Approximation of controllability and inverse problems for PDE

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PART 3
PARABOLIC CASE
\begin{align*}
\Omega \subset \mathbb{R}^N; \quad Q_T = \Omega \times (0, T); \quad q_T = \omega \times (0, T) \\
\begin{cases}
y_t - \nabla \cdot (c(x)\nabla y) + d(x, t)y = v 1_\omega, & \text{in } Q_T, \\
y = 0, & \text{in } \Sigma_T, \\
y(x, 0) = y_0(x), & \text{in } \Omega.
\end{cases}
\end{align*}

(1)

c := (c_{i,j}) \in C^1(\Omega; \mathcal{M}_N(\mathbb{R})); \quad (c(x)\xi, \xi) \geq c_0|x|^2 \text{ in } \overline{\Omega} \ (c_0 > 0),

d \in L^\infty(Q_T), \quad y_0 \in L^2(\Omega);

v = v(x, t) \text{ is the control} \quad y = y(x, t) \text{ is the associated state.}

We introduce the linear manifold

\[ C(y_0, T) = \{ (y, v) : v \in L^2(q_T), \ y \text{ solves } (1) \text{ and satisfies } y(T, \cdot) = 0 \}. \]

non empty (see Furikov-Imanuvilov’96, Robbiano-Lebeau’95).

notations - 
Ly := y_t - \nabla \cdot (c(x)\nabla y) + d(x, t)y; \quad L^*\varphi := -\varphi_t - \nabla \cdot (c(x)\nabla \varphi) + d(x, t)\varphi
Introduction

\[ \Omega \subset \mathbb{R}^N; \, Q_T = \Omega \times (0, T); \, q_T = \omega \times (0, T) \]

\[
\begin{cases}
    y_t - \nabla \cdot (c(x)\nabla y) + d(x, t)y = v \, 1_\omega, & \text{in } Q_T, \\
    y = 0, & \text{in } \Sigma_T, \\
    y(x, 0) = y_0(x), & \text{in } \Omega.
\end{cases}
\]

(1)

\[ c := (c_{i,j}) \in C^1(\overline{\Omega}; \mathcal{M}_N(\mathbb{R})); \, (c(x)\xi, \xi) \geq c_0 |\xi|^2 \text{ in } \overline{\Omega} \, (c_0 > 0), \]

\[ d \in L^\infty(Q_T), \, y_0 \in L^2(\Omega); \]

\[ v = v(x, t) \text{ is the control} \quad y = y(x, t) \text{ is the associated state.} \]

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non empty (see FURSIKOV-IMANUVILOV’96, ROBBIANO-LEBEAU’95).

**Notations**

\[ L^*\varphi := -\varphi_t - \nabla \cdot (c(x)\nabla \varphi) + d(x, t)\varphi \]
PART I
Control of minimal $L^2(q_T)$-norm

$$\begin{align*}
(P) \quad \inf_{(y,v) \in C(y_0,T)} J(v, y) &= \frac{1}{2} \|v\|_{L^2(q_T)}^2
\end{align*}$$
$N = 1 - L^2(q_T)$-norm of the HUM control with respect to time

Figure: $y_0(x) = \sin(\pi x) - T = 1 - \omega = (0.2, 0.8) - t \rightarrow \|v(\cdot,t)\|_{L^2(0,1)}$ in $[0, T]$
$N = 1 - L^2$-norm of the HUM control with respect to time: Zoom near $T$

Figure: $y_0(x) = \sin(\pi x) - T = 1 - \omega = (0.2, 0.8) - t \rightarrow \|v(\cdot, t)\|_{L^2(0,1)}$ in $[0.92T, T]$
Minimal $L^2$ norm control

Since it is difficult to construct pairs $(\nu, y) \in C(y_0, T)$ (a fortiori minimizing sequences for $J$!), it is standard to consider the corresponding dual:

$$\inf_{(y, \nu) \in C(y_0, T)} J(y, \nu) = - \inf_{\phi_T \in H} J^*(\phi_T), \quad J^*(\phi_T) = \frac{1}{2} \int_{Q_T} \phi^2 \, dx \, dt + \int_{\Omega} \phi(0, \cdot) y_0 \, dx$$

where $\phi$ solves the backward system

$$\begin{cases}
L^* \phi = 0 & Q_T = (0, T) \times \Omega, \\
\phi = 0 & \Sigma_T = (0, T) \times \partial \Omega, \quad \phi(T, \cdot) = \phi_T \quad \Omega.
\end{cases}$$

The Hilbert space $H$ is defined as the completion of $D(\Omega)$ with respect to the norm

$$\| \phi_T \|_H = \left( \int_{Q_T} \phi^2(t, x) \, dx \, dt \right)^{1/2}.$$

From the observability inequality

$$C(T, \omega) \| \phi(0, \cdot) \|^2_{L^2(\Omega)} \leq \| \phi_T \|^2_H \quad \forall \phi_T \in L^2(\Omega),$$

$J^*$ is coercive on $H$. The HUM control is given by $\nu = \phi \chi_\omega$ on $Q_T$. 
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Since it is difficult to construct pairs $(v, y) \in C(y_0, T)$ \textit{(a fortiori} minimizing sequences for $J^*$), it is standard to consider the corresponding dual:

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\inf_{(y, v) \in C(y_0, T)} J(y, v) = - \inf_{\phi_T \in H} J^*(\phi_T), \quad J^*(\phi_T) = \frac{1}{2} \int_{q_T} \phi^2 \, dx \, dt + \int_{\Omega} \phi(0, \cdot) y_0 \, dx
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\end{cases}
$$

The Hilbert space $H$ is defined as the completion of $\mathcal{D}(\Omega)$ with respect to the norm

$$
\|\phi_T\|_H = \left( \int_{q_T} \phi^2(t, x) \, dx \, dt \right)^{1/2}.
$$

From the observability inequality

$$
C(T, \omega) \|\phi(0, \cdot)\|_{L^2(\Omega)}^2 \leq \|\phi_T\|_H^2 \quad \forall \phi_T \in L^2(\Omega),
$$

$J^*$ is coercive on $H$. The HUM control is given by $v = \phi \tilde{x}_\omega$ on $Q_T$. 
Minimal $L^2$ norm control

Since it is difficult to construct pairs $(v, y) \in C(y_0, T)$ (a fortiori minimizing sequences for $J$!), it is standard to consider the corresponding dual:

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The Hilbert space $H$ is defined as the completion of $D(\Omega)$ with respect to the norm

$$\|\phi_T\|_H = \left( \int_{q_T} \phi^2(t, x) dx dt \right)^{1/2}.$$ 

From the observability inequality

$$C(T, \omega)\|\phi(0, \cdot)\|_{L^2(\Omega)}^2 \leq \|\phi_T\|_H^2 \quad \forall \phi_T \in L^2(\Omega),$$

$J^*$ is coercive on $H$. The HUM control is given by $v = \phi \chi_\omega$ on $Q_T$. 
Ill-posedness

- The completed space $H$ is huge:

$$H^{-s} \subset H \quad \forall s > 0!$$

($H$ may also contain elements which are not distribution !!):
Micu \(^{1}\) proved in 1D that

the set of initial data $y_0$, for which the corresponding $\phi_T$, minimizer of $J^*$, does not belong to any negative Sobolev spaces, is dense in $L^2(0, 1)$ !!!

- The dual variable $\phi_T$ is the Lagrange multiplier for the constraint $y(\cdot, T) = 0$ may belong to a "large" dual space, much larger than $L^2(\Omega)$:

$$< y(\cdot, T), \phi_T >= 0$$

- Ill-posedness here is therefore related to the hugeness of $H$, poorly approximated numerically.

- This phenomenon is unavoidable (unless $\omega = \Omega !$) and is independent of the choice of the norm !

\(^{1}\)S. Micu, *Regularity issues for the null-controllability of the linear 1-d heat equation*, 2011
Optimal backward solution $\phi$ on $\partial \omega \times [0, T]$

$$T = 1, \quad y_0(x) = \sin(\pi x), \quad a(x) = a_0 = 1/10, \quad \omega = (0.2, 0.8)$$

Figure: $T = 1 - \omega = (0.2, 0.8) - \phi^N(\cdot, 0.8)$ for $N = 80$ on $[0, T]$ (Left) and on $[0.92 T, T]$ (Right).

[Carthel-Glowinski-Lions’94, JOTA], [AM-Zuazua’11, Inverse Problems]
Regularization

For any $\epsilon > 0$, consider $J_\epsilon(y, v) = J(y, v) + \frac{\epsilon^{-1}}{2} \|y(T)\|^2_{H^{-s}(0,1)}$ and

$$\inf_{\phi_T, \epsilon \in L^2(0,1)} J_\epsilon^*(\phi_T, \epsilon), \quad J_\epsilon^*(\phi_T, \epsilon) = \frac{1}{2} \int_{q_T} \phi^2 \, dx \, dt + \int_{\Omega} \phi(0, \cdot) y_0 \, dx + \frac{\epsilon}{2} \|\phi_T, \epsilon\|_{H^s(0,1)}^2$$

and minimize in $L^2$ the quadratic and strictly convex function $J_\epsilon^*$ by a conjugate gradient algorithm as initially proposed in Carthel-Glowinski-Lions’94\(^2\).

$$\phi_T(x) = \sum_{k \geq 1} a_k \sin(k\pi x) \iff y_T(x) = \sum_{p \geq 1} b_p \sin(p\pi x), \quad x \in \Omega$$

and taking $y_0 = 0$ (for simplicity), we obtain the relation

$$b_p = \sum_{k \geq 1} \left(c_{p,k}(\omega) g_{p,k}(T) + \epsilon(k\pi)^{2s} \delta_{p,k}\right) a_{k,\epsilon}, \quad s = 0, 1.$$

$$c_{p,k}(\omega) = 2 \int_{\omega} \sin(k\pi x) \sin(p\pi x) \, dx, \quad g_{p,k}(T) = \frac{1 - e^{-c(\lambda_p + \lambda_k)T}}{\lambda_k + \lambda_p}, \quad \lambda_k = (k\pi)^2$$


