_0, \mathbf{p}_0) \in H$, $(f, \mathbf{F}) \in L^1(0, T; H)$ and let $(v, \mathbf{p})$ be the mild solution to* (1.16)*. Then, there exists $T^\star = T^\star(\Omega)$ such that for any $T > T^\star$ the following observability estimate holds*

$$\|(v_0, \mathbf{p}_0)\|_H \leq C(\Omega, T)(\|\mathbf{p} \cdot \nu\|_{L^2(\Sigma_T^+)} + \|(f, \mathbf{F})\|_{L^1(0,T;H)}). \tag{2.14}$$

Proof. We can split the solution as $(v, \mathbf{p}) = (\hat{v}, \hat{\mathbf{p}}) + (\tilde{v}, \tilde{\mathbf{p}})$, where $(\hat{v}, \hat{\mathbf{p}})$ solves (1.16) with $(v_0, \mathbf{p}_0) = 0$ and $(\tilde{v}, \tilde{\mathbf{p}})$ solves (1.16) with $(f, \mathbf{F}) = 0$. By Lemma 4 we have

$$\|(v_0, \mathbf{p}_0)\|_H \leq C(\Omega, T) \|\tilde{\mathbf{p}} \cdot \nu\|_{L^2(\Sigma_T^+)},$$

and using Lemma 3 we get

$$\|\tilde{\mathbf{p}} \cdot \nu\|_{L^2(\Sigma_T^+)} \leq \|\mathbf{p} \cdot \nu\|_{L^2(\Sigma_T^+)} + \|\hat{\mathbf{p}} \cdot \nu\|_{L^2(\Sigma_T^+)} \leq \|\mathbf{p} \cdot \nu\|_{L^2(\Sigma_T^+)} + C(\Omega, T)\|(f, \mathbf{F})\|_{L^1(0,T;H)}.$$

□

Corollary 1. *Assume that $(w_0, \mathbf{q}_0) \in H$, $(f, \mathbf{F}) \in L^1(0, T; H)$ and let $(w, \mathbf{q})$ be the mild backward solution to* (1.18)*. Then, there exists $T^\star = T^\star(\Omega)$ such that for any $T > T^\star$ the following observability estimate holds*

$$\|(w_0, \mathbf{q}_0)\|_H \leq C(\Omega, T)(\|\mathbf{q} \cdot \nu\|_{L^2(\Sigma_T^+)} + \|(f, \mathbf{F})\|_{L^1(0,T;H)}). \tag{2.15}$$

Remark 3. *i) Since any $(w, \mathbf{q}) \in W$ is a mild solution to* (1.18) *for some $(w_T, \mathbf{q}_T) \in H$, $(g, \mathbf{G}) \in L^2(0, T; H)$, by Lemma 3 the normal trace $\mathbf{q} \cdot \nu \in L^2(\Sigma_T)$ is well-defined for any $(w, \mathbf{q}) \in W$.*

ii) Let $(w, \mathbf{q}) \in W$, then there exists two unique pairs $(\hat{g}, \hat{\mathbf{G}}) \in L^2(0, T; H)$, $(\hat{w}_0, \hat{\mathbf{q}}_0)) \in H$ such that $(w, \mathbf{q}) = U((\hat{g}, \hat{\mathbf{G}}), (\hat{w}_0, \hat{\mathbf{q}}_0))$. Then, for any $(w, \mathbf{q}) \in W$ we will denote $M(w, \mathbf{q}) = (\hat{g}, \hat{\mathbf{G}})$. With this notation, we have $W_0 = \{(v, \mathbf{q}) \in W,\ M(v, \mathbf{q}) = 0\}$.

Remark 4. *Observe that the generalized observability inequality* (2.14) *implies*

$$\|(w_0, \mathbf{q}_0)\|_H \leq C(\Omega, T) \left(\|\mathbf{q} \cdot \nu\|_{L^2(\Sigma_T^+)} + \|M(w, \mathbf{q})\|_{L^2(0,T;H)} \right) \quad \forall (w, \mathbf{q}) \in W. \tag{2.16}$$

For any fixed $\eta > 0$, we endow W with the inner product

$$((w, \mathbf{q}), (\bar{w}, \bar{\mathbf{q}}))_W = \int_{\Sigma_T^+} (\mathbf{q} \cdot \nu)\, (\bar{\mathbf{q}} \cdot \nu)\, d\sigma\, dt + \eta \int_0^T (M(w, \mathbf{q}), M(\bar{w}, \bar{\mathbf{q}}))_H\, dt. \tag{2.17}$$

We denote $\|\cdot\|_W$ the associated norm such that

$$\|(w, \mathbf{q})\|_W^2 = ((w, \mathbf{q}), (w, \mathbf{q}))_W, \quad \text{for any } (v, \mathbf{q}) \in W.$$

Lemma 6. *The space W endowed with the inner product $(\cdot, \cdot)_W$ is a Hilbert space.*

Proof. We first have to check that $\|\cdot\|_W$ is indeed a norm. If $(w, \mathbf{q}) \in L^2(0, T; H)$ is such that $\|(w, \mathbf{q})\|_W = 0$, then $M(v, \mathbf{q}) = 0$, therefore, the observability inequality (2.16) implies $(w, \mathbf{q}) = 0$; thus $\|\cdot\|_W$ is a norm.

We now check that W is closed with respect to the norm $\|\cdot\|_W$. Let $\{(w_k, \mathbf{q}_k)\}_{k=0}^{+\infty} \subseteq W$ be a sequence converging to some $(w, \mathbf{q})$ in the norm $\|\cdot\|_W$. Then $M(w_k, \mathbf{q}_k)$ converges to some $(\hat{f}, \hat{\mathbf{F}}) \in L^2(0, T; H)$ and the observability inequality (2.16) implies that $(w_k(T), \mathbf{q}_k(T))$

converges to some $(\hat{w}, \hat{\mathbf{q}}) \in H$ as $k \to +\infty$. By the mapping properties of the semigroup which defines the mild solutions, we have that $(w_k, \mathbf{q}_k)$ converge to the mild solution of

$$\begin{cases} w_t - div\,\mathbf{q} = \hat{f}, & \text{in } Q_T, \\ \mathbf{q}_t - \nabla w = \hat{\mathbf{F}}, & \text{in } Q_T \\ w = 0, & \text{on } \Sigma_T, \\ w(\cdot, T) = \hat{w},\ \ \mathbf{q}(\cdot, T) = \hat{\mathbf{q}}, & \text{in } \Omega, \end{cases}$$

therefore $\lim_{k\to+\infty}(w_k, \mathbf{q}_k) \in W$, so W is closed. □

3. Mixed formulation

The minimization problem (1.19) is a constrained minimization problem in W where the constraint is the equation solved by $(w, \mathbf{q})$, i.e. $M(w, \mathbf{q}) = 0$. We take into account this constraint by introducing Lagrange multipliers $(\lambda, \boldsymbol{\mu}) \in L^2(0, T; H)$, which lead us to a mixed formulation: find $((w, \mathbf{q}), (\lambda, \boldsymbol{\mu})) \in W \times L^2(0, T; H)$ solving

$$\begin{cases} a((w, \mathbf{q}), (\bar{w}, \bar{\mathbf{q}})) + b((\bar{w}, \bar{\mathbf{q}}), (\lambda, \boldsymbol{\mu})) = l(\bar{w}, \bar{\mathbf{q}}), & \forall\ (\bar{w}, \bar{\mathbf{q}}) \in W, \\ b((w, \mathbf{q}), (\bar{\lambda}, \bar{\boldsymbol{\mu}})) = 0, & \forall\ (\bar{\lambda}, \bar{\boldsymbol{\mu}}) \in L^2(0, T; H), \end{cases} \tag{3.1}$$

where

$$a : W \times W \to \mathbb{R}, \quad a((w, \mathbf{q}), (\bar{w}, \bar{\mathbf{q}})) \triangleq \int_{\Sigma_T^+} (\mathbf{q} \cdot \nu)\,(\bar{\mathbf{q}} \cdot \nu)\, d\sigma\, dt,$$

$$b : W \times L^2(0, T; H) \to \mathbb{R}, \quad b((w, \mathbf{q}), (\lambda, \boldsymbol{\mu})) \triangleq \int_0^T (M(w, \mathbf{q}), (\lambda, \boldsymbol{\mu}))_H\, dt,$$

$$l : W \to \mathbb{R}, \quad l(w, \mathbf{q}) \triangleq -(u_0, w(0))_{L^2(\Omega)} + (\nabla(-\Delta)^{-1} u_1, \mathbf{q}(0))_{L^2(\Omega)}.$$

Theorem 1. i) *The mixed formulation* (3.1) *is well posed.*

ii) *Let* $\mathcal{L} : W \times L^2(0, T; H) \to \mathbb{R}$ *be the Lagrangian defined by*

$$\mathcal{L}((w, \mathbf{q}), (\lambda, \mu)) = \frac{1}{2} a((w, \mathbf{q}), (w, \mathbf{q})) + b((w, \mathbf{q}), (\lambda, \mu)) - l((w, \mathbf{q})).$$

Then, the unique solution $((w, \mathbf{q}), (\lambda, \boldsymbol{\mu})) \in W \times L^2(0, T; H)$ *to* (3.1) *is a solution to the saddle point problem*

$$\sup_{(\lambda, \boldsymbol{\mu}) \in L^2(0,T;H)}\ \inf_{(w, \mathbf{q}) \in W} \mathcal{L}((w, \mathbf{q}), (\lambda, \boldsymbol{\mu})).$$

iii) *The solution* $(w, \mathbf{q})$ *to* (3.1) *is the minimizer of* $\widehat{J}^\star$ *over* W_0. *The Lagrange multiplier* λ *is the controlled state.*

Remark 5. *We can also consider the augmented Lagrangian* $\mathcal{L}_r$*, which for any* $r > 0$ *is defined by*

$$\begin{cases} \mathcal{L}_r((w, \mathbf{q}), (\lambda, \boldsymbol{\mu})) = \frac{1}{2} a_r((w, \mathbf{q}), (w, \mathbf{q})) + b((w, \mathbf{q}), (\lambda, \boldsymbol{\mu})) - l((w, \mathbf{q})), \\ a_r((w, \mathbf{q}), (w, \mathbf{q})) = a((w, \mathbf{q}), (w, \mathbf{q})) + r \int_{Q_T} |M(w, \mathbf{q})|^2\, dx\, dt. \end{cases} \tag{3.2}$$

Since $a_r((w, \mathbf{q}), (w, \mathbf{q})) = a((w, \mathbf{q}), (w, \mathbf{q}))$ *in* W_0*,* $\mathcal{L}$ *and* $\mathcal{L}_r$ *share the same saddle point.*

Proof. i) In virtue of [5, Theorem 4.2.1], we have to check:

1) a and b are continuous: this is obvious by the definition of $\|\cdot\|_W$.
2) l is continuous: this is a direct consequence of the generalized observability inequality (2.16) and the energy estimate (1.17).

3) a is coercive on the kernel

$$\mathcal{N}(b)=\left\{(w,\mathbf{q})\in W,\ b((w,\mathbf{q}),(\lambda,\mu))=0\ \ \forall(\lambda,\boldsymbol{\mu})\in L^2(0,T;H)\right\}.$$

This is clear from the definition of a: if $(w,\mathbf{q})\in\mathcal{N}(b)$, then $a((w,\mathbf{q}),(w,\mathbf{q}))=\|(w,\mathbf{q})\|_W^2$.