Regularization

For any $\epsilon > 0$, consider $J_\epsilon(y,v) = J(y,v) + \frac{\epsilon^{-1}}{2} \| y(T) \|_{H^{-s}(0,1)}^2$ and

$$\inf_{\phi_T, \epsilon \in L^2(0,1)} J_\epsilon^*(\phi_T, \epsilon), \quad J_\epsilon^*(\phi_T, \epsilon) = \frac{1}{2} \int_{q_T} \phi^2 dx dt + \int_{\Omega} \phi(0, \cdot) y_0 dx + \frac{\epsilon}{2} \|\phi_T, \epsilon\|_{H^s(0,1)}^2$$

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$$b_p = \sum_{k \geq 1} \left( c_{p,k}(\omega) g_{p,k}(T) + \epsilon (k \pi)^{2s} \delta_{p,k} \right) a_{k, \epsilon}, \quad s = 0, 1.$$

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Lack of uniform observability vs. ill-posedness

Discrete observability inequalities: Behavior of $C_{obs,h}$ w.r.t. $h$?

$$\|\phi_h(0)\|^2_{L^2(\Omega)} \leq C_{obs,h} \int_0^T \int_\omega \phi_h^2(t,x)dxdt, \quad \forall \phi_{Th} \in L^2(\Omega)$$

Labbe-Trelat\(^5\), Boyer\(^6\) analyzed the case $\epsilon = h^\alpha$ and proposes some schemes leading to

$$\|\phi_h(0)\|^2_{L^2(\Omega)} \leq C_{obs} \left( \int_0^T \int_\omega \phi_h^2(t,x)dxdt + \epsilon \|\phi_{Th}\|^2_{L^2(\Omega)} \right), \quad \forall \phi_{Th} \in L^2(\Omega)$$

Remark- The condition number of the discrete HUM operator: $\Lambda_h: \phi_{Th} \rightarrow y_{Th}$ is estimated by

$$\text{cond}(\Lambda_h) \leq C_{obs,h} C_2 h^{-2}\quad$$

where

$$\int_0^T \int_\omega \phi_h^2(t,x)dxdt \leq C_2 h \|\phi_h(0)\|^2_{L^2(\Omega)}, \quad \forall \phi_{Th} \in L^2(\Omega)$$

We have that

$$C_2 h \rightarrow \infty \quad h \rightarrow 0$$


\(^6\)F. Boyer, On the penalized HUM approach and its applications to the numerical approximation of null-controls for parabolic problems (2013)
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Discrete observability inequalities: Behavior of $C_{obs,h}$ w.r.t. $h$?

$$\|\phi_h(0)\|_{L^2(\Omega)}^2 \leq C_{obs,h} \int_0^T \int_{\omega} \phi_h^2(t,x)\,dx\,dt, \quad \forall \phi_{Th} \in L^2(\Omega)$$

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We have that

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\textsuperscript{5}S. Labbe, E. Trelat, Uniform controllability of semidiscrete approximations of parabolic control systems. (2006)

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Discrete observability inequalities: Behavior of $C_{obs,h}$ w.r.t. $h$?

\[ \| \phi_h(0) \|_{L^2(\Omega)}^2 \leq C_{obs,h} \int_0^T \int_\omega \phi_h^2(t,x) \, dx \, dt, \quad \forall \phi_h \in L^2(\Omega) \]

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**Remark**- The condition number of the discrete HUM operator: $\Lambda_h : \phi_h \rightarrow y_h$ is estimated by

\[ \text{cond}(\Lambda_h) \leq C_{obs,h} C_{2h} h^{-2} \]

where

\[ \int_0^T \int_\omega \phi_h^2(t,x) \, dx \, dt \leq C_{2h} \| \phi_h(0) \|_{L^2(\Omega)}^2, \quad \forall \phi_h \in L^2(\Omega) \]

We have that

\[ C_{2h} \rightarrow \infty \quad h \rightarrow 0 \]

\textsuperscript{5}S.Labbe, E. Trelat, Uniform controllability of semidiscrete approximations of parabolic control systems. (2006)

\textsuperscript{6}F. Boyer, On the penalized HUM approach and its applications to the numerical approximation of null-controls for parabolic problems (2013)
Regular and singular perturbation of the controllability problem

Other regularization / perturbation are considered in [AM-Zuazua’10] 7

1- Replace the heat equation by the hyperbolic equation

\[ y_{\epsilon,t} - c\Delta y_{\epsilon} + \epsilon y_{\epsilon,tt} = v_{\epsilon} 1_\omega, \quad \text{in} \quad Q_T, \]

2- Singular (non uniformly controllable w.r.t. \( \epsilon \)) perturbation

\[ y_{\epsilon,t} - c\Delta y_{\epsilon} - \epsilon \partial_t \Delta y_{\epsilon} = v_{\epsilon} 1_\omega \quad \text{in} \quad Q_T. \]

Question:

\[ \frac{\partial^\alpha y_\alpha}{\partial t^\alpha} - \Delta y_\alpha = v_\alpha 1_\omega \quad \text{in} \quad Q_T, \quad \alpha < 1 ??? \]

PART II
Transmutation method ⁸

⁸L. Miller, The control transmutation method and the cost of fast controls 2006
The control transmutation method

Let $L > 0$ and $y_0 \in H^1_0(\Omega)$. IF $f \in L^2([0, L] \times \omega)$ is a null-control for $w$, solution of the wave equation

\[
\begin{cases}
  w_{ss} - w_{xx} = f & (s, x) \in (0, L) \times \Omega, \\
  w = 0 & (0, L) \times \partial\Omega,
\end{cases}
\]

AND if $H \in C^0([0, T], \mathcal{M}(\mathbb{R}^m - L, L])$ is a fundamental controlled solution for the heat equation

\[
\begin{cases}
  \partial_t H - \partial_s^2 H = 0 & \text{in } \mathcal{D}'([0, T[ \times ] - L, L[), \\
  H(t = 0) = \delta, \quad H(t = T) = 0
\end{cases}
\]

THEN the function

\[
v(t, x) = 2 \int_0^L H(t, s)f(s, x)ds 1_\omega(x), \quad (0, T) \times \Omega
\]

is a null control in $L^2(q_T)$ for $y(t, x) = 2 \int_0^L H(t, s)w(s, x)ds$ solution of the heat equation

\[
\begin{cases}
  y_t - y_{xx} = v & (0, T) \times \Omega, \\
  y = 0 & (0, T) \times \partial\Omega, \\
  y(0) = y_0
\end{cases}
\]
The control transmutation method

Let \( L > 0 \) and \( y_0 \in H^1_0(\Omega) \). **IF** \( f \in L^2([0, L] \times \omega) \) is a null-control for \( w \), solution of the wave equation

\[
\begin{cases}
    w_{ss} - w_{xx} = f 1_\omega & (s, x) \in (0, L) \times \Omega, \\
    w = 0 & (0, L) \times \partial\Omega, \\
    (w(0), w_s(0)) = (y_0, 0) \implies (w(L), w_s(L)) = (0, 0)
\end{cases}
\]

**AND** if \( H \in C^0([0, T], \mathcal{M}(-L, L]) \) is a fundamental controlled solution for the heat equation

\[
\begin{cases}
    \partial_t H - \partial_s^2 H = 0 & \text{in } D'(]0, T[ \times -L, L[), \\
    H(t = 0) = \delta, \quad H(t = T) = 0
\end{cases}
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**THEN** the function

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Computation of the fundamental solution for the heat equation

Jones \(^9\), Rouchon \(^{10}\). Let \( \delta \in (0, T) \). For \( t \in (0, \delta) \), \( H \) is taken as the Gaussian:

\[
H(t, s) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{s^2}{4t}}, \quad (t, s) \in (0, \delta) \times \mathbb{R}.
\]

so that it remains to join \( H(\delta, s) \) to 0 at time \( T \). For any \( a > 0 \) and any \( \alpha \geq 1 \), we consider the bump function

\[
h(n) = \exp\left(-\frac{a}{((n - \delta)(T - n))^\alpha}\right), \quad n \in (\delta, T)
\]

and then the function

\[
p(t) = \frac{1}{\sqrt{4\pi t}} \begin{cases} 1, & t \in (0, \delta) \\ \int_\delta^T h(n)dn \int_{\delta}^{T} h(n)dn & t \in (\delta, T) \end{cases}
\]

so that \( p(T) = 0 \). \( h \in C^\infty([\delta, T]) \) and \( p \in C^\infty([0, T]) \). \( h \) and \( p \) are both Gevrey functions of order \( 1 + 1/\alpha \in (1, 2] \) so that the serie

\[
H(t, s) = \sum_{k \geq 0} p^{(k)}(t) \frac{s^{2k}}{(2k)!}
\]

is convergent. (2) defines a solution of the heat equation and satisfies \( H(T, s) = 0 \) for all \( s \in \mathbb{R} \) and

\[
\lim_{t \to 0^+} H(t, s) = \delta_{s=0}.
\]

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\(^9\) B. Jones, A fundamental solution for the heat equation which is supported in a strip, J. Math. Anal. Appl. 1977

\(^{10}\) B. Laroche, P. Martin and P. Rouchon, Motion planning for the heat equation, Int. Journal of Robust and Nonlinear Control, (2000)
Computation of the fundamental solution for the heat equation

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so that \(p(T) = 0\). \(h \in C_{c}^{\infty}([\delta, T])\) and \(p \in C^{\infty}([0, T])\). \(h\) and \(p\) are both Gevrey functions of order \(1 + 1/\alpha \in (1, 2]\) so that the serie