4) b satisfies the *inf-sup* condition over $W\times L^2(0,T;H)$: there exists $\delta>0$ such that

$$\inf_{(\lambda,\boldsymbol{\mu})\in L^2(0,T;H)}\ \sup_{(w,\mathbf{q})\in W}\frac{b((w,\mathbf{q}),(\lambda,\boldsymbol{\mu}))}{\|(w,\mathbf{q})\|_W\|(\lambda,\boldsymbol{\mu})\|_{L^2(0,T;H)}}\geq\delta. \tag{3.3}$$

To prove this, we fix $(\hat\lambda,\hat{\boldsymbol{\mu}})\in L^2(0,T;H)$ and we take the unique $(\hat w,\hat{\mathbf{q}})\in W$ such that $M(\hat w,\hat{\mathbf{q}})=(\hat\lambda,\hat{\boldsymbol{\mu}})$ and $(\hat w(T),\hat{\mathbf{q}}(T))=(0,0)$. The trace inequality (2.5) implies

$$\|\hat{\mathbf{q}}\cdot\nu\|_{L^2(\Sigma_T^+)}\leq C(\Omega,T)\|(\hat\lambda,\hat{\boldsymbol{\mu}})\|_{L^2(0,T;H)}=C(\Omega,T)\|M(\hat w,\hat{\mathbf{q}})\|_{L^2(0,T;H)}. \tag{3.4}$$

We also have $b((\hat w,\mathbf{q}),(\hat\lambda,\hat{\boldsymbol{\mu}}))=\|(\hat\lambda,\hat{\boldsymbol{\mu}})\|^2_{L^2(0,T;H)}$, therefore

$$\begin{aligned}\sup_{(w,\mathbf{q})\in W}\frac{b((w,\mathbf{q}),(\hat\lambda,\hat{\boldsymbol{\mu}}))}{\|(w,\mathbf{q})\|_W\|(\hat\lambda,\hat{\boldsymbol{\mu}})\|_{L^2(0,T;H)}}&\geq\frac{\|(\hat\lambda,\hat{\boldsymbol{\mu}})\|_{L^2(0,T;H)}}{\|(\hat w,\hat{\mathbf{q}})\|_W}=\frac{\|M(\hat w,\hat{\mathbf{q}})\|_{L^2(0,T;H)}}{\|(\hat w,\hat{\mathbf{q}})\|_W}\\&=\frac{\|M(\hat w,\hat{\mathbf{q}})\|_{L^2(0,T;H)}}{\left(\eta\|M(\hat w,\hat{\mathbf{q}})\|^2_{L^2(0,T;H)}+\|\hat{\mathbf{q}}\cdot\nu\|^2_{L^2(\Sigma_T^+)}\right)^{\frac12}}\\&\geq\frac{1}{\sqrt{\eta+C(\Omega,T)^2}},\end{aligned}$$

where the last inequality is a consequence of (3.4). This shows that the inequality (3.3) holds with $\delta=(\eta+C(\Omega,T)^2)^{-\frac12}$.

ii) This is due to the symmetry and positivity of a. *iii)* If $((\hat w,\hat{\mathbf{q}}),(\hat\lambda,\hat{\boldsymbol{\mu}}))\in W\times L^2(0,T;H)$ solves the mixed formulation (3.1), then the second equation in (3.1) implies that $M(\hat w,\hat{\mathbf{q}})=0$, therefore $(\hat w,\hat{\mathbf{q}})\in W_0$ and $\mathcal{L}((\hat w,\hat{\mathbf{q}}),(\hat\lambda,\hat{\boldsymbol{\mu}}))=\widehat{\mathcal{J}}^\star(\hat w,\hat{\mathbf{q}})$.

The first equation (3.1) holds for any $(\bar w,\bar{\mathbf{q}})\in W$, in particular, if we set $v=\hat{\mathbf{q}}\cdot\nu$ we have

$$\int_{\Sigma_T^+}v\,\bar{\mathbf{q}}\cdot\nu\,d\sigma\,dt+\int_{Q_T}g\lambda\,dx\,dt=-(u_0,w(0))_{L^2(\Omega)}+(\nabla(-\Delta)^{-1}u_1,\bar{\mathbf{q}}(\cdot,0))_{L^2(\Omega)} \tag{3.5}$$

for any $(\bar w,\bar{\mathbf{q}})\in W$ such that $M(\bar w,\bar{\mathbf{q}})=(g,0)$ for some $g\in L^2(Q_T)$.

If $(\bar w,\bar{\mathbf{q}})\in W$ with $M(\bar w,\bar{\mathbf{q}})=(g,0)$ for some $g\in L^2(Q_T)$ and $\psi_T\in H_0^1(\Omega)$ is the unique function such that $\nabla\bar{\mathbf{q}}(\cdot,T)=\psi_T$, then it is easy to check that $(\bar w,\bar{\mathbf{q}})$ can be realized as $(\bar w,\bar{\mathbf{q}})=(\varphi_t,\nabla\varphi)$ where $\varphi\in C(0,T;H_0^1(\Omega))\cap C^1(0,T;L^2(\Omega))$ is the unique weak solution to

$$\begin{cases}\varphi_{tt}-\Delta\varphi=g, & \text{in } Q_T,\\ \varphi=0, & \text{on } \Sigma_T,\\ \varphi(\cdot,T)=\psi_T\ \ \varphi_t(\cdot,T)=\bar w(T). & \end{cases} \tag{3.6}$$

Hence, the formulation (3.5) is equivalent to

$$\int_{\Sigma_T^+}v\frac{\partial\varphi}{\partial\nu}\,d\sigma\,dt+\int_{Q_T}(\varphi_{tt}-\Delta\varphi)\lambda\,dx\,dt=-(u_0,\varphi_t(\cdot,0))_{L^2(\Omega)}+\langle u_1,\varphi(\cdot,0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$$

for any $\varphi\in C(0,T;H_0^1(\Omega))\cap C^1(0,T;L^2(\Omega))$ with $\varphi(T)=\varphi_t(T))=0$, but this means that λ is the unique transposition solution to (1.4) with $\mathcal{J}=\Sigma_T^+$, so λ is the controlled state with control $v|_{\Sigma_T^+}$. □

4. Dual extremal problem

Here we derive an extremal problem which is dual to (1.19) and only involves the variable $(\lambda, \boldsymbol{\mu})$.

For $r > 0$ we define the linear operator $A_r : L^2(0,T;H) \to L^2(0,T;H)$ by

$$A_r(\lambda, \boldsymbol{\mu}) \triangleq M(w, \mathbf{q}), \text{ for any } (\lambda, \boldsymbol{\mu}) \in L^2(0,T;H),$$

where $(w, \mathbf{q}) \in W$ is the unique solution to

$$a_r((w,\mathbf{q}),(\bar{w},\bar{\mathbf{q}})) = b((\bar{w},\bar{\mathbf{q}}),(\lambda,\boldsymbol{\mu})), \quad \text{for any } (\bar{w},\bar{\mathbf{q}}) \in W. \tag{4.1}$$

The condition $r > 0$ implies that the augmented bilinear form a_r is coercive, hence (4.1) has a unique solution.

Lemma 7. *The operator A_r is a strongly elliptic, symmetric isomorphism from $L^2(0,T;H)$ onto $L^2(0,T;H)$.*

Proof. The coercivity estimate for a_r and the Lax-Milgram Lemma imply that the solution to (4.1) satisfies $\|(w,\mathbf{q})\|_W \le \frac{1}{r}\|(\lambda,\boldsymbol{\mu})\|_{L^2(0,T;H)}$, therefore

$$\|A_r(\lambda,\boldsymbol{\mu})\|_{L^2(0,T;H)} \le \frac{1}{r}\|(\lambda,\boldsymbol{\mu})\|_{L^2(0,T;H)}.$$

Let $(\lambda',\boldsymbol{\mu}') \in W$ and denote $(w',\mathbf{q}') \in L^2(0,T;H)$ the corresponding unique solution to (4.1) such that $A_r(\lambda',\boldsymbol{\mu}') = M(w',\mathbf{q}')$. Then (4.1) with $(\bar{w},\bar{\mathbf{q}}) = (w',\mathbf{q}')$ gives

$$\int_0^T (A_r(\lambda',\boldsymbol{\mu}'),(\lambda,\boldsymbol{\mu}))_H \, dt = a_r((w,\mathbf{q}),(w',\mathbf{q}')), \tag{4.2}$$

which implies the positivity and symmetry of A_r. We now check by contradiction that there is a positive constant $C > 0$ such that

$$\int_0^T (A_r(\lambda,\boldsymbol{\mu}),(\lambda,\boldsymbol{\mu}))_H \, dt \ge C\|(\lambda,\boldsymbol{\mu})\|^2_{L^2(0,T;H)}, \tag{4.3}$$

for any $(\lambda,\boldsymbol{\mu}) \in L^2(0,T;H)$. Assume that (4.3) is false, then there exists a sequence $\{(\lambda_n,\boldsymbol{\mu}_n)\}_{n\ge 0}$ in $L^2(0,T;H)$ such that

$$\|(\lambda_n,\boldsymbol{\mu}_n)\|_{L^2(0,T;H)} = 1, \ \forall\, n \ge 0, \quad \lim_{n\to+\infty} \int_0^T (A_r(\lambda_n,\boldsymbol{\mu}_n),(\lambda_n,\boldsymbol{\mu}_n))_H \, dt = 0.$$

Let $(w_n,\mathbf{q}_n)$ the solution to (4.1) corresponding to $(\lambda_n,\boldsymbol{\mu}_n)$, then, from (4.2) we get

$$\lim_{n\to+\infty} \|M(w_n,\mathbf{q}_n)\|_{L^2(0,T;H)} = 0, \qquad \lim_{n\to+\infty} \|\mathbf{q}_n \cdot \nu\|_{L^2(\Sigma_T^+)} = 0. \tag{4.4}$$

Now let $(\lambda,\boldsymbol{\mu}) \in L^2(0,T;H)$ and $(w,\mathbf{q}) \in W$ be the corresponding solution to (4.1), then

$$\begin{aligned}\int_0^T (M(w,\mathbf{q}),(\lambda_n,\boldsymbol{\mu}_n))_H \, dt &= \int_0^T (A_r(\lambda,\boldsymbol{\mu}),(\lambda_n,\boldsymbol{\mu}_n))_H \, dt \\ &= \int_0^T ((\lambda,\boldsymbol{\mu}),A_r(\lambda_n,\boldsymbol{\mu}_n))_H \, dt = \int_0^T ((\lambda,\boldsymbol{\mu}),M(w_n,\mathbf{q}_n))_H \, dt,\end{aligned}$$

and (4.4) implies

$$\lim_{n\to+\infty} \int_0^T (M(w,\mathbf{q}),(\lambda_n,\boldsymbol{\mu}_n))_H \, dt = 0,$$

for any $(w,\mathbf{q}) \in W$, thus $(\lambda_n,\boldsymbol{\mu}_n)$ converges to 0 in the weak-$L^2(0,T;H)$ topology. Equation (4.1) implies, with $(w,\mathbf{q}) = (w_n,\mathbf{q}_n)$ and $(\lambda,\boldsymbol{\mu}) = (\lambda_n,\boldsymbol{\mu}_n)$, that

$$\int_0^T (rM(w_n,\mathbf{q}_n) - (\lambda_n,\boldsymbol{\mu}_n), M(\bar{w},\bar{\mathbf{q}}))_H \, dt + \int_{\Sigma_T^+} (\mathbf{q}_n\cdot\nu)(\bar{\mathbf{q}}\cdot\nu)\, d\sigma\, dt = 0 \tag{4.5}$$

for any $(\bar{w},\bar{\mathbf{q}}) \in W$.

We define the sequence $\{(\bar{w}_n, \bar{\mathbf{q}}_n)\}_{n\geq 0} \subseteq W$ of mild solutions to

$$M(\bar{w}_n, \bar{\mathbf{q}}_n) = rM(w_n, \mathbf{q}_n) - (\lambda_n, \boldsymbol{\mu}_n) \text{ in } Q_T, \quad \bar{w}_n = 0 \text{ on } \Sigma_T, \quad (\bar{w}_n(\cdot, T), \bar{\mathbf{q}}_n(\cdot, T)) = (0, 0).$$

Then (4.5) with this choice of $(\bar{w}_n, \bar{\mathbf{q}}_n)$ implies

$$\|rM(w_n, \mathbf{q}_n) - (\lambda_n, \mu_n)\|^2_{L^2(0,T;H)} \leq \|\mathbf{q}_n \cdot \nu\|_{L^2(\Sigma_T^+)} \|\bar{\mathbf{q}}_n \cdot \nu\|_{L^2(\Sigma_T^+)}.$$

The last inequality together with (2.5) yields

$$\|rM(w_n, \mathbf{q}_n) - (\lambda_n, \mu_n)\|_{L^2(0,T;H)} \leq C\|\mathbf{q}_n \cdot \nu\|_{L^2(\Sigma_T^+)}.$$

This inequality together with (4.4) gives $\|(\lambda_n, \boldsymbol{\mu}_n)\|_{L^2(0,T;H)} \to 0$ as $n \to +\infty$, which is a contradiction. Thus (4.3) holds for some $C > 0$. $\square$

The ellipticity of A_r allows us to introduce a coercive functional $J^{\star\star}$ in next Lemma.

Lemma 8. *For any $r > 0$, let $(w_0, \mathbf{q}_0) \in W$ be the unique solution to*

$$a_r((w_0, \mathbf{q}_0), (\bar{w}, \bar{\mathbf{q}})) = l(\bar{w}, \bar{\mathbf{q}}), \quad \forall (\bar{w}, \bar{q}) \in W,$$

and let $J^{\star\star} : L^2(0, T; H) \to \mathbb{R}$ be the functional defined by

$$J^{\star\star}(\lambda, \boldsymbol{\mu}) = \frac{1}{2}\int_0^T (A_r(\lambda, \boldsymbol{\mu}), (\lambda, \boldsymbol{\mu}))_H \, dt - b((w_0, \mathbf{q}_0), (\lambda, \boldsymbol{\mu})).$$

The following equality holds:

$$\sup_{(\lambda,\boldsymbol{\mu})\in L^2(0,T;H)} \inf_{(w,\mathbf{q})\in W} \mathcal{L}_r((w, \mathbf{q}), (\lambda, \boldsymbol{\mu})) = -\inf_{(\lambda,\boldsymbol{\mu})\in L^2(0,T;H)} J^{\star\star}(\lambda, \boldsymbol{\mu}) + \mathcal{L}_r((w_0, \mathbf{q}_0), 0).$$

Proof. For any $(\lambda, \boldsymbol{\mu}) \in L^2(0, T; H)$, let $(w, \mathbf{q})_{\lambda,\boldsymbol{\mu}} \in W$ be the minimizer of $\mathcal{L}_r((w, \mathbf{q}), (\lambda, \boldsymbol{\mu}))$, then $(w, \mathbf{q})_{\lambda,\boldsymbol{\mu}}$ satisfies the equation

$$a_r((w, \mathbf{q})_{\lambda,\boldsymbol{\mu}}, (\bar{w}, \bar{\mathbf{q}})) + b((\bar{w}, \bar{\mathbf{q}}), (\lambda, \boldsymbol{\mu})) = l(\bar{w}, \bar{\mathbf{q}}), \quad \forall (\bar{w}, \bar{\mathbf{q}}) \in W$$

and can be decomposed as $(w, \mathbf{q})_{\lambda,\boldsymbol{\mu}} = (w_0, \mathbf{q}_0) + (\widetilde{w}, \widetilde{\mathbf{q}})$, where $(\widetilde{w}, \widetilde{\mathbf{q}}) \in W$ solves

$$a_r((\widetilde{w}, \widetilde{\mathbf{q}}), (\bar{w}, \bar{\mathbf{q}})) + b((\bar{w}, \bar{\mathbf{q}}), (\lambda, \boldsymbol{\mu})) = 0, \quad \forall (\bar{w}, \bar{\mathbf{q}}) \in W.$$

We then have

$$\begin{aligned}\inf_{(w,\mathbf{q})\in W} \mathcal{L}_r((w, \mathbf{q}), (\lambda, \boldsymbol{\mu})) &= \mathcal{L}_r((w, \mathbf{q})_{\lambda,\boldsymbol{\mu}}, (\lambda, \boldsymbol{\mu})) = \mathcal{L}_r((w_0, \mathbf{q}_0) + (\widetilde{w}, \widetilde{\mathbf{q}}), (\lambda, \boldsymbol{\mu})) \\ &\triangleq X_1 + X_2 + X_3,\end{aligned}$$

with

$$\begin{cases} X_1 = \frac{1}{2}a_r((\widetilde{w}, \widetilde{\mathbf{q}}), (\widetilde{w}, \widetilde{\mathbf{q}})) + b((\widetilde{w}, \widetilde{\mathbf{q}}), (\lambda, \boldsymbol{\mu})) + b((w_0, \mathbf{q}_0), (\lambda, \boldsymbol{\mu})), \\ X_2 = a_r((\widetilde{w}, \widetilde{\mathbf{q}}), (w_0, \mathbf{q}_0)) - l((\widetilde{w}, \widetilde{\mathbf{q}})), \\ X_3 = \frac{1}{2}a_r((w_0, \mathbf{q}_0)(w_0, \mathbf{q}_0)) - l((w_0, \mathbf{q}_0)). \end{cases}$$

From the definition of $(w_0, \mathbf{q}_0)$, $X_2 = 0$ and $X_3 = \mathcal{L}_r((w_0, \mathbf{q}_0), 0)$. Finally, the definition of $(\tilde{w}, \tilde{\mathbf{q}})$ implies

$$\begin{aligned}X_1 = -\frac{1}{2}a_r((\widetilde{w}, \widetilde{\mathbf{q}}), (\widetilde{w}, \widetilde{\mathbf{q}})) &+ b((w_0, \mathbf{q}_0), (\lambda, \boldsymbol{\mu})) \\ &= -\frac{1}{2}\int_0^T (A_r(\lambda, \boldsymbol{\mu}), (\lambda, \boldsymbol{\mu}))_H \, dt + b((w_0, \mathbf{q}_0), (\lambda, \boldsymbol{\mu}))\end{aligned}$$

and the result follows. $\square$

From the ellipticity of the operator A_r, the minimization of $J^{\star\star}$ in H is well posed. We observe that, in contrast with the initial problem of finding the control of minimal L^2- norm, the minimization of $J^{\star\star}$ in H does not entail any constraint.

Due to the symmetry and ellipticity of the operator A_r, the *conjugate gradient* method is well-suited for the numerical minimization of $J^{\star\star}$. The Polak-Ribière version of the conjugate gradient method read as follows: for a given $(w_0, \mathbf{q}_0) \in W$ (see Lemma 8) and $\epsilon > 0$ (convergence criterion), we follow the steps 1), 2) and 3) below:

1) *Initialization.* An arbitrary initial candidate $(\lambda_0, \mu_0) \in L^2(Q_T)$ is chosen and then we set:

$$(g_0, \mathbf{h}_0) = A_r(\lambda_0, \boldsymbol{\mu}_0) - M(w_0, \mathbf{q}_0), \quad (v_0, \mathbf{z}_0) = (g_0, \mathbf{h}_0).$$

2) *Iteration.* For $n \geq 0$, we compute

$$\alpha_n = \frac{\|(g_n, \mathbf{h}_n)\|_H^2}{\int_0^T ((v_n, \mathbf{z}_n), A_r(v_n, \mathbf{z}_n))_H \, dt}$$

and update:

$$(\lambda_{n+1}, \boldsymbol{\mu}_{n+1}) = (\lambda_n, \boldsymbol{\mu}_n) - \alpha_n (v_n, \mathbf{z}_n), \quad (g_{n+1}, \mathbf{h}_{n+1}) = (g_n, \mathbf{h}_n) - \alpha_n A_r(v_n, \mathbf{z}_n).$$

3) *Convergence test.* If $\|(g_{n+1}, \mathbf{h}_{n+1})\|_H \leq \varepsilon \|g_0, \mathbf{h}_0\|_H$, stop and set $(\bar{\lambda}, \bar{\boldsymbol{\mu}})$ as the approximation of the minimizer of $J^{\star\star}$. Else, compute

$$\beta_n = \frac{\int_0^T ((g_{n+1}, \mathbf{h}_{n+1}), (g_{n+1} - g_n, \mathbf{h}_{n+1} - \mathbf{h}_n))_H \, dt}{\|(g_n, \mathbf{h}_n)\|_H^2},$$

and set

$$(v_{n+1}, \mathbf{z}_{n+1}) = (g_{n+1}, \mathbf{h}_{n+1}) + \beta_n (v_n, \mathbf{z}_n).$$

Do $n = n + 1$ and return to step 2).

5. Stabilized mixed formulation

In this section we introduce a stabilized mixed formulation equivalent to (3.1). It adds some uniform coercivity property with respect to the multiplier variables, property which allows to bypass the Babuška-Brezzi *inf-sup* condition.

We define the Hilbert space

$$\Lambda = \left\{(\lambda, \boldsymbol{\mu}) \in C([0,T]; H) \cap C^1([0,T]; V'), \ M(\lambda, \boldsymbol{\mu}) \in L^2(0,T; V'), \ \lambda \in L^2(\Sigma_T)\right\}$$

endowed with the norm given by the following scalar product

$$((\lambda, \boldsymbol{\mu}), (\bar{\lambda}, \bar{\boldsymbol{\mu}}))_\Lambda = \int_0^T (M(\lambda, \boldsymbol{\mu}), M(\bar{\lambda}, \bar{\boldsymbol{\mu}}))_H \, dt + \int_{\Sigma_T} \lambda \bar{\lambda} \, d\sigma \, dt.$$

Let $r > 0$ and $\alpha \in (0,1)$, then we want to find $(w, \mathbf{q}) \in W$, $(\lambda, \boldsymbol{\mu}) \in \Lambda$ such that

$$\begin{cases} a_{r,\alpha}((w, \mathbf{q}), (\bar{w}, \bar{\mathbf{q}})) + b_\alpha((\bar{w}, \bar{\mathbf{q}}), (\lambda, \boldsymbol{\mu})) = l(\bar{w}, \bar{\mathbf{q}}), & \forall \ (\bar{w}, \bar{\mathbf{q}}) \in W, \\ b_\alpha((w, \mathbf{q}), (\bar{\lambda}, \bar{\boldsymbol{\mu}})) - c_\alpha((\lambda, \boldsymbol{\mu}), (\bar{\lambda}, \bar{\boldsymbol{\mu}})) = 0, & \forall \ (\bar{\lambda}, \bar{\boldsymbol{\mu}}) \in \Lambda, \end{cases} \tag{5.1}$$

where the forms $a_{r,\alpha} : W \times W \to \mathbb{R}$, $b_\alpha : W \times \Lambda \to \mathbb{R}$, $c_\alpha : \Lambda \times \Lambda \to \mathbb{R}$ are defined by

$$a_{r,\alpha}((w, \mathbf{q}), (\bar{w}, \bar{\mathbf{q}})) \triangleq (1 - \alpha) \int_{\Sigma_T^+} (\mathbf{q} \cdot \nu) \, (\bar{\mathbf{q}} \cdot \nu) \, d\sigma \, dt + r \int_0^T (M(w, \mathbf{q}), M(\bar{w}, \bar{\mathbf{q}})_H \, dt,$$

$$b_\alpha((w, \mathbf{q}), (\bar{\lambda}, \bar{\boldsymbol{\mu}})) \triangleq \int_0^T (M(w, \mathbf{q}), (\bar{\lambda}, \bar{\boldsymbol{\mu}}))_H \, dt + \alpha \int_{\Sigma_T^+} \bar{\lambda} \mathbf{q} \cdot \nu \, d\sigma \, dt,$$

$$c_\alpha((\lambda, \boldsymbol{\mu}), (\bar{\lambda}, \bar{\boldsymbol{\mu}})) \triangleq \alpha((\lambda, \boldsymbol{\mu}), (\bar{\lambda}, \bar{\boldsymbol{\mu}}))_\Lambda.$$

In view of [5, Remark 4.3.1], if $(w,\mathbf{q}) \in W$ and $(\lambda,\mu) \in \Lambda$ solve (5.1), then they are solutions of the saddle-point problem

$$\inf_{(w,\mathbf{q})\in W}\sup_{(\lambda,\boldsymbol{\mu})\in\Lambda} \mathcal{L}_r((w,\mathbf{q}),(\lambda,\boldsymbol{\mu})) - \frac{\alpha}{2}\|M(\lambda,\boldsymbol{\mu})\|^2_{L^2(0,T;V')} - \frac{\alpha}{2}\|\lambda - \mathbf{q}\cdot\nu\|^2_{L^2(\Sigma_T)}, \tag{5.2}$$

where $\mathcal{L}_r$ is the augmented Lagrangian defined in (3.2).

Theorem 2. *The stabilized mixed formulation* (5.1) *is well-posed for any* $r > 0$ *and* $\alpha \in (0,1)$.