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Computation of the fundamental solution for the heat equation

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so that it remains to join $H(\delta, s)$ to 0 at time $T$. For any $a > 0$ and any $\alpha \geq 1$, we consider the bump function

$$h(n) = \exp \left( - \frac{a}{((n - \delta)(T - n))^\alpha} \right), \quad n \in (\delta, T)$$ 

and then the function

$$p(t) = \frac{1}{\sqrt{4\pi t}} \begin{cases} 1 & t \in (0, \delta) \\ \frac{\int_t^T h(n)dn}{\int_\delta^T h(n)dn} & t \in (\delta, T) \end{cases}$$ 

so that $p(T) = 0$. $h \in C^\infty_c([\delta, T])$ and $p \in C^\infty ([0, T])$. $h$ and $p$ are both Gevrey functions of order $1 + 1/\alpha \in (1, 2]$ so that the serie

$$H(t, s) = \sum_{k \geq 0} p^{(k)}(t) \frac{s^{2k}}{(2k)!}$$

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$$\lim_{t \to 0^+} H(t, s) = \delta_{s=0}.$$

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Fundamental solution for the heat equation: example

\[ a_0 = 1 \] by the change of variable \((\tilde{x}, \tilde{t}) = (a_0 t, x)\)

**Figure**: \(L = 0.5 - T = 0.1 - (a, \alpha, \delta) = (10^{-2}, 1, T/5)\) - **Left**: fundamental solution \(H\) on \((0, T) \times (0, L)\) - **Right**: \(H(t, L)\) vs. \(t \in (0, T)\).
Fundamental solution for the heat equation: example

Figure: $L = 0.5 - T = 0.1 - (a, \alpha, \delta) = (10^{-2}, 1, T/2)$ - Left: fundamental solution $H$ on $(0, T) \times (0, L)$ - Right: $H(t, L)$ vs. $t \in (0, T)$. 
Control by the transmutation method

Figure: $y_0(x) = \sin(\pi x)$, $L = 0.5$ - Controlled wave solution $w$ (Left) and corresponding HUM control $f$ (Right) on $(0, L) \times \Omega$.

Figure: $y_0(x) = \sin(\pi x)$, $T = 1$, $a_0 = 1/10$, $(\delta, \alpha) = (T/5, 1)$ - Controlled heat solution $y$ (Left) and corresponding transmuted control $v$ (Right) on $(0, T) \times \Omega$.
Control by the transmutation method

Figure: $L^2(\omega)$ norm of the control $v$ vs $t \in [0, T]$ for $(y_0(x), T, a_0) = (\sin(\pi x), 1, 1/10)$
Transmutation to HUM?

\[ \|v\|_{L^2(Q_T)} \leq 2\|f\|_{L^2((0,L) \times \omega)} \|H\|_{L^2((0,T) \times (0,L))} \]

\[ \|H\|_{L^2((0,T) \times (0,L))} \] is reduced if \( \delta \) is small (reduce the time period where the dissipation is governed by the gaussian), and \( \alpha_1 > 1 \) (allows to take \( \delta \) small) and \( \alpha_2 < 1 \) (increase the magnitude of the control near \( T \)).

\[ h(s) = \exp \left( -\frac{a}{(s-\delta)^{\alpha_1} (T-s)^{\alpha_2}} \right) \]

Figure: \((y_0(x), a_0) = (\sin(\pi x), 1/10)\) - Heat fundamental solution \( H(t, L) \) vs. \( t \in [0, \tilde{T}] \) (Left) and \( L^2(\Omega) \)-norm of corresponding control \( v \) (Right).

\( \alpha_1 = 1.1, \alpha_2 = 0.7 \|g\|_{L^2(Q_T)} \approx 5.67 \times 10^{-1} \)
-The transmuted control $v_h = (v)_{h>0}$ ensures that $\|y_h(T, \cdot)\|_{L^2(\Omega)} \approx 10^{-5}$

-Once a solution $H$ in the one dimensional is constructed, we can take

$$H_n(t, x_1, x_2, \cdots, x_n) = H(t, x_1) \times H(t, x_2) \times \cdots \times H(t, x_n)$$

as a fundamental control solution for $(t, x) \in (0, T) \times [-L, L]^n$. Consequently, the transmutation provides also a control in any dimension, provided some geometric condition on the support $\omega$.

- The transmutation method provides uniformly bounded discrete control $\{v_h\}$ discretization of

$$v(t, x) = 2 \sum_{k \geq 0} p^{(k)}(t) \int_0^L s^{2k} f(s, x) ds \, 1_{\omega}(x)$$

- The main difficulty is the robust evaluation of $p^{(k)}$.

---

11 P. Martin, P. Rouchon, L. Rosier, Null controllability of one-dimensional parabolic equations by the flatness approach (2016)
PART III
Change of the norm : framework of Fursikov-Imanuvilov’96

\[
\begin{align*}
&\text{Minimize } J(y, v) = \frac{1}{2} \int \int_{Q_T} \rho^2 |y|^2 \, dx \, dt + \frac{1}{2} \int \int_{q_T} \rho_0^2 |v|^2 \, dx \, dt \\
&\text{Subject to } (y, v) \in C(y_0, T).
\end{align*}
\]

where \( \rho, \rho_0 \) are non-negative continuous weights functions such that \( \rho, \rho_0 \in L^\infty(Q_{T-\delta}) \forall \delta > 0. \)

Primal (direct) approach

Following Fursikov-Imanuvilov’96, we assume Carleman type weights:

\[
\rho(x, t) = \exp \left( \frac{\beta(x)}{T - t} \right), \quad \rho_0(x, t) = (T - t)^{3/2} \rho(x, t), \quad \beta(x) = K_1 \left( e^{K_2} - e^{\beta_0(x)} \right)
\]

where the \( K_i \) are sufficiently large positive constants (depending on \( T, c_0 \) and \( \|c\|_{C^1} \)) and \( \beta_0 \in C^\infty(\overline{\Omega}), \beta_0 > 0 \) in \( \Omega \), \( (\beta_0)|_{\partial \Omega} = 0 \), \( |\nabla \beta_0| > 0 \) outside \( \omega \).

We introduce

\[
P_0 = \{ q \in C^2(\overline{Q_T}) : q = 0 \ \text{on} \ \Sigma_T \}.
\]

In this linear space, the bilinear form

\[
(p, q)_P := \int \int_{Q_T} \rho^{-2} L^* p L^* q \, dx \, dt + \int \int_{\Omega_T} \rho_0^{-2} p q \, dx \, dt
\]

is a scalar product (unique continuation property).

Let \( P \) be the completion of \( P_0 \) for this scalar product.
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\end{align*}
\]

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P_0 = \{ q \in C^2(\overline{Q_T}) : q = 0 \text{ on } \Sigma_T \}.
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is a scalar product (unique continuation property).

Let \( P \) be the completion of \( P_0 \) for this scalar product.
Carleman estimates

Lemma (Fursikov-Imanuvilov'96)
Let $\rho$ and $\rho_0$ be given by (4) and let $(\rho_1, \rho_2) = ((T - t)^{1/2}, (T - t)^{-1/2}) \rho$. Then there exists $C > 0$, only depending on $\omega$, $T$, $c_0$ and $\|c\|_{C^1}$, such that

$$
\int\int_{Q_T} \left[ \rho_2^{-2} \left( |q_t|^2 + |\Delta q|^2 \right) + \rho_1^{-2} |\nabla q|^2 + \rho_0^{-2} |q|^2 \right] \, dx \, dt
\leq C \left( \int\int_{Q_T} \rho^{-2} |L^* q|^2 \, dx \, dt + \int\int_{Q_T} \rho_0^{-2} |q|^2 \, dx \, dt \right), \forall q \in P.
$$

(5)

Lemma (Fursikov-Imanuvilov'96, Fernández-Cara-Guerrero'06)
Under the same assumptions, for any $\delta > 0$, one has

$$
P \hookrightarrow C^0([0, T - \delta]; H^1_0(\Omega)),
$$

where the embedding is continuous. In particular, there exists $C > 0$, only depending on $\omega$, $T$, $a_0$ and $\|a\|_{C^1}$, such that, for all $q \in P$,

$$
\|q(\cdot, 0)\|_{H^1_0(\Omega)}^2 \leq C \left( \int\int_{Q_T} \rho^{-2} |L^* q|^2 \, dx \, dt + \int\int_{Q_T} \rho_0^{-2} |q|^2 \, dx \, dt \right).
$$

(6)
Carleman estimates

**Lemma (Fursikov-Imanuvilov’96)**

Let $\rho$ and $\rho_0$ be given by (4) and let $(\rho_1, \rho_2) = ((T - t)^{1/2}, (T - t)^{-1/2}) \rho$. Then there exists $C > 0$, only depending on $\omega$, $T$, $c_0$ and $\|c\|_{C^1}$, such that

$$\int_0^T \int_{Q_T} \left[ \rho_2^{-2} \left( |q_t|^2 + |\Delta q|^2 \right) + \rho_1^{-2} |\nabla q|^2 + \rho_0^{-2} |q|^2 \right] \, dx \, dt \leq C \left( \int_0^T \int_{Q_T} \rho^{-2} |L^* q|^2 \, dx \, dt + \int_0^T \int_{q_T} \rho_0^{-2} |q|^2 \, dx \, dt \right), \, \forall q \in P.$$  

(5)