Proof. This is a straightforward consequence of [5, Proposition 4.2.1], the boundedness of $a_{r,\alpha}$, b_α and c_α and the coercivity of $a_{r,\alpha}$ and c_α in W and Λ respectively. □

Proposition 1. *The solutions of* (5.1) *and of the augmented formulation associated to* (3.1) *coincide.*

Proof. For any $r > 0$, let us check that the saddle-point $((w_r,\mathbf{q}_r),(\lambda_r,\boldsymbol{\mu}_r)) \in W\times L^2(0,T;H)$ of $\mathcal{L}_r$ is also a saddle point of $\mathcal{L}_{r,\alpha}$. We observe that by Theorem 1, λ_r is a controlled solution of the wave equation with boundary control in $L^2(\Sigma_T)$, thus $\lambda_r \in L^2(\Sigma_T)$ and therefore $((w_r,\mathbf{q}_)),(\lambda_r,\boldsymbol{\mu}_r)) \in W\times\Lambda$. Moreover, for any $(\lambda,\boldsymbol{\mu}) \in \Lambda$,

$$\begin{aligned}\mathcal{L}_{r,\alpha}((w_r,q_r),(\lambda,\boldsymbol{\mu})) \leq& \mathcal{L}_r((w_r,q_r),(\lambda,\boldsymbol{\mu})) \leq \mathcal{L}_r((w_r,q_r),(\lambda_r,\boldsymbol{\mu}_r))\\ =&\mathcal{L}_{r,\alpha}((w_r,q_r),(\lambda_r,\boldsymbol{\mu}_r)) + \frac{\alpha}{2}\|\lambda_r - \mathbf{q}_r\cdot\nu\|^2_{L^2(\Sigma_T)}\\ &+ \frac{\alpha}{2}\|M(\lambda_r,\boldsymbol{\mu}_r)\|^2_{L^2(0,T;V')} = \mathcal{L}_{r,\alpha}((w_r,q_r),(\lambda_r,\boldsymbol{\mu}_r))\end{aligned}$$

since $((w_r,\mathbf{q}_r),(\lambda_r,\boldsymbol{\mu}_r))$ solves the augmented formulation associated to (3.1). Therefore $(\lambda_r,\boldsymbol{\mu}_r)$ maximizes $(\lambda,\boldsymbol{\mu}) \to \mathcal{L}_{r,\alpha}((w_r,\mathbf{q}_r),(\lambda,\boldsymbol{\mu}))$. Conversely, the functional $F : W \to \mathbb{R}$, given by $F(w,\mathbf{q}) = \mathcal{L}_{r,\alpha}((w,\mathbf{q}),(\lambda_r,\boldsymbol{\mu}_r))$, admits a unique extremal point for any $r > 0$ and any $\alpha \in (0,1)$ due to the ellipticity of $a_{r,\alpha}$. Moreover, for any $(\hat{w},\hat{\mathbf{q}}) \in W$ we have

$$\begin{aligned}&\frac{d}{d\varepsilon}F((w,\mathbf{q})+\varepsilon(\hat{w},\hat{\mathbf{q}}))|_{\varepsilon=0}\\ &\qquad = a_r((w,\mathbf{q}),(\hat{w},\hat{\mathbf{q}})) + b((w,\mathbf{q}),(\hat{w},\hat{\mathbf{q}})) - l(\hat{w},\hat{\mathbf{q}}) - \alpha\int_{\Sigma_T}(\lambda_r - \mathbf{q}\cdot\nu)\hat{\mathbf{q}}\cdot\nu\, d\sigma\, dt,\end{aligned}$$

therefore, by (3.1) we have

$$\frac{d}{d\varepsilon}F((w_r,\mathbf{q}_r)+\varepsilon(\hat{w},\hat{\mathbf{q}}))|_{\varepsilon=0} = 0 \quad \forall(\hat{w},\hat{\mathbf{q}}) \in W,$$

thus, $(w_r,\mathbf{q}_r)$ minimizes $(w,\mathbf{q}) \to \mathcal{L}_{r,\alpha}((w,\mathbf{q}),(\lambda_r,\boldsymbol{\mu}_r))$. Consequently, the pair $(w_r,\mathbf{q}_r)$, $(\lambda_r,\boldsymbol{\mu}_r)$ is also a saddle-point for $\mathcal{L}_{r,\alpha}$. The result follows from the uniqueness of the saddle point. □

6. Finite dimensional conformal approximation

We now focus on the discretization of the mixed formulation (3.1) with the form a replaced by the augmented quadratic form a_r introduced in (3.2) and assuming $r > 0$.

Let W_h and V_h be two finite dimensional spaces parametrized by the variable $h > 0$ such that

$$W_h \subseteq W, \quad M_h \subseteq L^2(0,T;H),\ \forall h > 0. \tag{6.1}$$

Then, we can introduce the following approximated problem: find $(w_h,\mathbf{q}_h) \in W_h$, $(\lambda_h,\mu_h) \in M_h$ solution of

$$\begin{cases} a_r((w_h,\mathbf{q}_h),(\bar{w}_h,\bar{\mathbf{q}}_h)) + b((\bar{w}_h,\bar{\mathbf{q}}_h),(\lambda_h,\boldsymbol{\mu}_h)) = l(\bar{w}_h,\bar{\mathbf{q}}_h), & \forall\ (\bar{w}_h,\bar{\mathbf{q}}_h) \in W_h,\\ b((w_h,\mathbf{q}_h),(\bar{\lambda}_h,\bar{\boldsymbol{\mu}}_h)) = 0, & \forall\ (\bar{\lambda}_h,\bar{\boldsymbol{\mu}}_h) \in M_h.\end{cases} \tag{6.2}$$

The well-posedness of this mixed formulation is again a consequence of two properties: the coercivity of the bilinear form a_r on the subset

$$\mathcal{N}_h(b) = \left\{ (w_h, q_h) \in W_h;\ b((w_h, \mathbf{q}_h), (\lambda_h, \boldsymbol{\mu}_h)) = 0\ \forall \lambda_h \in M_h \right\}$$

and a discrete *inf-sup* condition.

Actually, from the relation

$$a_r((w, \mathbf{q}), (w, \mathbf{q})) \geq \frac{r}{\eta} \|(w, \mathbf{q})\|_W^2, \quad \forall (w, \mathbf{q}) \in W \tag{6.3}$$

the form a_r is coercive on the whole space W, therefore, also on $\mathcal{N}_h(b) \subseteq W_h \subseteq W$. We emphasize that for $r = 0$, the discrete formulation (6.2) may not be well-posed over $W_h \times Z_h$ because the form $a_{r=0}$ may not be coercive over the discrete kernel of b: the equality $b((w_h, \mathbf{q}_h), (\lambda_h, \boldsymbol{\mu}_h)) = 0$ for all $(\lambda_h, \boldsymbol{\mu}_h) \in M_h$ does not imply in general that $M(w_h, \mathbf{q}_h)$ vanishes. Therefore, the term $r\|M(w, \mathbf{q})\|_{L^2(Q_T)}$ which appears in the Lagrangien $\mathcal{L}_r$ may be understood as a stabilization term: for any $h > 0$, it ensures the uniform coercivity of the form a_r and vanishes at the limit in h. We also emphasize that this term is not a regularization term as it does not add any regularity on the variable $(w, \mathbf{q})$.

The discrete inf-sup condition reads as follows; for any $h > 0$,

$$\delta_h := \inf_{(\lambda_h, \boldsymbol{\mu}_h) \in M_h} \sup_{(w_h, \mathbf{q}) \in W_h} \frac{b((w_h, \mathbf{q}_h), (\lambda_h, \boldsymbol{\mu}_h))}{\|(w_h, \mathbf{q}_h)\|_{W_h} \|(\lambda_h, \boldsymbol{\mu}_h)\|_{M_h}} > 0 \tag{6.4}$$

Let us assume that this condition holds, so that for any fixed $h > 0$, there exists a unique couple $(w_h, \mathbf{q}_h), (\lambda_h, \mu_h)$ solution of (6.2). Taking $\eta = r$ in (2.17), we then have the following estimate which follows from the classical theory of approximations of saddle point problems (see [5, Theorem 5.2.2]).

Proposition 2. *Let $h > 0$. Let $(w, \mathbf{q}), (\lambda, \boldsymbol{\mu})$ and $(w_h, \mathbf{q}_h), (\lambda_h, \boldsymbol{\mu}_h)$ be the solution of (3.1) and (6.2). Let δ_h be the discrete inf-sup constant defined by (6.4). Then,*

$$\begin{aligned} &\left\|(w, \mathbf{q}) - (w_h, \mathbf{q}_h)\right\|_W \leq 2\left(1 + \frac{1}{\sqrt{r}\delta_h}\right) d((w, \mathbf{q}), W_h) + \frac{1}{\sqrt{r}} d((\lambda, \boldsymbol{\mu}), M_h), \\ &\left\|(\lambda, \boldsymbol{\mu}) - (\lambda_h, \boldsymbol{\mu}_h)\right\|_{L^2(0,T;H)} \leq \left(2 + \frac{1}{\sqrt{r}\delta_h}\right) \frac{1}{\delta_h} d((w, \mathbf{q}), W_h) + \frac{3}{\sqrt{r}\delta_h} d((\lambda, \boldsymbol{\mu}), M_h) \end{aligned} \tag{6.5}$$

where $d((w, \mathbf{q}), W_h) := \inf_{(w, \mathbf{q}) \in W_h} \|(w, \mathbf{q}) - (w_h, \mathbf{q}_h)\|_W$ given by (2.17) and similarly $d((\lambda, \boldsymbol{\mu}), M_h) := \inf_{(\lambda, \boldsymbol{\mu}) \in M_h} \|(\lambda, \boldsymbol{\mu}) - (\lambda_h, \boldsymbol{\mu}_h)\|_{L^2(0,T;H)}$.

Let $n_h = \dim W_h$, $m_h = \dim M_h$ and let the real matrices $A_{r,h} \in \mathbb{R}^{n_h, n_h}$, $B_h \in \mathbb{R}^{m_h, n_h}$, $J_h \in \mathbb{R}^{m_h, m_h}$ and $L_h \in \mathbb{R}^{n_h}$ be defined by

$$\begin{cases} a_r((w_h, \mathbf{q}_h), (\bar{w}_h, \bar{\mathbf{q}}_h)) = \{\bar{w}_h, \bar{\mathbf{q}}_h\}^t A_{r,h} \{w_h, \mathbf{q}_h\}, & \forall (w_h, \mathbf{q}_h), (\bar{w}_h, \bar{\mathbf{q}}_h) \in W_h, \\ b_r((w_h, \mathbf{q}_h), (\lambda_h, \boldsymbol{\mu}_h)) = \{\lambda_h, \boldsymbol{\mu}_h\}^t B_h \{w_h, \mathbf{q}_h\}, & \forall (w_h, \mathbf{q}_h) \in W_h,\ (\lambda_h, \boldsymbol{\mu}_h) \in M_h, \\ \int_0^T ((\lambda_h, \boldsymbol{\mu}_h), (\bar{\lambda}_h, \bar{\boldsymbol{\mu}}_h))_H, dt = \left\{\bar{\lambda}_h, \bar{\boldsymbol{\mu}}_h\right\}^t J_h \{\lambda_h, \boldsymbol{\mu}_h\}, & \forall (\lambda_h, \boldsymbol{\mu}_h),\ (\bar{\lambda}_h, \bar{\boldsymbol{\mu}}_h) \in M_h \\ l((w_h, \mathbf{q}_h)) = L_h \{w_h, \mathbf{q}_h\}, & \forall (w_h, \mathbf{q}_h) \in W_h, \end{cases}$$

where $\{w_h, \mathbf{q}_h\} \in \mathbb{R}^{n_h}$ denotes the (column) vector associated to $(w_h, \mathbf{q}_h)$ and the same notation holds for $\{\lambda_h, \boldsymbol{\mu}_h\} \in \mathbb{R}^{n_h}$.

Taking this into account, the discrete mixed formulation (6.2) reads as follows: find $(w_h, \mathbf{q}_h) \in \mathbb{R}^{n_h}$ and $(\lambda_h, \boldsymbol{\mu}_h) \in \mathbb{R}^{m_h}$ such that

$$\begin{pmatrix} A_{r,h} & B_h^T \\ B_h & 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h, n_h+m_h}} \begin{pmatrix} \{w_h, \mathbf{q}_h\} \\ \{\lambda_h, \boldsymbol{\mu}_h\} \end{pmatrix}_{\mathbb{R}^{n_h+m_h}} = \begin{pmatrix} L_h \\ 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h}}. \tag{6.6}$$

The matrix $A_{r,h}$ as well as the mass matrix J_h are symmetric and positive definite for any $h > 0$ and any $r > 0$. On the other hand, the matrix of order $m_h + n_h$ in (6.6) is symmetric but not necessarily positive definite.

We recall (see [5, Theorem 3.2.1]) that the inf-sup property (6.4) is equivalent to the injective character of the matrix B_h^T of size $n_h \times m_h$, that is $Ker(B_h^T) = 0$. If a necessary condition is given by $m_h \leq n_h$, this property strongly depends on the structure of the space W_h and M_h. This issue will be numerically analyzed in Section 8.1 and will highlight definitively the role of the parameter r.