**Lemma (Fursikov-Imanuvilov’96, Fernández-Cara-Guerrero’06)**

Under the same assumptions, for any $\delta > 0$, one has

$$P \hookrightarrow C^0([0, T - \delta]; H^1_0(\Omega)), $$

where the embedding is continuous. In particular, there exists $C > 0$, only depending on $\omega$, $T$, $a_0$ and $\|a\|_{C^1}$, such that, for all $q \in P$,

$$\|q(\cdot, 0)\|^2_{H^1_0(\Omega)} \leq C \left( \int_0^T \int_{Q_T} \rho^{-2} |L^* q|^2 \, dx \, dt + \int_0^T \int_{q_T} \rho_0^{-2} |q|^2 \, dx \, dt \right).$$  

(6)
Primal (direct) approach

Proposition

Let \( \rho \) and \( \rho_0 \) be given by (4). Let \((y, v)\) be the corresponding optimal pair for \( J \). Then there exists \( p \in P \) such that

\[
y = \rho^{-2}L^* p, \quad v = -\rho_0^{-2} p_{|q_T}.
\] (7)

The function \( p \) is the unique solution in \( P \) of

\[
\int\!\!\int_{Q_T} \rho^{-2}L^* p L^* q \, dx \, dt + \int\!\!\int_{q_T} \rho_0^{-2} p q \, dx \, dt = \int_0^1 y_0(x) q(x, 0) \, dx, \quad \forall q \in P
\] (8)

Remark

\( p \) solves, at least in \( \mathcal{D}' \), the following differential problem, that is second order in time and fourth order in space:

\[
\begin{cases}
L(\rho^{-2}L^* p) + \rho_0^{-2} p 1_\omega = 0, & (x, t) \in (0, 1) \times (0, T) \\
p(x, t) = 0, \quad (-\rho^{-2}L^* p)(x, t) = 0 & (x, t) \in \{0, 1\} \times (0, T) \\
(-\rho^{-2}L^* p)(x, 0) = y_0(x), \quad (-\rho^{-2}L^* p)(x, T) = 0, & x \in (0, 1).
\end{cases}
\] (9)
Primal (direct) approach

Proposition
Let $\rho$ and $\rho_0$ be given by (4). Let $(y, v)$ be the corresponding optimal pair for $J$. Then there exists $p \in P$ such that

$$y = \rho^{-2} L^* p, \quad v = -\rho_0^{-2} p|_{q_T}. \quad (7)$$

The function $p$ is the unique solution in $P$ of

$$\int \int_{Q_T} \rho^{-2} L^* p L^* q \, dx \, dt + \int \int_{q_T} \rho_0^{-2} p q \, dx \, dt = \int_0^1 y_0(x) q(x, 0) \, dx, \quad \forall q \in P \quad (8)$$

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(-\rho^{-2} L^* p)(x, 0) = y_0(x), \quad (-\rho^{-2} L^* p)(x, T) = 0, & x \in (0, 1).
\end{cases} \quad (9)$$
Conformal approximation

For large integers $N_x$ and $N_t$, we set $\Delta x = 1/N_x$, $\Delta t = T/N_t$ and $h = (\Delta x, \Delta t)$. Let us introduce the associated uniform triangulation $\mathcal{T}_h$, with

$$Q_T = \bigcup_{K \in \mathcal{T}_h} K.$$ 

The following (conformal) finite element approximations of the space $P$ are introduced:

$$P_h = \{ q_h \in C^{1,0}_{x,t}(Q_T) : q_h|_K \in (\mathbb{P}_3, x) \otimes (\mathbb{P}_1, t)(K) \quad \forall K \in \mathcal{T}_h, \quad q_h|_{\Sigma_T} \equiv 0 \},$$

where $C^{1,0}_{x,t}(Q_T)$ is the space of the functions in $C^0(Q_T)$ that are continuously differentiable with respect to $x$ in $Q_T$.

The variational equality (28) is approximated as follows:

$$\begin{cases}
\iint_{Q_T} \rho^{-2} L^* \rho L^* q_h \, dx \, dt + \iiint_{Q_T} \rho_0^{-2} p_h \, q_h \, dx \, dt = \int_0^1 y_0(x) \, q_h(x, 0) \, dx \\
\forall q_h \in P_h; \quad p_h \in P_h.
\end{cases}$$

(10)
Conformal approximation

For large integers $N_x$ and $N_t$, we set $\Delta x = 1/N_x$, $\Delta t = T/N_t$ and $h = (\Delta x, \Delta t)$. Let us introduce the associated uniform triangulation $\mathcal{T}_h$, with

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$$
\begin{align*}
\left\{ \int_{Q_T} \rho^{-2} L^* p_h L^* q_h \, dx \, dt + \int_{Q_T} \rho_0^{-2} p_h q_h \, dx \, dt = \int_{0}^{1} y_0(x) \, q_h(x,0) \, dx \right. \\
\forall q_h \in P_h; \quad p_h \in P_h.
\end{align*}
$$

(10)
Conformal approximation

**Theorem (Fernandez-Cara, AM)**

Let \( p_h \in P_h \) be the unique solution to (11). Let us set

\[
y_h := \rho^{-2}L^* p_h, \quad v_h := -\rho_0^{-2} p_h 1_{qT}.
\]

Then one has

\[
\|y - y_h\|_{L^2(Q_T)} \to 0 \quad \text{and} \quad \|v - v_h\|_{L^2(q_T)} \to 0, \quad \text{as} \quad h \to 0
\]

where \((y, v)\) is the minimizer of \(J\).

In practice, we introduce the variable

\[
m_h := \rho_0^{-1} p_h \in \rho_0^{-1} P_h \subset \rho_0^{-1} P \subset C([0, T], H^1_0(\Omega)) \quad \text{and we solve}
\]

\[
\int\int_{Q_T} \rho^{-2}L^*(\rho_0 m_h) L^*(\rho_0 \overline{m_h}) \, dx \, dt + \int\int_{q_T} m_h \overline{m_h} \, dx \, dt = \int_{0}^{1} y_0 \rho_0(\cdot, 0) \overline{m_h}(\cdot, 0) \, dx
\]

\[
\forall m_h \in \rho_0^{-1} P_h; \quad \overline{m_h} \in \rho_0^{-1} P_h.
\]

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\]

and we solve

\[
\begin{aligned}
\int_Q \rho^{-2} L^*(\rho_0 m_h) L^*(\rho_0 \overline{m_h}) \, dx \, dt + \int_{q_T} m_h \overline{m_h} \, dx \, dt &= \int_0^1 \rho_0(\cdot, 0) \overline{m_h}(\cdot, 0) \, dx \\
\forall m_h &\in \rho_0^{-1} P_h; \quad \overline{m_h} \in \rho_0^{-1} P_h.
\end{aligned}
\]

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In practice, we introduce the variable

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m_h := \rho_0^{-1} p_h \in \rho_0^{-1} P_h \subset \rho_0^{-1} P \subset C([0, T], H^1_0(\Omega))
\]

and we solve

\[
\begin{aligned}
\int_0^T \int_{Q_T} \rho^{-2} L^*(\rho_0 m_h) L^*(\rho_0 \overline{m_h}) \, dx \, dt + \int_0^T \int_{q_T} m_h \, \overline{m_h} \, dx \, dt &= \int_0^1 y_0 \rho_0(\cdot, 0) \overline{m_h} (\cdot, 0) \, dx \\
\forall m_h \in \rho_0^{-1} P_h; \quad \overline{m_h} \in \rho_0^{-1} P_h.
\end{aligned}
\]
Experiment with $\omega = (0.2, 0.8)$

<table>
<thead>
<tr>
<th>$\Delta x = \Delta t$</th>
<th>1/20</th>
<th>1/40</th>
<th>1/80</th>
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<tr>
<td>conditioning</td>
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<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>$|p_h(\cdot, T)|_{L^2(0,1)}$</td>
<td>$1.33 \times 10^{14}$</td>
<td>$1.76 \times 10^{22}$</td>
<td>$7.86 \times 10^{32}$</td>
<td>$2.17 \times 10^{44}$</td>
<td>$2.30 \times 10^{54}$</td>
</tr>
<tr>
<td>$|y_h(\cdot, T)|_{L^2(0,1)}$</td>
<td>$2.85 \times 10^{1}$</td>
<td>$2.04 \times 10^{2}$</td>
<td>$1.59 \times 10^{3}$</td>
<td>$4.70 \times 10^{4}$</td>
<td>$6.12 \times 10^{6}$</td>
</tr>
<tr>
<td>$|v_h|_{L^2(qT)}$</td>
<td>$4.37 \times 10^{-2}$</td>
<td>$2.18 \times 10^{-2}$</td>
<td>$1.09 \times 10^{-2}$</td>
<td>$5.44 \times 10^{-3}$</td>
<td>$2.71 \times 10^{-3}$</td>
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</table>

Table: $T = 1/2$, $y_0(x) \equiv \sin(\pi x)$, $a(x) \equiv 10^{-1}$. $\|y_h(\cdot, T)\|_{L^2(0,1)} = O(h)$.