7. One-dimensional discretization

We observe that when $n = 1$, we have $L^2(0,T;H) = (L^2(Q_T))^2$ with $Q_T = (0,1) \times (0,T)$.

Similarly to what was done in [11], the finite dimensional and conformal space W_h must be chosen so that $M(w_h, \mathbf{q}_h) \in L^2(0,T;H)$ for any $(w_h, \mathbf{q}_h) \in W_h$; in the one-dimensional setting, this is guaranteed if both w_h and q_h have first order space and time derivatives in $L^2_{loc}(Q_T)$, therefore, in order to have a conformal approximation based on standard triangulations of Q_T it will be enough to consider $H^1(Q_T)$ functions, that is, functions having first order weak derivatives —in both space and time variables— in $L^2(Q_T)$. This is, at the practical level of the implementation, the main advantages with respect to [11].

We introduce a triangulation $\mathcal{T}_h$ such that $\overline{Q_T} = \cup_{K \in \mathcal{T}_h} K$ and we assume that $\{\mathcal{T}_h\}_{h>0}$ is a regular family. Then we introduce the space W_h as follows:

$$W_h = W_h^1 \times W_h^2 \subseteq W, \tag{7.1}$$

with W_h^1 and W_h^2 are defined by:

$$W_h^1 = \left\{ w_h \in H^1(Q_T),\ w_h|_K \in \mathbb{P}_2(K)\ \forall K \in \mathcal{T}_h,\ w_h = 0 \text{ on } \partial\Omega \times (0,T) \right\},$$

$$W_h^2 = \left\{ q_h \in H^1(Q_T),\ q_h|_K \in \mathbb{P}_2(K)\ \forall K \in \mathcal{T}_h \right\}.$$

The space $\mathbb{P}_2(K)$ denotes the space of second order polynomial functions in x and t defined in a triangle K; it involves 6 degrees of freedom, namely, the values of w_h —or $\mathbf{q}_h$— on the vertices and on the midpoints of the edges of the triangle.

We also define the finite dimensional space

$$M_h = M_h^1 \times M_h^2 \subseteq (L^2(Q_T))^2 \tag{7.2}$$

with M_h^1 and M_h^2 defined by

$$M_h^1 = \left\{ \lambda_h \in H^1(Q_T),\ \lambda_h|_K \in \mathbb{P}_1(K)\ \forall K \in \mathcal{T}_h \right\},$$

$$M_h^2 = \left\{ \mu_h \in H^1(Q_T),\ \mu_h|_K \in \mathbb{P}_1(K)\ \forall K \in \mathcal{T}_h \right\}.$$

The space $\mathbb{P}_1(K)$ denotes the space of first order polynomial functions in x and t defined in a triangle K; it involves 3 degrees of freedom: the values of λ_h —or $\boldsymbol{\mu}_h$— on the vertices of the triangle.

Let us now describe the meshes we use in the $1D$ —in space— setting. We will use uniform and non-uniform regular meshes (see Figure 1):

- Uniform triangular mesh: each element is a right isosceles triangle whose equal sides have length $\Delta t, \Delta x > 0$;
- Non uniform but regular triangular mesh obtained by Delaunay triangulation. Each triangle of the mesh having a side on the boundaries $(0,1) \times \{0,T\}$ has the side on the boundary of lenght Δx; similarly, each triangle having a side on the boundaries $\{0,1\} \times (0,T)$ has the side on that boundary of length Δt.

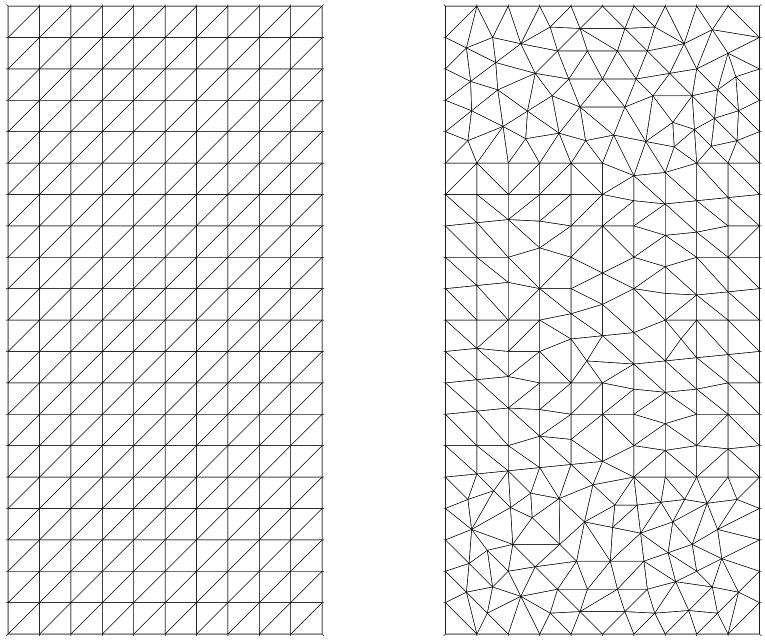

FIGURE 1. Regular meshes for Q_T ; Left: uniform mesh - $h = 1.41 \times 10^{-1}$. Right: non uniform mesh - $h = 1.52 \times 10^{-1}$.

In the numerical experiments we shall consider 5 levels of meshes, that is, we consider $(\Delta x, \Delta t) = (1/N, 1/N)$ for $N = 10 \cdot 2^k$ for $0 \leq k \leq 4$. We denote

$$h \triangleq \max(diam(K), K \in \mathcal{T}_h),$$

where $diam(K)$ is the diameter of K, the parameter h measures the level of fineness of the mesh and decreases as N increases.

N	10	20	40	80	160
card($\mathcal{T}_h$) - uniform	400	1 600	6 400	25 600	102 400
card($\mathcal{T}_h$) - non uniform	446	1 784	7 136	28 544	114 176
$\sharp$ nodes - uniform	861	3 321	13 041	56 681	205 761
$\sharp$ nodes - non uniform	953	3 689	14 513	57 569	229 313
h - uniform	1.41×10^{-1}	7.01×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
h - non uniform	1.52×10^{-1}	7.60×10^{-2}	3.80×10^{-2}	1.90×10^{-2}	9.50×10^{-3}

TABLE 1. Number of elements for the uniform/non uniform meshes and value of h for each type of mesh w.r.t. N with $T = 2$.

We finally remark that other choices for the finite element spaces W_h-M_h could be considered, however, the numerical experiments suggest that choosing $\mathbb{P}_1$-based finite element spaces for both W_h and M_h is not a good choice. Nevertheless, when we consider stabilized formulations (see Section 8.4) we can bypass the inf-sup condition and good approximations of the control are obtained using $\mathbb{P}_1$-based finite element spaces for both W_h and M_h.

8. Numerical experiments

We now comment some computations performed with the FreeFem++ package developed at the University Paris 6 (see [17]), which is very well adapted to our space-time setting. Section 11 provides one of the code we have used.

8.1. **The discrete inf-sup condition.** Before showing the numerical experiments, we test numerically the discrete inf-sup condition for the choice of finite elements spaces W_h and M_h in (7.1)-(7.2). Taking $\eta = r > 0$ so that $a_r((w,\mathbf{q}),(\bar{w},\bar{\mathbf{q}})) = ((w,\mathbf{q}),(\bar{w},\bar{\mathbf{q}}))_W$ for any $(w,\mathbf{q}),(\bar{w},\bar{\mathbf{q}}) \in W$, it is readily seen ([8]) that the discrete inf-sup constant satisfies

$$\delta_h = \inf\left\{\sqrt{\delta}:\ B_h A_{r,h}^{-1} B_h^T \{\lambda_h,\boldsymbol{\mu}_h\} = \delta\{\lambda_h,\boldsymbol{\mu}_h\},\ \ \forall\{\lambda_h,\boldsymbol{\mu}_h\} \in \mathbb{R}^{m_h}\setminus\{0\}\right\}. \tag{8.1}$$

The matrix $B_h A_{r,h}^{-1} B_h^T$ enjoys the same properties than the matrix $A_{r,h}$: it is symmetric and positive definite so that the scalar δ_h defined in terms of the (generalized) eigenvalue problem (8.1) is strictly positive. This eigenvalue problem is solved using the power iteration algorithm (assuming that the lower eigenvalue is simple): for any $\{v_h^0\} \in \mathbb{R}^{n_h}$ such that $\|\{v_h^0\}\|_2 = 1$, compute for any $n \geq 0$, $\{w_h^n,\mathbf{q}_h^n\} \in \mathbb{R}^{n_h}$, $\{\lambda_h^n,\boldsymbol{\mu}_h^n\} \in \mathbb{R}^{m_h}$ and $\{v_h^{n+1}\} \in \mathbb{R}^{n_h}$ iteratively as follows:

$$\begin{cases} A_{r,h}\{w_h^n,\mathbf{q}_h^n\} + B_h^T\{\lambda_h^n,\boldsymbol{\mu}_h^n\} = 0, \\ B_h\{w_h^n,\mathbf{q}_h^n\} = -J_h\{v_h^n\}, \end{cases} \qquad \{v_h^{n+1}\} = \frac{\{w_h^n,\mathbf{q}_h^n\}}{\|\{w_h^n,\mathbf{q}_h^n\}\|_2}.$$

The scalar δ_h defined by (8.1) is then given by

$$\delta_h = \lim_{h\to 0} \|\{\lambda_h^n,\boldsymbol{\mu}_h^n\}\|_2^{-\frac{1}{2}}.$$

We now give some numerical values of δ_h with respect to N (equivalently with respect to h)

h	1.41×10^{-1}	7.01×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$r=1$	0.264	0.197	0.132	0.099	0.070
$r=10^{-1}$	0.751	0.569	0.412	0.310	0.222
$r=10^{-2}$	1.881	1.478	1.112	0.839	0.627
$r=h$	0.652	0.660	0.660	0.679	0.661
$r=h^2$	1.397	1.934	2.642	3.636	5.031

TABLE 2. δ_h w.r.t. r and h, $T=2$, for the W_h-M_h finite elements and uniform mesh.

h	1.52×10^{-1}	7.60×10^{-2}	3.80×10^{-2}	1.90×10^{-2}	9.50×10^{-3}
$r=1$	0.426	0.316	0.229	0.155	0.106
$r=10^{-1}$	0.991	0.868	0.698	0.489	0.339
$r=10^{-2}$	2.269	1.738	1.373	1.099	0.896
$r=h$	0.885	0.927	0.929	0.921	0.908
$r=h^2$	1.612	2.154	2.974	4.115	5.733

TABLE 3. δ_h w.r.t. r and h, $T=2$, for the W_h-M_h finite elements and non uniform mesh.

for the chosen finite element spaces $W_h - M_h$ with uniform and non uniform meshes. Tables 2 and 3 are concerned with the uniform and non uniform meshes respectively. We observe (see Tables 2 and 3 and Figure 2) that for some fixed values of $r > 0$, the quantity δ_h is not bounded by below as $h \to 0$, in that case, it can be concluded that the pair of finite element spaces considered do not *pass* the inf-sup test.

However, if r is chosen to be equal h^α with $\alpha \geq 1$, then we observe that for both the uniform and non uniform meshes, the discrete inf-sup constant δ_h remains bounded by below. We remark that similar observations were done in [11, Section 4.2], but in that case it was necessary to choose $r = h^{2\alpha}$ with $\alpha \geq 1$ in order to have a lower bound for the discrete inf-sup constant, while in the present case a smaller power of h suffices.

With the non-uniform mesh we observe (Figure 2) that the inf-sup constant δ_h behaves as:

$$\delta_h \approx C_r \sqrt{\frac{h}{r}} \quad \text{as } h \to 0^+, \tag{8.2}$$

for some constant $C_r > 0$ uniformly bounded with respect to r.

Taking $r = h$, δ_h is therefore of order one so that the general estimate of Proposition 2 leads to

$$\big\|(w,\mathbf{q}) - (w_h,\mathbf{q}_h)\big\|_W + \big\|(\lambda,\boldsymbol{\mu}) - (\lambda_h,\boldsymbol{\mu}_h)\big\|_W \le h^{-1/2}\Big(d((w,\mathbf{q}),W_h) + d((\lambda,\boldsymbol{\mu}),M_h)\Big)$$

from which, using standard interpolation estimates for elliptic problems (given in [9, Chapter III]) and assuming regularity on the solution of (3.1), we can obtain the strong convergence of the approximation with respect to h (we refer to [10, Section 4]). Without additional regularity on the solution, precisely on the pair $(w,\mathbf{q}) \in C([0,T],H)$, the convergence to zero of $d((w,\mathbf{q}),W_h)$ requires more care and will be detailed in a distinct work.

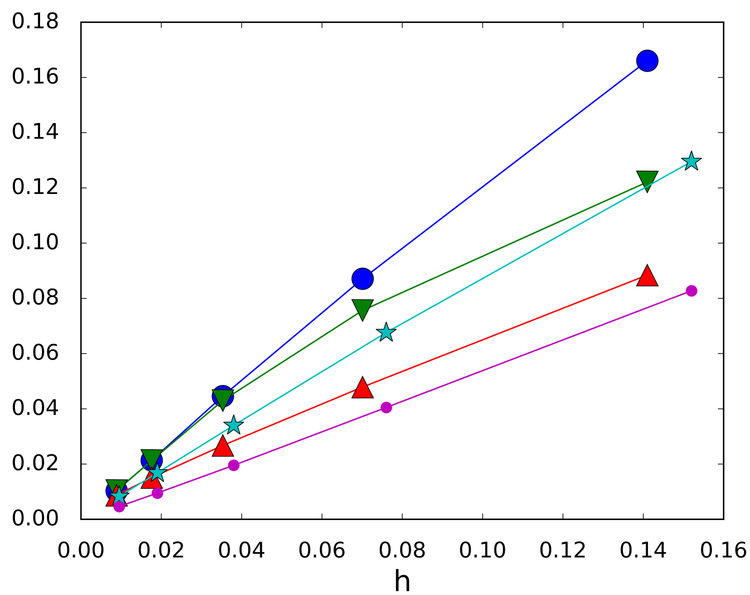

FIGURE 2. Non uniform mesh - Evolution of $\sqrt{hr}\delta_h$ with respect to h (see Table 3) for $r = 1$ ($\bigcirc$), $r = 10^{-1}$ ($\bigtriangledown$), $r = 10^{-2}$ ($\triangle$) , $r = h$ ($\star$), $r = h^2$ ($\circ$).

On the other hand, when W_h is replaced by $\widetilde{W}_h$ based on $\mathbb{P}_1$ approximation (see (8.5)), our simulation suggests that the discrete inf-sup constant is much smaller and not uniformly bounded by below with respect to h. As shown in Table 4, this property seems independent of r.

8.2. **Numerical experiments.** We consider the control problem

$$\begin{cases} u_{tt} - u_{xx} = 0, & \text{in } (0,1)\times(0,T), \\ u(0,t) = 0,\ u(1,t) = v(t), & \text{in } (0,T), \\ u(\cdot,0) = u_0,\ u_t(\cdot,0) = u_1, & \text{in } (0,1), \end{cases} \tag{8.3}$$

with

$$u_0(x) = 4x\,1_{(0,1/2)}(x), \quad u_1(x) = 0, \quad T = 2. \tag{8.4}$$

h	1.41×10^{-1}	7.01×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$r = 1$	5.77×10^{-5}	1.41×10^{-10}	1.8×10^{-10}	4.07×10^{-9}	1.97×10^{-10}
$r = 10^{-1}$	2.45×10^{-9}	5.17×10^{-10}	2.23×10^{-10}	2.05×10^{-9}	1.63×10^{-9}
$r = 10^{-2}$	1.51×10^{-8}	1.71×10^{-9}	3.88×10^{-9}	1.77×10^{-8}	8.11×10^{-9}
$r = h$	2.4×10^{-9}	1.05×10^{-9}	7.77×10^{-10}	6.73×10^{-9}	1.64×10^{-9}
$r = h^2$	4.92×10^{-9}	4.19×10^{-9}	2.6×10^{-9}	3.33×10^{-9}	1.44×10^{-9}

TABLE 4. δ_h w.r.t. r and h, $T = 2$, for the $\widetilde{W_h}$-M_h finite elements and uniform mesh.

This is a well-known stiff problem (considered in [21, 11, 7]) where the initial position belongs to $L^2(0,1)$ but is discontinuous. The corresponding control of minimal L^2-norm, discontinuous as well, is given by $v(t) = 2(1-t)1_{(1/2,3/2)}(t)$, $t \in (0,T)$, so that $\|v\|_{L^2(0,T)} = 1/\sqrt{3} \approx 0.5773$.

We recall the reader that in the continuous problem the Lagrange multiplier λ coincides with the controlled solution of (8.3) and $\mathbf{q} = \varphi_x$, where φ is the solution to the minimization problem (1.13), therefore, both $\lambda(1,t)$ and $\mathbf{q}(1,t)$ coincide with the theoretical control v and the restriction of the discrete solutions λ_h and $\mathbf{q}_h$ to $x = 1$ should approximate v.

Tables 5 and 6 collect some values of $\|\mathbf{q}_h\|_{L^2(0,T)}$, $\|v - \mathbf{q}_h\|_{L^2(0,T)}$, $\|\lambda_h\|_{L^2(Q_T)}$ and $\|M(w_h, \mathbf{q}_h)\|_{L^2(Q_T)}$ for different values of r and h with uniform and non uniform meshes.

Figure 3 depicts the dependence of $\|v - \mathbf{q}_h(1,\cdot)\|_{L^2(0,T)}$ with respect to h for $r = 1$ and $r = h$. We observe similar rates of convergence for λ_h and $\mathbf{q}_h$ are similar, however, the error is smaller for $r = h$. We also observe (see Table 5) that the error is smaller for the non uniform mesh than for the uniform mesh.

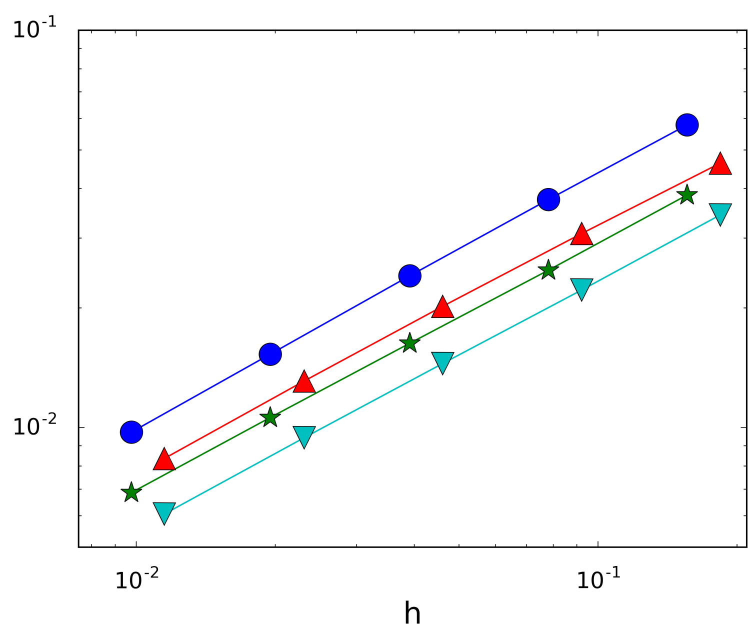

FIGURE 3. Evolution of $\|v - \mathbf{q}_h(1,\cdot)\|_{L^2(0,T)}$ w.r.t h for uniform mesh with $r = 1$ ($\circ$), $r = h$ ($\star$) and non uniform mesh with $r = 1$ ($\triangle$) and $r = h$ ($\bigtriangledown$).

We observe the following rates of convergence (see Figure 3) for $R_1(h) = \|v - \mathbf{q}_h\|_{L^2(0,T)}$ and $R_2(h) = \|M(w_h, \mathbf{q}_h)\|_{L^2(Q_T)}$ using the uniform mesh:

$$
\begin{aligned}
r = 1: &\qquad R_1(h) \approx e^{-4.38} h^{0.64}, \ \ R_2(h) \approx e^{-2.87} h^{0.39},\\
r = 10^{-2}: &\qquad R_1(h) \approx e^{-5.35} h^{0.7}, \ \ R_2(h) \approx e^{0.89} h^{0.51},\\
r = h: &\qquad R_1(h) \approx e^{-4.74} h^{0.62}, \ \ R_2(h) \approx e^{-1.57} h^{0.47},\\
r = h^2: &\qquad R_1(h) \approx e^{-5.19} h^{0.66}, \ \ R_2(h) \approx e^{0.21} h^{0.51}.
\end{aligned}
$$

For the non uniform mesh we observe the following rates of convergence:

$$
\begin{aligned}
r = 1: &\qquad R_1(h) \approx e^{-4.53} h^{0.62}, \ \ R_2(h) \approx e^{-2.99} h^{0.39},\\
r = 10^{-2}: &\qquad R_1(h) \approx e^{-5.28} h^{0.74}, \ \ R_2(h) \approx e^{0.63} h^{0.61},\\
r = h: &\qquad R_1(h) \approx e^{-4.86} h^{0.63}, \ \ R_2(h) \approx e^{-1.84} h^{0.52},\\
r = h^2: &\qquad R_1(h) \approx e^{-5.18} h^{0.7}, \ \ R_2(h) \approx e^{0.19} h^{0.6}.
\end{aligned}
$$

h	1.41×10^{-1}	7.01×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\|\mathbf{q}_h(1,\cdot)\|_{L^2(0,T)}$	0.523	0.543	0.556	0.564	0.569
$\|v - \mathbf{q}_h(1,\cdot)\|_{L^2(0,T)}$	3.85×10^{-2}	2.49×10^{-2}	1.63×10^{-2}	1.06×10^{-2}	6.86×10^{-3}
$\|\lambda_h(1,\cdot)\|_{L^2(0,T)}$	0.538	0.555	0.564	0.57	0.573
$\|v - \lambda_h(1,\cdot)\|_{L^2(0,T)}$	5.27×10^{-2}	3.37×10^{-2}	2.18×10^{-2}	1.41×10^{-2}	8.97×10^{-3}
$\|M(w_h, \mathbf{q}_h)\|_{L^2(Q_T)}$	0.645	0.462	0.331	0.239	0.174

TABLE 5. $r = h$ - uniform mesh.

h	1.52×10^{-1}	7.60×10^{-2}	3.80×10^{-2}	1.90×10^{-2}	9.50×10^{-3}
$\|\mathbf{q}_h(1,\cdot)\|_{L^2(0,T)}$	0.535	0.549	0.559	0.566	0.57
$\|v - \mathbf{q}_h(1,\cdot)\|_{L^2(0,T)}$	3.43×10^{-2}	2.22×10^{-2}	1.45×10^{-2}	9.43×10^{-3}	6.06×10^{-3}
$\|\lambda_h(1,\cdot)\|_{L^2(0,T)}$	0.545	0.558	0.566	0.57	0.573
$\|v - \lambda_h(1,\cdot)\|_{L^2(0,T)}$	3.89×10^{-2}	2.3×10^{-2}	1.46×10^{-2}	9.35×10^{-3}	6×10^{-3}
$\|M(w_h, \mathbf{q}_h)\|_{L^2(Q_T)}$	0.561	0.388	0.265	0.184	0.131

TABLE 6. $r = h$ - non uniform mesh.

Figure 4 shows an initial non uniform but regular mesh and two adaptative refinements of it. We observe that the elements get concentrated along the jumps displayed by the primal variable λ, which are generated by the initial data u_0, discontinuous at the point $x = 1/2$. The adaptative refinement together with the use of $\mathbb{P}_1$ elements for the variable λ leads to an excellent approximation of the control (Figure 6) defined as the restriction of λ_h at $x = 1$. Figure 5 depicts a level sel of λ_h. We have used $r = 10^{-6}$ here.