Figure: $\omega = (0.2, 0.8)$. The adjoint state $p_h$ and its restriction to $(0, 1) \times \{T\}$. 
Experiments with $\omega = (0.2, 0.8)$

$T = 1/2, \ y_0(x) \equiv \sin(\pi x), \ a(x) \equiv 10^{-1}.$

Figure: $\omega = (0.2, 0.8)$. The state $y_h$ (Left) and the control $v_h$ (Right).
Experiments with $\omega = (0.3, 0.4)$

<table>
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<tr>
<th>$\Delta x = \Delta t$</th>
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</thead>
<tbody>
<tr>
<td>conditioning</td>
<td>$3.06 \times 10^{14}$</td>
<td>$5.24 \times 10^{22}$</td>
<td>$2.13 \times 10^{33}$</td>
<td>$5.11 \times 10^{44}$</td>
<td>$4.03 \times 10^{54}$</td>
</tr>
<tr>
<td>$| p_h(\cdot, T) |_{L^2(0,1)}$</td>
<td>$1.37 \times 10^{3}$</td>
<td>$5.51 \times 10^{3}$</td>
<td>$5.12 \times 10^{4}$</td>
<td>$2.16 \times 10^{6}$</td>
<td>$3.90 \times 10^{6}$</td>
</tr>
<tr>
<td>$| y_h(\cdot, T) |_{L^2(0,1)}$</td>
<td>$1.55 \times 10^{-1}$</td>
<td>$9.46 \times 10^{-2}$</td>
<td>$6.12 \times 10^{-2}$</td>
<td>$3.91 \times 10^{-2}$</td>
<td>$2.41 \times 10^{-2}$</td>
</tr>
<tr>
<td>$| v_h |_{L^2(qT)}$</td>
<td>5.813</td>
<td>8.203</td>
<td>10.68</td>
<td>13.20</td>
<td>15.81</td>
</tr>
</tbody>
</table>

**Table:** $T = 1/2$, $y_0(x) \equiv \sin(\pi x)$, $a(x) \equiv 10^{-1}$. $\| y_h(\cdot, T) \|_{L^2(0,1)} = O(h^{0.66})$.

**Figure:** $\omega = (0.3, 0.4)$. The adjoint state $p_h$ in $Q_T$ (Left) and its restriction to $(0, 1) \times \{T\}$ (Right).
Experiments with $\omega = (0.3, 0.4)$

Figure: $\omega = (0.3, 0.4)$. The state $y_h$ (Left) and the control $v_h$ (Right).
E. Fernández-Cara and A. Münch, 
*Numerical null controllability of the 1-d heat equation: primal algorithms*, SéMA, (2013), Available at http://hal.archives-ouvertes.fr/hal-00687884.

E. Fernández-Cara and A. Münch, 
*Numerical null controllability of the 1-d heat equation: Carleman weights and duality*, JOTA, (2013) - Available at http://hal.archives-ouvertes.fr/hal-00687887.
PART III

\[
\begin{align*}
\min \quad & J(y, v) = \frac{1}{2} \int \int_{q_T} \rho_0^2 |v|^2 \, dx \, dt \\
\text{Subject to} \quad & (y, v) \in C(y_0, T).
\end{align*}
\]

(12)

where \( \rho_0 \) are non-negative continuous weights functions such that 
\( \rho, \rho_0 \in L^\infty(Q_{T-\delta}) \quad \forall \delta > 0. \)

\[
\min_{\varphi \in \hat{W}_{\rho_0, \rho}} \hat{J}^*(\varphi) = \frac{1}{2} \int \int_{q_T} \rho_0^{-2} |\varphi(x, t)|^2 \, dx \, dt + (y_0, \varphi(\cdot, 0))_{L^2(\Omega)}.
\]

(13)

\[
\hat{W}_{\rho_0, \rho} = \{ \varphi \in \hat{\Phi}_{\rho_0, \rho} : \rho^{-1} L^* \varphi = 0 \text{ in } L^2(Q_T) \}
\]

---

\(^{\text{13}}\) D. Araujo de Souza, AM, A mixed formulation for the direct approximation of the control of minimal \( L^2 \)-weighted norm for the linear heat equation. Advances in Computational Mathematics, (2016)
\textbf{PART III}

\[
\left\{\begin{array}{l}
\text{Minimize } J(y, v) = \frac{1}{2} \int \int_{Q_T} \rho_0^2 |v|^2 \, dx \, dt \\
\text{Subject to } (y, v) \in C(y_0, T).
\end{array}\right.
\tag{12}
\]

where $\rho_0$ are non-negative continuous weights functions such that $\rho, \rho_0 \in L^\infty(Q_{T-\delta}) \quad \forall \delta > 0$.

\[
\min_{\varphi \in \tilde{W}_{\rho_0, \rho}} \hat{J}^* (\varphi) = \frac{1}{2} \int \int_{Q_T} \rho_0^{-2} |\varphi(x, t)|^2 \, dx \, dt + (y_0, \varphi(\cdot, 0))_{L^2(\Omega)}. \tag{13}
\]

\[
\tilde{W}_{\rho_0, \rho} = \{ \varphi \in \Phi_{\rho_0, \rho} : \rho^{-1}L^* \varphi = 0 \text{ in } L^2(Q_T) \}\]

\[\text{---}\]

\[13\text{D. Araujo de Souza, AM, A mixed formulation for the direct approximation of the control of minimal } L^2\text{-weighted norm for the linear heat equation. Advances in Computational Mathematics, (2016)}\]
PART IV
Semi-linear case

E. Fernández-Cara and A. Münch,
*Numerical null controllability of a semi-linear 1D heat equation via a least squares reformulation*,

E. Fernández-Cara and A. Münch,
*Numerical null controllability of semi-linear 1D heat equations: fixed point, least squares and Newton methods*,
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$\omega \subset \Omega$, $a \in C^1(\overline{\Omega}, \mathbb{R}^+_\ast)$, $y_0 \in L^2(\Omega)$, $Q_T = \Omega \times (0, T)$, $q_T = \omega \times (0, T)$, $v \in L^\infty(q_T)$

\[
\begin{cases}
  y_t - (c(x)y_x)_x + f(y) = v1_\omega, & (x, t) \in Q_T \\
  y(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T) \\
  y(x, 0) = y_0(x), & x \in (0, 1).
\end{cases}
\] (14)

$f : \mathbb{R} \rightarrow \mathbb{R}$ is, at least, locally Lipschitz-continuous.

\[|f'(s)| \leq C(1 + |s|^p) \quad \text{a.e., with } p \leq 5.\] (15)

Under this condition, (14) possesses exactly one local in time solution.

Under the growth condition [Cazenave-Haraux’89]

\[|f(s)| \leq C(1 + |s| \log(1 + |s|)) \quad \forall s \in \mathbb{R},\] (16)

the solutions to (14) are globally defined in $[0, T]$ and one has

\[y \in C^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1_0(0, 1)).\] (17)
\[ \begin{align*}
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The goal is to analyze numerically the null controllability properties of (14), in particular when blow-up occurs. Without a growth condition of the kind (16), the solutions to (14) can blow up before \( t = T \); in general, the blow-up time depends on the sizes of \( \|y_0\|_{L^2(\Omega)} \) and \( \|c\|_{L^\infty} \).

Assume \( f(0) = 0 \). The system (14) is said to be "null-controllable" at time \( T \) if, for any \( y_0 \in L^2(\Omega) \), there exist controls \( v \in L^\infty(\mathbb{Q}_T) \) and associated states \( y \) that are again globally defined in \([0, T]\) and satisfy (17) and

\[
y(x, T) = 0, \quad x \in (0, 1).
\]
Controllability results

The first one states that, if \( f \) is “too super-linear” at infinity, then the control cannot compensate the blow-up phenomena occurring in \((0, 1) \setminus \overline{\omega} \):

\[
\text{Theorem (Fernandez-Cara and Zuazua’00)}
\]

There exist locally Lipschitz-continuous functions \( f \) with \( f(0) = 0 \) and

\[
|f(s)| \sim |s| \log^p(1 + |s|) \quad \text{as} \quad |s| \to \infty, \quad p > 2, \quad \quad (19)
\]

such that (14) fails to be null-controllable for all \( T > 0 \).

The second result provides conditions under which (14) is null-controllable:

\[
\text{Theorem (Fernandez-Cara and Zuazua’00, Barbu’00)}
\]

Let \( T > 0 \) be given. Assume that \( f : \mathbb{R} \to \mathbb{R} \) is locally Lipschitz-continuous and satisfies (15) and

\[
\frac{f(s)}{|s| \log^{3/2}(1 + |s|)} \to 0 \quad \text{as} \quad |s| \to \infty. \quad \quad (20)
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Then (14) is null-controllable at time \( T \).
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The proof in [Fernandez-Cara & Zuazua, 2000] is based on

- a linearization of the eq.

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with

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g(z) = \frac{f(z)}{z}
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- a fixed point argument: it is shown that the operator \( \Lambda_0 : z \rightarrow y \) is continuous compact from \( L^2(Q_T) \) to \( L^2(Q_T) \) and maps the closed ball \( B(0, M) \subset L^2(Q_T) \) into itself. Then, Schauder Theorem provides the existence of at least one fixed point for \( \Lambda_0 \).
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Step 1: A linear control problem

First, we deal with the controllability properties of the following linear system

\[
\begin{aligned}
L_A y := y_t - (c(x)y_x)_x + A(x, t)y &= \nu 1_\omega + B(x, t), \quad (x, t) \in Q_T \\
y(x, t) &= 0, \quad (x, t) \in \Sigma_T \\
y(x, 0) &= y_0(x),
\end{aligned}
\]

(23)

that arises naturally after linearization of (14). From Lebeau-Robbiano’95 and Fursikov-Imanuvilov’96, (23) is null-controllable.

We give some numerical methods to address the extremal problem

\[
\begin{aligned}
\text{Minimize} & \quad J(y, \nu) = \frac{1}{2} \int_{Q_T} \rho^2 |y|^2 \, dx \, dt + \frac{1}{2} \int_{q_T} \rho_0^2 |\nu|^2 \, dx \, dt \\
\text{Subject to} & \quad (y, \nu) \in C_{\text{lin}}(y_0, T)
\end{aligned}
\]

(24)

where \( C_{\text{lin}}(y_0, T) \) is the linear manifold

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C_{\text{lin}}(y_0, T) = \{ (y, \nu) : \nu \in L^2(q_T), \ y \text{ solves (23) and satisfies } y(T) = 0 \}.
\]

We assume that \( A \in L^\infty(Q_T) \) and \( B \in L^2(Q_T) \) and, also, that \( B \) vanishes at \( t = T \) in an appropriate sense (i.e. \( \int_{Q_T} \rho_0^2 |B|^2 \, dx \, dt < +\infty \)).
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Numerical solution via a dual method [Fernandez-Cara, AM-2010]

\[
\begin{aligned}
\text{Minimize } & J^*(\mu, \varphi_T) = \frac{1}{2} \left( \iint_{Q_T} \rho^{-2}|\mu|^2 \, dx \, dt + \iint_{q_T} \rho_0^{-2}|\varphi|^2 \, dx \, dt \right) \\
& + \iint_{Q_T} B(x, t) \varphi \, dx \, dt + \int_0^1 y_0(x) \varphi(x, 0) \, dx
\end{aligned}
\] (25)

Subject to \((\mu, \varphi_T) \in \mathcal{V}\)

where \(\mathcal{V}\) is defined as the completion of \(\mathcal{D}(Q_T) \times \mathcal{D}(0, 1)\) with respect to the norm

\[
(\mu, \varphi_T) \to \left( \iint_{Q_T} \rho^{-2}|\mu|^2 \, dx \, dt + \iint_{q_T} \rho_0^{-2}|\varphi|^2 \, dx \, dt \right)^{1/2}.
\]

where \(\psi\) solves

\[
L_A^* \varphi = \mu \quad Q_T, \quad \varphi = 0 \quad \Sigma_T, \quad \varphi(\cdot, T) = \varphi_T \quad (0, 1). \quad (26)
\]
Primal (direct) approach

Proposition

Let $\rho$ and $\rho_0$ be given by (4). Let $(y, v)$ be the corresponding optimal pair for $J$. Then there exists $p \in P$ such that

$$y = \rho^{-2} L_A^* p, \quad v = -\rho_0^{-2} p|_{qT}.$$  \hfill (27)

The function $p$ is the unique solution in $P$ of

$$(p, q)_P = \int_0^1 y_0(x) q(x, 0) \, dx + \iint_{Q_T} Bq \, dx \, dt, \quad \forall q \in P$$ \hfill (28)

Remark

$p$ solves, at least in $D'$, the following differential problem, that is second order in time and fourth order in space:

$$\begin{cases} L_A(\rho^{-2} L_A^* p) + \rho_0^{-2} p \, 1_\omega = B, & (x, t) \in (0, 1) \times (0, T) \\ p(x, t) = 0, \quad (\rho^{-2} L_A^* p)(x, t) = 0 & (x, t) \in \{0, 1\} \times (0, T) \\ (\rho^{-2} L_A^* p)(x, 0) = y_0(x), \quad (\rho^{-2} L_A^* p)(x, T) = 0, & x \in (0, 1). \end{cases}$$ \hfill (29)

The "boundary" conditions at $t = 0$ and $t = T$ appear as Neumann conditions.
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Step 2: Fixed points
Step 2 : Back to the nonlinear problem

For simplicity, we will assume that $y_0 \in L^\infty(0, 1)$ and $f \in C^1(\mathbb{R})$ and is globally Lipschitz-continuous. Let us introduce the function $g$, with

$$g(s) = \begin{cases} \frac{f(s)}{s} & \text{if } s \neq 0, \\ f'(0) & \text{otherwise.} \end{cases}$$

Then $g \in C^0_b(\mathbb{R})$ and $f(s) = g(s) s$ for all $s$ (recall that $f(0) = 0$). We will set $G_0 = \|g\|_{L^\infty(\mathbb{R})}$.

For any $z \in L^1(Q_T)$, let us introduce the bilinear form

$$m(z; p, q) = \int\int_{Q_T} \rho^{-2} L^*_g(z) p L^*_g(z) q \, dx \, dt + \int\int_{Q_T} \rho^{-2} 0 p q \, dx \, dt \quad \forall p, q \in P_0. \quad (30)$$

Then $m(z; \cdot, \cdot)$ is a scalar product in $P_0$ and can be used to construct a Hilbert space $P$ that, in principle, may depend on $z$. We will use the following result, which is a direct consequence of the Carleman estimates:

**Lemma**

Under the previous conditions, if the constants $K_i$ in (4) are large enough (depending on $\omega$, $T$, $c_0$, $\|c\|_{C^1}$, and $G_0$), then there exist $C_1, C_2 > 0$ such that

$$C_1 m(0; p, p) \leq m(z; p, p) \leq C_2 m(0; p, p) \quad \forall p \in P_0 \quad (31)$$

for all $z \in L^1(Q_T)$. 
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$$m(z; p, q) = \iiint_{Q_T} \rho^{-2} L_{g(z)}^* p L_{g(z)}^* q \, dx \, dt + \iiint_{Q_T} \rho_0^{-2} p \, q \, dx \, dt \quad \forall p, q \in P_0. \quad (30)$$

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*for all $z \in L^1(Q_T)$.***
Back to the nonlinear problem

Accordingly, all the spaces $P$ provided by the bilinear forms $m(z; \cdot, \cdot)$ are the same and, in fact, (31) holds for all $p \in P$:

$$C_1 m(0; p, p) \leq m(z; p, p) \leq C_2 m(0; p, p) \quad \forall p \in P.$$  \hspace{1cm} (32)

We will fix the following norm in $P$:

$$\|p\|_P = m(0; p, p)^{1/2} \quad \forall p \in P.$$ \hspace{1cm} (33)
The operator $\Lambda_0$

Let us introduce the mapping $\Lambda_0 : L^2(Q_T) \mapsto L^2(Q_T)$ where, for any $z \in L^2(Q_T)$, $y_z = \Lambda_0(z)$ is, together with $v_z$, the unique solution to the linear extremal problem

$$\text{Minimize } J(z; y, v) := \frac{1}{2} \iint_{Q_T} \rho^2 |y|^2 \, dx \, dt + \frac{1}{2} \iint_{q_T} \rho_0^2 |v|^2 \, dx \, dt$$

subject to $v \in L^2(q_T)$ and

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\end{align*}$$

such that $y(\cdot, T) = 0$. $\Lambda_0 : L^2(Q_T) \mapsto L^2(Q_T)$ is well defined. Furthermore, applying proposition 2 with $A = g(z)$ and $B = 0$, we obtain that $y_z$ and $v_z$ are characterized as follows:

$$y_z = \Lambda_0(z) = \rho^{-2} L^*_{g(z)} p_z, \quad v_z = -\rho_0^{-2} p_z|_{q_T},$$

where $p_z \in P$ is the unique solution to the linear problem

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A fixed point method

In order to solve the null controllability problem for (14), it suffices to find a solution to the fixed point equation

\[ y = \Lambda_0(y), \quad y \in L^2(Q_T). \]  

(38)

**ALG 1 (fixed point):**

\[ y^0 \in L^2(Q_T), \quad y^{n+1} = \Lambda_0(y^n), \quad n \geq 0 \]  

(39)

If \((y^n, v^n) \rightharpoonup (y, v)\) in \(L^2(Q_T) \times L^2(q_T)\), then \((y, v)\) solves the nonlinear null controllability problem. Indeed, since the \(g(y^n)\) are uniformly bounded in \(L^\infty(Q_T)\), after extraction of a subsequence it can be assumed that \(y^n\) (resp. \(y^n_t\)) converges weakly in \(L^2(0, T; H^1_0(0, 1))\) (resp. \(L^2(0, T; H^{-1}(0, 1))\)). Therefore, \(y^n\) converges strongly in \(L^2(Q_T)\) and a.e., \(g(y^n)\) converges to \(g(y)\) weakly-* in \(L^\infty(Q_T)\) and we can take limits and deduce that \(y\) solves, together with \(v\), the nonlinear system.

This fixed point formulation has been used in [Fernandez-Cara, Zuazua, 2000] to prove Theorem 3. Precisely, it is shown there that \(\Lambda_0 : L^2(Q_T) \mapsto L^2(Q_T)\) is continuous and compact and, also that there exists \(M > 0\) such that \(\Lambda_0\) maps the whole space \(L^2(Q_T)\) inside the ball \(B(0; M)\). Then, Schauder's Theorem provides the existence of at least one fixed point for \(\Lambda_0\).

It is however important to note that this does not imply the convergence of the sequence \(\{y^n\}\) defined by \(y^{n+1} = \Lambda_0(y^n)\).
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A fixed point : a numerical application

\[ f(s) = C_f s \log^p(1 + |s|) \quad \forall s \in \mathbb{R}. \] (40)

We consider the following data:

\[ a(x) = 1/10, \quad p = 1.4, \quad C_f = -5, \quad T = 1/2, \quad y_0(x) = \alpha \sin(\pi x) \]

In the uncontrolled situation, these data lead to the blow-up of the solution of (14) at time \( t_c \approx 0.406, 0.367, 0.339, 0.318 \) for \( \alpha = 10, 20, 40 \) and 80, respectively.