8.3. **Conjugate gradient for $J^{\star\star}$.** We illustrate here Section 4 and minimize the functional $J^{\star\star} : L^2(Q_T) \to \mathbb{R}$ with respect to the variable λ. This minimization corresponds to the resolution of the mixed formulation (3.1) by an iterative Uzawa type procedure. The conjugate gradient algorithm is given at the end of Section 4. In practice, each iteration amounts to solve a linear system involving the matrix $A_{r,h}$ (see (6.6)) which is sparse, symmetric and positive definite. We use the Cholesky method. The performances of the algorithm are related to the condition number of operator A_r restricted to M_h, which coincides here (see [5]) with the condition number, denoted by $\nu(B_h A_{r,h}^{-1} B_h^T)$ of the symmetric and positive definite matrix $B_h A_{r,h}^{-1} B_h^T$ introduced in (8.1). Arguing as in [11], Section 4.4, we obtain that $\nu(B_h A_{r,h}^{-1} B_h^T) = r^{-1}\delta_h^{-2}$ and therefore, in view of (8.2), $\nu(B_h A_{r,h}^{-1} B_h^T) \approx C_r^{-2} h^{-1}$, a

$\sharp$ triangles	110	1197	2880	5113	8636
$\Vert \mathbf{q}_h(1,\cdot)\Vert_{L^2(0,T)}$	0.46	0.57	0.574	0.576	0.577
$\Vert v-\mathbf{q}_h(1,\cdot)\Vert_{L^2(0,T)}$	8.24×10^{-2}	1.55×10^{-2}	3.72×10^{-3}	1.24×10^{-3}	5.18×10^{-4}
$\Vert \lambda_h(1,\cdot)\Vert_{L^2(0,T)}$	0.451	0.569	0.574	0.576	0.577
$\Vert v-\lambda_h(1,\cdot)\Vert_{L^2(0,T)}$	8.04×10^{-2}	1.52×10^{-2}	3.88×10^{-3}	1.23×10^{-3}	4.48×10^{-4}
$\Vert M(w_h,\mathbf{q}_h)\Vert_{L^2(Q_T)}$	1.13×10^{5}	4.45×10^{4}	1.48×10^{4}	5.63×10^{3}	2.86×10^{3}

TABLE 7. $r=10^{-6}$ - 5 adaptive meshes. Figure 4 displays the 1st, 3rd and 5th adaptive meshes used.

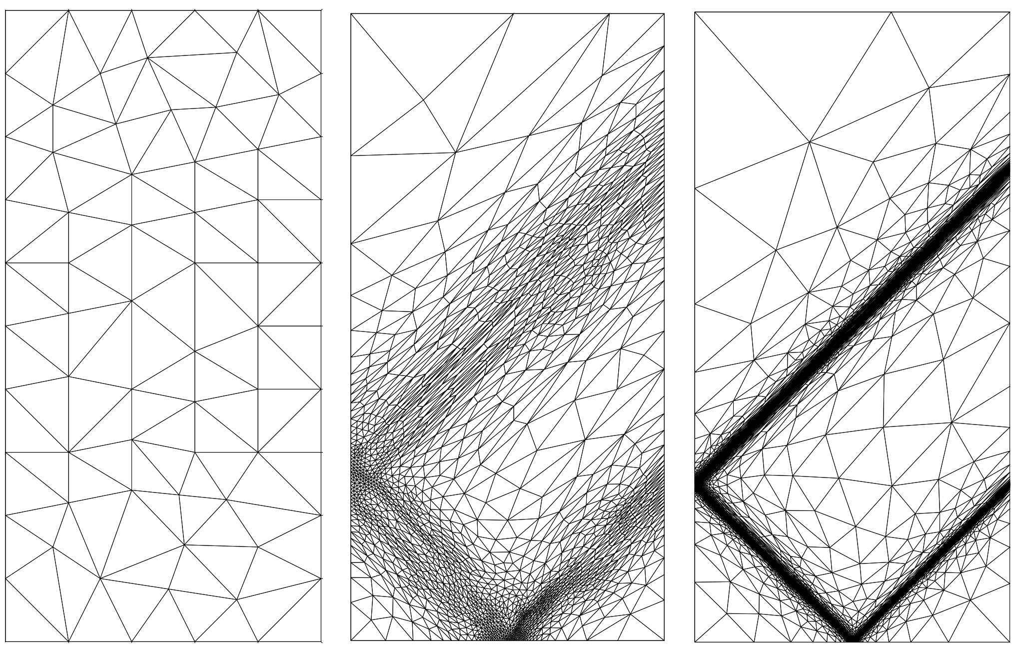

FIGURE 4. Iterative refinement of the triangular mesh over Q_T with respect to the variable λ_h: 110, 2 880 and 8 636 triangles.

better situation than the quadratic behavior observed in general for elliptic problems, in particular in [11].

We take $\varepsilon=10^{-10}$ as a stopping threshold for the algorithm. The algorithm is initialized with $(\lambda^0,\boldsymbol{\mu}^0)=(0,0)$ in Q_T. Tables 8 display the results for $r=1$, $r=10^{-2}$ and $r=h$. We check that the minimization of $J^{\star\star}$ leads exactly to the same result. We recall that the norm of the control is $\Vert v\Vert_{L^2(0,T)}\approx 0.5773$. Moreover, we observe that the number of iterates is sub-linear with respect to h, with a low influence with respect to the value of r. This is in contrast with the behavior of the conjugate gradient algorithm when this latter is used to minimize $J^\star$ with respect to (φ_0,φ_1) (see [7, 21]).

Eventually, additional experiments not reported here suggest the divergence of the solution of (6.2) as h decreases when only $\mathbb{P}_1$ approximation is used. This is in agreement with the behavior of δ_h in that case, see Table 4.

8.4. **Discretization of the stabilized formulation.** We now discuss the numerical discretization of (5.1). The implementation of the discrete stabilized mixed formulation follows

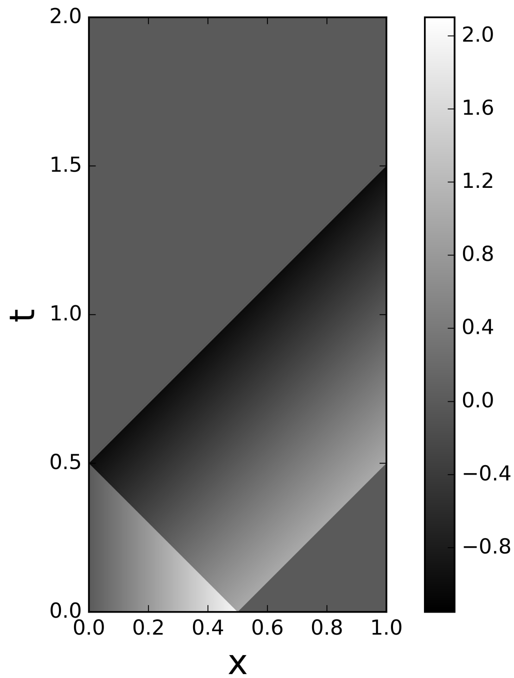

FIGURE 5. The primal variable λ_h in Q_T - Third adapted mesh in Figure 4, $r = 10^{-6}$.

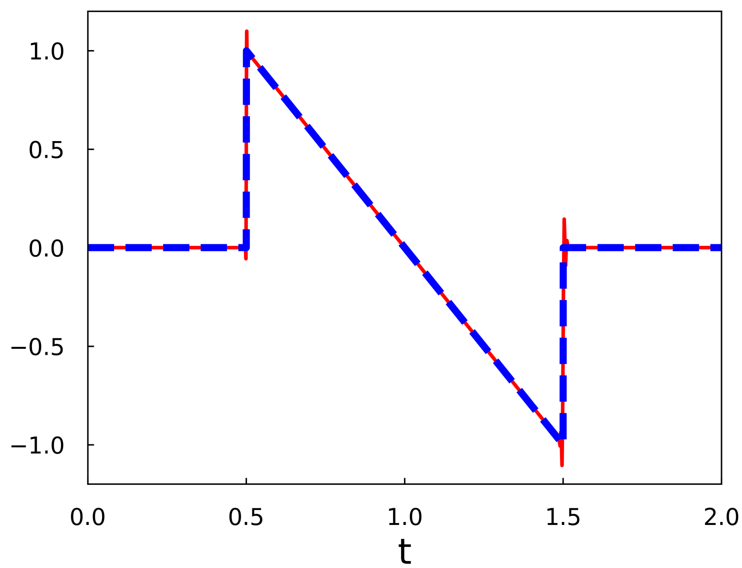

FIGURE 6. Control of minimal L^2-norm v (dashed blue line) and its approximation $\lambda_h(1,\cdot)$ (red line) on $(0,T)$. Third adapted mesh in Figure 4, $r = 10^{-6}$.

the lines of Section 6. There are only two important changes: the appearance of the parameter α in (5.1) (compare with (3.1)) and the possibility of choosing different finite elements. In fact, the latter feature is the main motivation for introducing a stabilized formulation, since it allows us to work with finite element spaces which may not satisfy the inf-sup property, uniformly with respect to h. The stabilization formulation (5.1) coupled with the $\mathbb{P}_1/\mathbb{P}_2$

h	1.52×10^{-1}	7.60×10^{-2}	3.80×10^{-2}	1.90×10^{-2}
$\sharp$ iterates	31	41	54	77
$\Vert\lambda_h(1,\cdot)\Vert_{L^2(0,T)}$	0.469	0.576	0.589	0.586
$\Vert v-\lambda_h(1,\cdot)\Vert_{L^2(0,T)}$	3.21×10^{-1}	1.72×10^{-1}	1.43×10^{-1}	1.25×10^{-1}

h	1.52×10^{-1}	7.60×10^{-2}	3.80×10^{-2}	1.90×10^{-2}
$\sharp$ iterates	46	103	125	133
$\Vert\lambda_h(1,\cdot)\Vert_{L^2(0,T)}$	0.55	0.566	0.569	0.571
$\Vert v-\lambda_h(1,\cdot)\Vert_{L^2(0,T)}$	2.05×10^{-1}	1.47×10^{-1}	1.12×10^{-1}	8.71×10^{-2}

h	1.52×10^{-1}	7.60×10^{-2}	3.80×10^{-2}	1.90×10^{-2}
$\sharp$ iterates	36	43	56	80
$\Vert\lambda_h(1,\cdot)\Vert_{L^2(0,T)}$	0.523	0.566	0.574	0.573
$\Vert v-\lambda_h(1,\cdot)\Vert_{L^2(0,T)}$	2.39×10^{-1}	1.46×10^{-1}	1.19×10^{-1}	9.54×10^{-2}

Table 8. Non uniform mesh - Conjugate gradient method - Number of iterates for $r = 1$ (top), $r = 10^{-2}$ and $r = h$ (bottom).

approximation (7.1)-(7.2) leads exactly to the same results. This is expected since such $\mathbb{P}_1/\mathbb{P}_2$ discretization passes the inf-sup test.

Accordingly, we only discuss a $\mathbb{P}_1/\mathbb{P}_1$ based approximation and we define

$$\widetilde{W}_h = \widetilde{W}_h^1 \times \widetilde{W}_h^2 \subseteq W, \tag{8.5}$$

with $\widetilde{W}_h^1$ and $\widetilde{W}_h^2$ as follows

$$\widetilde{W}_h^1 = \left\{ w_h \in H^1(Q_T),\ w_h|_K \in \mathbb{P}_1(K)\ \forall K \in \mathcal{T}_h,\ w_h = 0 \text{ on } \partial\Omega \times (0,T) \right\},$$

$$\widetilde{W}_h^2 = \left\{ q_h \in H^1(Q_T),\ q_h|_K \in \mathbb{P}_1(K)\ \forall K \in \mathcal{T}_h \right\}.$$

We also define the finite dimensional space

$$\widetilde{M}_h = \widetilde{M}_h^1 \times \widetilde{M}_h^2 \subseteq (L^2(Q_T))^2 \tag{8.6}$$

with $\widetilde{M}_h^1 = M_h^1$ and $\widetilde{M}_h^2 = M_h^2$ given in (7.2).

Tables 9 and 10 give the results for $\alpha = 0.5$ and $\alpha = h^2$ respectively. As expected, the stabilized α-terms allows to recover the convergence of the approximation with respect to h.