We first take \( \omega = (0.2, 0.8) \) and initialize \textbf{ALG 1} with

\[ y^0(x, t) = y_0(x)(1 - t/T)^2. \]
A fixed point : a numerical application - Lack of convergence

Figure: Fixed point method - $h = (1/60, 1/60)$ - $y_0(x) = \alpha \sin(\pi x)$ - Evolution of $\log_{10}(\|\Lambda_0(y^n_h) - y^n_h\|_{L^2(Q_T)}/\|y^n_h\|_{L^2(Q_T)})$ vs. $n$ for $\alpha = 10, 20, 40$ and $80$. 
A fixed point: a numerical application

<table>
<thead>
<tr>
<th></th>
<th>$|v_h|_{L^2(q_T)}$</th>
<th>$|v_h|_{L^\infty(q_T)}$</th>
<th>$|y_h|_{L^2(Q_T)}$</th>
<th># iterates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 10$</td>
<td>$3.531 \times 10^1$</td>
<td>$2.542 \times 10^2$</td>
<td>$1.742$</td>
<td>20</td>
</tr>
<tr>
<td>$\alpha = 40$</td>
<td>$2.142 \times 10^2$</td>
<td>$2.053 \times 10^3$</td>
<td>$6.654$</td>
<td>14</td>
</tr>
<tr>
<td>$\alpha = 80$</td>
<td>$5.109 \times 10^2$</td>
<td>$7.021 \times 10^3$</td>
<td>$14.410$</td>
<td>81</td>
</tr>
</tbody>
</table>

**Table:** Fixed point method - $h = (1/60, 1/60)$ - $y_0(x) = \alpha \sin(\pi x)$ - Norms for $\alpha = 10, 40$ and $\alpha = 80$. 
A fixed point : a numerical application

Figure: Fixed point method - $h = (1/60, 1/60)$ - $y_0(x) = 40 \sin(\pi x)$ - Control $v_h$ (Left) and corresponding controlled solution $y_h$ (Right) in $Q_T$. 
A fixed point : a numerical application : $p \geq 3/2$

Figure: Fixed point method - $h = (1/60, 1/60)$ - $y_0(x) \equiv 40 \sin(\pi x)$ - $p = 1.4$ -
Evolution of $\log_{10}(\|A_0(y^n_h) - y^n_h\|_{L^2(Q_T)}/\|y^n_h\|_{L^2(Q_T)})$ for $p = 1.4, 1.5, 1.6$ and $p = 1.7$. 
Least squares reformulation

We now introduce the function \( \zeta(t) = (T - t)^{-1/2} \) for all \( t \) in \((0, T)\) and the space \( Z := L^2(\zeta^2, Q_T) \). We will denote by \( \Lambda \) the restriction to \( Z \) of the mapping \( \Lambda_0 \). Obviously, \( \Lambda(z) \in Z \) for all \( z \in Z \).

Let us consider the following least squares reformulation of (38):

\[
\begin{align*}
\text{Minimize } & \quad R(z) := \frac{1}{2} \| z - \Lambda(z) \|_Z^2 \\
\text{Subject to } & \quad z \in Z.
\end{align*}
\]

(41)

Any solution to (38) solves (41). Conversely, if \( y \) solves (41), we necessarily have \( R(y) = 0 \) (because (14) is null controllable with control-states \((y, v)\) such that \( J(z; y, v) < +\infty \)); hence, \( y \) also solves (38). This shows that (38) and (41) are, in the present context, equivalent.
Proposition

Let us assume that $g \in C^1_b(\mathbb{R})$. Then $R \in C^1(\mathbb{Z})$. Moreover, for any $z \in \mathbb{Z}$, the gradient of $R$ with respect to the inner product of $Z$ is given by

$$DR(z) = (1 - \rho^{-2}g'(z)p_z)(z - y_z) + \zeta^{-2}g'(z)(y_z\lambda_z + p_z\mu_z), \quad (42)$$

where $p_z$ is the unique solution to (37), $y_z = \rho^{-2}L^*_g(z)p_z$, $\lambda_z$ is the unique solution to the linear (adjoint) problem

$$m(z; q, \lambda_z) = (z - y_z, \rho^{-2}L^*_g(z)q)z \quad \forall q \in P; \quad \lambda_z \in P \quad (43)$$

and, finally, $\mu_z = \rho^{-2}L^*_g(z)\lambda_z$. 

Least squares reformulation
Let the assumptions in proposition 3 be satisfied and let us introduce $G_1 := \|g\|_{L^\infty(R)}$. There exists a constant $K$ that depends on $\omega$, $T$, $c_0$, $\|c\|_{C^1}$ and $G_0$ but is independent of $z$ and $y_0$, such that the following holds for all $z \in Z$:

$$\|DR(z)\|_Z \geq \left(1 - K G_1 \|y_0\|_{L^2}\right) \|z - \Lambda(z)\|_Z.$$  (44)
Least squares : Gradient method for $R$

**ALG 2 (Least-squares):**

$$ z^0 \in L^2(Q_T), \quad (z^{n+1}, h)_Z = (z^n, h)_Z - \eta (DR(z^n), h)_Z, \quad n \geq 0 $$
Least squares: a numerical application

\[ f_\eta(s) = C_f s \log^\beta(1 + |s|\eta) \quad \forall s \in \mathbb{R}, \quad |s|_\eta := \sqrt{s^2 + \eta^2} - \eta \quad (45) \]

so that, for all \( \eta > 0 \), \( g_\eta := C_f \log^\beta(1 + |s|_\eta) \) belongs to \( C^1_b(\mathbb{R}) \). We have \( f_\eta(0) = 0 \) and Theorem applies for \( f_\eta \), since \( f_\eta \) and \( f \) are equivalent at infinity. We take \( \eta = 10^{-1} \).

Figure: Least squares method - \( h = (1/60, 1/60) \) - Evolution of
\[ \log_{10}(\|\Lambda(z^n_h) - z^n_h\|_{L^2(Q_T)}/\|z^n_h\|_{L^2(Q_T)}) \] - \( \alpha = 10, 20, 40, 80 \) and algorithm ALG 2'.
Least squares reformulation: numerical application

\[ f_\eta(s) = C_f s \log^p(1 + |s|_\eta) \quad \forall s \in \mathbb{R}, \quad |s|_\eta := \sqrt{s^2 + \eta^2 - \eta} \quad (46) \]

so that, for all \( \eta > 0 \), \( g_\eta := C_f \log^p(1 + |s|_\eta) \) belongs to \( C^1_b(\mathbb{R}) \). Moreover, we have \( f_\eta(0) = 0 \) and Theorem applies for \( f_\eta \), since \( f_\eta \) and \( f \) are equivalent at infinity. We shall take \( \eta = 10^{-1} \).

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( | V_h |_{L^2(Q_T)} )</th>
<th>( | V_h |_{L^\infty(Q_T)} )</th>
<th>( | Z_h |_{L^2(Q_T)} )</th>
<th>( | R'(Z_h) |_{L^2(Q_T)} )</th>
<th>( e^n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>3.507 \times 10^1</td>
<td>2.532 \times 10^2</td>
<td>1.753</td>
<td>1.27 \times 10^{-3}</td>
<td>1.43 \times 10^{-3}</td>
</tr>
<tr>
<td>20</td>
<td>8.781 \times 10^1</td>
<td>7.323 \times 10^2</td>
<td>3.180</td>
<td>1.44 \times 10^{-3}</td>
<td>1.54 \times 10^{-3}</td>
</tr>
<tr>
<td>40</td>
<td>2.137 \times 10^2</td>
<td>2.048 \times 10^3</td>
<td>6.651</td>
<td>5.42 \times 10^{-3}</td>
<td>3.39 \times 10^{-3}</td>
</tr>
<tr>
<td>80</td>
<td>2.526 \times 10^2</td>
<td>3.299 \times 10^3</td>
<td>14.73</td>
<td>2.23 \times 10^{-1}</td>
<td>7.89 \times 10^{-1}</td>
</tr>
</tbody>
</table>

**Table:** Least squares method approach after 100 iterates - \( h = (1/60, 1/60) \) - \( y_0(x) \equiv \alpha \sin(\pi x) \) - \( p = 1.4 \) - Norms for \( \alpha = 10, 20, 40, 80 \). Here, \( e^n = \| \Lambda(Z^n_h) - Z^n_h \|_{L^2(Q_T)}/\| Z^n_h \|_{L^2(Q_T)} \).
Newton-Raphson Algorithm (a different way to linearize $f(y^{n+1})$)

**ALG 3’ (Newton):**

1. Choose $(y^0, z^0) \in Y$.
2. Then, given $n \geq 0$ and $(y^n, v^n) \in Y$, solve in $(y^{n+1}, v^{n+1}) \in Y$ the linear problem

   \[ F'(y^n, v^n) \cdot (y^{n+1} - y^n, v^{n+1} - v^n) = -F(y^n, v^n), \]

   i.e. find $y^{n+1}$ and $v^{n+1}$ such that $(y^{n+1}, v^{n+1}) \in Y$ and

   \[
   \begin{cases}
   y^{n+1}_t - (c(x)y^{n+1}_x)_x + f'(y^n) y^{n+1} = v^{n+1} + f'(y^n) y^n - f(y^n), \\
   y^{n+1} = 0, \\
   y^{n+1}(\cdot, 0) = y_0,
   \end{cases}
   \]

   \[
   \begin{aligned}
   Q_T & \\
   \Sigma_T & = (0, 1).
   \end{aligned}
   \]
Newton-Raphson Algorithm (a different way to linearize $f(y^{n+1})$)

**ALG 3’ (Newton):**

1. Choose $(y^0, z^0) \in Y$.
2. Then, given $n \geq 0$ and $(y^n, v^n) \in Y$, solve in $(y^{n+1}, v^{n+1}) \in Y$ the linear problem
   \[
   F'(y^n, v^n) \cdot (y^{n+1} - y^n, v^{n+1} - v^n) = -F(y^n, v^n),
   \]
i.e. find $y^{n+1}$ and $v^{n+1}$ such that $(y^{n+1}, v^{n+1}) \in Y$ and
   \[
   \begin{cases}
   y_t^{n+1} - (c(x)y_x^{n+1})_x + f'(y^n) y^{n+1} = v^{n+1} 1_\omega + f'(y^n) y^n - f(y^n), \\
y^{n+1} = 0, \\
y^{n+1}(:, 0) = y_0,
   \end{cases}
   \]
\[(47)\]
smaller support $\omega$

**Figure:** Fixed point method - $h = (1/60, 1/60)$ - $y_0(x) \equiv 10 \sin(\pi x)$ - $p = 1.4$ - $\omega = (0.2, 0.5)$ - Control $v_h$ (Left) and corresponding controlled solution $y_h$ (Right) in $Q_T$. 
Least-squares approach

**PART IV**

A variational least-squares approach

14

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Least-squares approach

We define the non-empty set

\[ \mathcal{A} = \left\{ (u, f); \ u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)); u' \in L^2(0, T, H^{-1}(\Omega)), \right. \]

\[ \left. u(\cdot, 0) = u_0, u(\cdot, T) = 0, f \in L^2(q_T) \right\} \]

and find \((u, f) \in \mathcal{A}\) solution of the heat eq. !