Compared with the richer $\mathbb{P}_1/\mathbb{P}_2$ approximation discussed in the previous sections, we also check that the rate of convergence is reduced : precisely, we observe the following rates of convergence for $R_1(h) = \Vert v - \mathbf{q}_h\Vert_{L^2(0,T)}$ and $R_2(h) = \Vert M(w_h, \mathbf{q}_h)\Vert_{L^2(Q_T)}$ using the non uniform mesh and with $\alpha = 0.5$:

$$\begin{aligned}
r = 1: &\qquad R_1(h) \approx e^{-3.38} h^{0.52},\ \ R_2(h) \approx e^{-2.19} h^{0.26},\\
r = 10^{-2}: &\qquad R_1(h) \approx e^{-4.48} h^{0.6},\ \ R_2(h) \approx e^{-0.46} h^{0.38},\\
r = h: &\qquad R_1(h) \approx e^{-3.83} h^{0.49},\ \ R_2(h) \approx e^{-1.52} h^{0.31}.
\end{aligned}$$

For $\alpha = h^2$, we observe:

$$\begin{aligned}
r = 1: &\qquad R_1(h) \approx e^{-3.37} h^{0.53},\ \ R_2(h) \approx e^{-2.18} h^{0.25},\\
r = 10^{-2}: &\qquad R_1(h) \approx e^{-4.46} h^{0.62},\ \ R_2(h) \approx e^{-0.5} h^{0.18},\\
r = h: &\qquad R_1(h) \approx e^{-3.82} h^{0.51},\ \ R_2(h) \approx e^{-1.53} h^{0.28}.
\end{aligned}$$

h	1.52×10^{-1}	7.60×10^{-2}	3.80×10^{-2}	1.90×10^{-2}	9.50×10^{-3}
$\Vert\mathbf{q}_h(1,\cdot)\Vert_{L^2(0,T)}$	0.444	0.494	0.522	0.539	0.551
$\Vert v-\mathbf{q}_h(1,\cdot)\Vert_{L^2(0,T)}$	7.47×10^{-2}	5.21×10^{-2}	3.65×10^{-2}	2.56×10^{-2}	1.81×10^{-2}
$\Vert\lambda_h(1,\cdot)\Vert_{L^2(0,T)}$	0.525	0.543	0.554	0.561	0.566
$\Vert v-\lambda_h(1,\cdot)\Vert_{L^2(0,T)}$	1.26×10^{-1}	6.5×10^{-2}	4.2×10^{-2}	2.79×10^{-2}	1.89×10^{-2}
$\Vert M(w_h,\mathbf{q}_h)\Vert_{L^2(Q_T)}$	0.423	0.343	0.281	0.235	0.197

TABLE 9. $r = h$ - non uniform mesh - stabilized formulation with $\alpha = 0.5$.

h	1.52×10^{-1}	7.60×10^{-2}	3.80×10^{-2}	1.90×10^{-2}	9.50×10^{-3}
$\Vert\mathbf{q}_h(1,\cdot)\Vert_{L^2(0,T)}$	0.456	0.498	0.523	0.54	0.551
$\Vert v-\mathbf{q}_h(1,\cdot)\Vert_{L^2(0,T)}$	6.99×10^{-2}	4.99×10^{-2}	3.56×10^{-2}	2.54×10^{-2}	1.8×10^{-2}
$\Vert\lambda_h(1,\cdot)\Vert_{L^2(0,T)}$	0.511	0.536	0.55	0.559	0.565
$\Vert v-\lambda_h(1,\cdot)\Vert_{L^2(0,T)}$	7.2×10^{-2}	5.06×10^{-2}	3.59×10^{-2}	2.55×10^{-2}	1.8×10^{-2}
$\Vert M(w_h,\mathbf{q}_h)\Vert_{L^2(Q_T)}$	0.474	0.363	0.29	0.238	0.199

TABLE 10. $r = h$ - non uniform mesh - stabilized formulation with $\alpha = h^2$.

9. Conclusion - Perspective

We have introduced a space-time variational formulation which allows to characterize null controls for the wave equation posed in $\mathbb{R}^n$, restate as a first order system of $n+1$ variables. A generalized observability inequality holds true for this system, leading to the well-posedness of the mixed formulation. Appropriate conformal finite dimensional discretization based on H^1 approximation both in time and space provides a convergent approximation of the solution and therefore allows to construct a sequence $\lambda_h(1,\cdot)$ strongly convergent in $L^2(0,T)$ toward the null control of minimal L^2 norm. An example of appropriate conformal approximation is obtained by considering piecewise $\mathbb{P}_2$ finite element spaces for the primal variables and piecewise $\mathbb{P}_1$ finite elements spaces for the dual (or lagrange multipliers) variables. Moreover, a stabilized procedure *à la Barbosa-Hugues* allows to consider any conformal approximation, for instance the one based on piecewise $\mathbb{P}_1$ finite elements, both for the primal and dual variables. The numerical experiments based on a very stiff example for which the initial condition to be controlled is discontinuous support the analysis and show the robustness of the approach.

Much of the methods presented here could be applied to the numerical controllability of more general wave equations. For instance, we must mention that wave equations of the form $u_{tt} - div(A(x)\nabla u) = f$, can be handled in an analogous way as long as its corresponding observability observability is available. In that case the natural choice for the variables $(v, \mathbf{p})$ in order to rewrite the equation as a first order system would be $v = u_t$ and $\mathbf{p} = A\nabla u$; in that case, we would have the following equivalent system $v_t - div\,\mathbf{p} = f, \mathbf{p}_t - A\nabla v = 0$. In this direction, we mention the elasticity system for which an observability holds true (see [20] chapter 3) if n controls are introduced and for which first order formulations are available (see for instance [4]). Another direction in which we could generalize this work is in considering a wave equation with a zero order term $u_{tt} - \Delta u + qu = 0$ leading to the equivalent first order system

$$v_t - div\,\mathbf{p} + q\int_0^t v(\cdot,\tau)d\tau = q\,u_0, \quad \mathbf{p}_t - \nabla v = 0,$$

involving a non local term. Eventually, we mention that this kind of methodology based on conformal H^1 approximation may be employed to address inverse problems as done in [22] in a parabolic context, following [10].

10. Appendix - Semigroup generation

Let $\mathcal{A}$ be the closed and densely defined operator $\mathcal{A} : D(\mathcal{A}) \subseteq H \to H$ with $D(\mathcal{A}) = V$ given by $\mathcal{A}(v, \mathbf{p}) = (div\, \mathbf{p}, \nabla v)$.

Lemma 9. *$\mathcal{A}$ and $-\mathcal{A}$ generate strongly continuous semigroups of contractions.*

Proof. We follow the proof in [6, Section 7.4.3]. By the Hille-Yosida theorem we only need to check that $(0, +\infty)$ is contained in the resolvent set $\rho(\mathcal{A})$ and $\|R_\lambda\| \le \frac{1}{\lambda}$ for $\lambda > 0$, where $R_\lambda = (\lambda - \mathcal{A})^{-1}$.

i) Let $\lambda > 0$, then for any $G \in H$ we must show that there exists a unique $U \in D(\mathcal{A})$ such that $\lambda U - \mathcal{A}U = G$; this amounts to show that for any $\lambda > 0$ and $(f, \mathbf{F}) \in H$ there are $(v, \mathbf{p}) \in D(\mathcal{A})$ such that

$$\lambda v - div\, \mathbf{p} = f, \quad \lambda \mathbf{p} - \nabla v = \mathbf{F}. \tag{10.1}$$

If we can uniquely solve (10.1), then clearly $R_\lambda(f, \mathbf{F}) = (v, \mathbf{p})$.

We rewrite the second equation in (10.1) as $\mathbf{p} = \lambda^{-1}(\nabla v + \mathbf{F})$ and we plug it into the first equation to get

$$\lambda v - \lambda^{-1}\Delta v = f + \lambda^{-1} div\, \mathbf{F}. \tag{10.2}$$

Since $\lambda > 0$, there exists a unique weak solution $v \in H_0^1(\Omega)$ to (10.2). From the second equation in (10.1) we recover $\mathbf{p} = \lambda^{-1}(\nabla v + \mathbf{F})$.

Observe that (10.2) means that

$$\int_\Omega \lambda v \varphi \, dx + \lambda^{-1} \int_\Omega \nabla v \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx - \lambda^{-1} \mathbf{F} \cdot \nabla \varphi \, dx \tag{10.3}$$

holds for any $\varphi \in H_0^1(\Omega)$.

If we multiply the second equation in (10.1) by $\nabla\varphi$ and integrate in Ω, then the resulting expression together with (10.3) implies that

$$\int_\Omega \lambda v \varphi + \mathbf{p} \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \quad \text{for any } \varphi \in H_0^1(\Omega).$$

This means that $div\, \mathbf{p} = \lambda v - f \in L^2(\Omega)$, therefore $\mathbf{p} \in H(div)$ and the first equation in (10.1) holds in $L^2(\Omega)$. Thus, we have shown that $(0, +\infty) \subseteq \rho(\mathcal{A})$.

ii) In order to show that $\|R_\lambda\| \le \lambda^{-1}$, we multiply the equations in (10.1) by v and $\mathbf{p}$ respectively, we integrate in Ω and sum the two expressions to get

$$\lambda \int_\Omega v^2 + |\mathbf{p}|^2 \, dx - \int_\Omega div\, \mathbf{p} v + \nabla v \cdot \mathbf{p} \, dx = \int_\Omega f v + \mathbf{F} \cdot \mathbf{p} \, dx.$$

Integration by parts gives

$$\lambda \int_\Omega v^2 + |\mathbf{p}|^2 \, dx = \int_\Omega f v + \mathbf{F} \cdot \mathbf{p} \, dx,$$

and the Cauchy inequality finally yields

$$\lambda \|(v, \mathbf{p})\|_H \le \|(f, \mathbf{F})\|_H$$

which implies the uniqueness of $(v, \mathbf{p})$ and the desired estimate for R_λ.

The structure of the operator $\mathcal{A}$ allows to repeat the same proof for $-\mathcal{A}$. $\square$

11. Appendix - FreeFem++ code

In this Section we list a code written in FreeFem++[17], mainly in order to emphasize that the corresponding implementation of the augmented-stabilized mixed formulation (5.1) is very simple and short. We use the value $r = 10^{-2}$ in $Q_T = (0, L) \times (0, T)$ with $L = 1$ and $T = 2$ and initial data (8.4). It also uses the $W_h - M_h$ pair of finite elements defined at the beginning of Section 7 in a uniform mesh (see left mesh in Figure 1) with $N = 10$.

```
real L = 1; // Size of the spatial domain
int N = 10; // Fineness of the mesh
real T = 2; // Final time
mesh Th = square(N,2*N,[L*x,T*y]); // uniform space-time mesh
fespace Wh(Th,P2); // P2 Finite Element Space for the Dual variables
fespace Mh(Th,P1); // P1 Finite Element Space for the Primal variables
Wh w1, q1; //Declaration of the Dual variables (solution functions)
Wh w2, q2; //Declaration of the Dual variables (test functions)
Mh lambda1, mu1; //Declaration of the Primal variables (solution
    functions)
Mh lambda2, mu2; //Declaration of the Primal variables (test functions
    )
func u0 = 4*x*(x>=0 && x<0.5); //Initial data to be controlled
real r = 10e-2; // Augmentation parameter 'r'
real alpha = 10e-2; //  Stabilized parameter 'alpha'

// Definition of the stabilized mixed variational formulation
problem Stabilized([w1,q1,lambda1,mu1],[w2,q2,lambda2,mu2])=
        // Initial conjugate functional terms
        int1d(Th,2)(q1*q2)-int1d(Th,1)(u0*w2)

        // primal-dual bilinear terms
        +int2d(Th)((dy(w2)-dx(q2))*lambda1+(dy(q2)-dx(w2))*mu1
                    +(dy(w1)-dx(q1))*lambda2+(dy(q1)-dx(w1))*mu2)

        // augmentation terms
        +int2d(Th)(r*((dy(w1)-dx(q1))*(dy(w2)-dx(q2))
                +(dy(q1)-dx(w1))*(dy(q2)-dx(w2)))

        //stabilized terms
        -int1d(Th,2)(alpha*q1*q2+alpha*(q2*lambda1+q1*lambda2))
        -int1d(Th,2,4)(alpha*lambda1*lambda2)
        -int2d(Th)(alpha*((dy(lambda1)-dx(mu1))*(dy(lambda2)-dx(mu2))
                +(dy(mu1)-dx(lambda1))*(dy(mu2)-dx(lambda2)))

        //boundary conditions
        +on(2,w1=0)+on(4,w1=0) + on(4,lambda1=0);

// The following instruction solves the mixed formulation
Stabilized;
// The solution for the dual variables are stored in (w1,q1)
// The solution for the primal variables are stored in (lambda1,mu1)
```

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(Santiago Montaner) Laboratoire de Mathématiques, Université Clermont Auvergne, UMR CNRS 6620, Campus des Cézeaux, 63177 Aubière, France.

E-mail address: santiago.montaner@uca.fr

(Arnaud Münch) Laboratoire de Mathématiques, Université Clermont Auvergne, UMR CNRS 6620, Campus des Cézeaux, 63177 Aubière, France.

E-mail address: arnaud.munch@uca.fr