For any \((u, f) \in \mathcal{A}\), we define the "corrector" \(v = v(u, f) \in H^1(Q_T)\) solution of the \(Q_T\)-elliptic problem

\[
\begin{aligned}
- v_{tt} - \nabla \cdot (c(x) \nabla v) + (Lu - f 1_\omega) &= 0, & (x, t) \in Q_T, \\
v_t &= 0, & x \in \Omega, t \in \{0, T\} \\
v &= 0, & x \in \Sigma_T.
\end{aligned}
\]
Least-squares approach

We define the non-empty set

\[ \mathcal{A} = \left\{ (u, f); \ u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)); \ u' \in L^2(0, T, H^{-1}(\Omega)), \right. \]

\[ \left. \quad u(\cdot, 0) = u_0, \ u(\cdot, T) = 0, \ f \in L^2(\Omega) \right\} \]

and find \((u, f) \in \mathcal{A}\) solution of the heat eq. !

For any \((u, f) \in \mathcal{A}\), we define the "corrector" \(v = v(u, f) \in H^1(Q_T)\) solution of the \(Q_T\)-elliptic problem

\[
\begin{cases}
- v_{tt} - \nabla \cdot (c(x) \nabla v) + (Lu - f 1_\omega) = 0, & (x, t) \in Q_T, \\
v_t = 0, & x \in \Omega, t \in \{0, T\} \\
v = 0, & x \in \Sigma_T.
\end{cases}
\]
Least-squares approach (2)

**Theorem**

$u$ is a controlled solution of the heat eq. by the control function $f 1_\omega \in L^2(q_T)$ if and only if $(u, f)$ is a solution of the extremal problem

$$\inf_{(u, f) \in A} E(u, f) := \frac{1}{2} \iint_{Q_T} \left(|v_t|^2 + c(x)|\nabla v|^2\right) dx \, dt.$$  \hspace{1cm} (49)

**Proof.**

$\Leftarrow$ From the null controllability of the heat eq., the extremal problem is well-posed in the sense that the infimum, equal to zero, is reached by any controlled solution of the heat eq. (the minimizer is not unique).

$\Rightarrow$ Conversely, we check that any minimizer of $E$ is a solution of the (controlled) heat eq.:

We define the vector space

$$\mathcal{A}_0 = \left\{(u, f); \ u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)); u' \in L^2(0, T, H^{-1}(\Omega)), \right\}$$

$$u(\cdot, 0) = u(\cdot, T) = 0, \ x \in \Omega, \ f \in L^2(q_T)$$
Least-squares approach (2)

Theorem

\( u \) is a controlled solution of the heat eq. by the control function \( f 1_\omega \in L^2(q_T) \) if and only if \((u, f)\) is a solution of the extremal problem

\[
\inf_{(u,f)\in \mathcal{A}} E(u, f) := \frac{1}{2} \iint_{Q_T} (|v_t|^2 + c(x)|\nabla v|^2) \, dx \, dt.
\]

(49)

Proof.

\( \leftarrow \) From the null controllability of the heat eq., the extremal problem is well-posed in the sense that the infimum, equal to zero, is reached by any controlled solution of the heat eq. (the minimizer is not unique).

\( \implies \) Conversely, we check that any minimizer of \( E \) is a solution of the (controlled) heat eq.

We define the vector space

\[
\mathcal{A}_0 = \left\{ (u, f); \ u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)); \ u' \in L^2(0, T, H^{-1}(\Omega)), \ u(\cdot, 0) = u(\cdot, T) = 0, x \in \Omega, f \in L^2(q_T) \right\}
\]
Least-squares approach (2)

Theorem

$u$ is a controlled solution of the heat eq. by the control function $f \ 1_{\omega} \in L^2(q_T)$ if and only if $(u, f)$ is a solution of the extremal problem

$$
\inf_{(u,f) \in \mathcal{A}} E(u,f) := \frac{1}{2} \int_Q \left( |v_t|^2 + c(x)|\nabla v|^2 \right) dx \ dt.
$$

(49)

Proof.

$\Leftarrow$ From the null controllability of the heat eq., the extremal problem is well-posed in the sense that the infimum, equal to zero, is reached by any controlled solution of the heat eq. (the minimizer is not unique).

$\implies$ Conversely, we check that any minimizer of $E$ is a solution of the (controlled) heat eq.:

We define the vector space

$$
\mathcal{A}_0 = \left\{ (u, f); \ u \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1_0(\Omega)); \ u' \in L^2(0,T, H^{-1}(\Omega)), \ u(\cdot,0) = u(\cdot,T) = 0, x \in \Omega, f \in L^2(q_T) \right\}
$$
Least-squares approach (2)

The first variation of $E$ at $(u, f)$ in the admissible direction $(U, F) \in \mathcal{A}_0$ defined by

$$
\langle E'(u, f), (U, F) \rangle = \lim_{\eta \to 0} \frac{E((u, f) + \eta(U, F)) - E(u, f)}{\eta}
$$

exists and is given by

$$
\langle E'(u, f), (U, F) \rangle = \iint_{Q_T} (v_t V_t + a(x) \nabla v \cdot \nabla V) \, dx \, dt,
$$

where the corrector $V \in H^1(Q_T)$ associated to $(U, F)$ is the solution of

\[
\begin{cases}
  U_t - V_{tt} - \nabla \cdot (a(x)(\nabla U + \nabla V)) - F 1_\omega = 0, & (x, t) \in Q_T, \\
  V_t(x, 0) = V_t(x, T) = 0, & x \in \Omega, \\
  V(0, t) = V(1, t) = 0, & t \in (0, T).
\end{cases}
\]

Using that

$$
- \int_0^T \langle U_t, v \rangle_{H^{-1}(\Omega), H^1(\Omega)} \, dt = \iint_{Q_T} U v_t \, dx \, dt - \int_0^1 [Uv]_0 \, dx = \iint_{Q_T} U v_t \, dx \, dt,
$$

we get that

$$
\langle E'(u, f), (U, F) \rangle = \iint_{Q_T} (U v_t - a(x) \nabla U \cdot \nabla v + F v 1_\omega) \, dx \, dt, \quad \forall (U, F) \in \mathcal{A}_0
$$
 Least-squares approach (2)

Therefore, if \((u, f)\) minimizes \(E\), the equality \(< E'(u, f), (U, F) > = 0\) for all \((U, F) \in A_0\) implies that the corrector \(v = v(u, f)\) satisfies

\[
\begin{aligned}
- v_t - \nabla \cdot (c(x) \nabla v) + dv &= 0, \quad (x, t) \in Q_T, \\
v &= 0, \quad (x, t) \in q_T
\end{aligned}
\]

in addition to the boundary conditions: \(v = 0\) on \(\Sigma_T\) and \(v_t = 0\) on \(\Omega \times \{0, T\}\). Unique continuation property implies that \(v = 0\) in \(Q_T\) and so \(E(u, f) = 0\) and so \((u, f) \in \mathcal{A}\) solves the heat eq.

Remark The proposition educes the search of ONE control \(f\) distributed in \(\omega\) to the minimization of the functional \(E\) over \(\mathcal{A}\).

Remark Least squares terminology:

\[
E(u, f) := \frac{1}{2} \|u_t - \nabla \cdot (a(x) \nabla u) + d u - f1_\omega\|^2_{H^{-1}(Q_T)}
\]
Least-squares approach (2)

Therefore, if \((u, f)\) minimizes \(E\), the equality \(\langle E'(u, f), (U, F) \rangle = 0\) for all \((U, F) \in A_0\) implies that the corrector \(v = v(u, f)\) satisfies

\[
\begin{cases}
- v_t - \nabla \cdot (c(x) \nabla v) + d v = 0, & (x, t) \in Q_T, \\
v = 0, & (x, t) \in q_T
\end{cases}
\]

in addition to the boundary conditions: \(v = 0\) on \(\Sigma_T\) and \(v_t = 0\) on \(\Omega \times \{0, T\}\).

Unique continuation property implies that \(v = 0\) in \(Q_T\) and so \(E(u, f) = 0\) and so \((u, f) \in A\) solves the heat eq.

Remark The proposition educes the search of ONE control \(f\) distributed in \(\omega\) to the minimization of the functional \(E\) over \(A\).

Remark Least squares terminology:

\[
E(u, f) := \frac{1}{2} \| u_t - \nabla \cdot (a(x) \nabla u) + d u - f1_\omega \|_{H^{-1}(Q_T)}^2
\]
Least-squares approach (2)

Therefore, if \((u, f)\) minimizes \(E\), the equality \(< E'(u, f), (U, F) > = 0\) for all \((U, F) \in A_0\) implies that the corrector \(v = v(u, f)\) satisfies

\[
\begin{align*}
- v_t - \nabla \cdot (c(x) \nabla v) + dv &= 0, \quad (x, t) \in Q_T, \\
v &= 0, \quad (x, t) \in q_T
\end{align*}
\]

in addition to the boundary conditions: \(v = 0\) on \(\Sigma_T\) and \(v_t = 0\) on \(\Omega \times \{0, T\}\).

**Unique continuation property** implies that \(v = 0\) in \(Q_T\) and so \(E(u, f) = 0\) and so \((u, f) \in \mathcal{A}\) solves the heat eq.

**Remark** The proposition educes the search of ONE control \(f\) distributed in \(\omega\) to the minimization of the functional \(E\) over \(\mathcal{A}\).

**Remark** Least squares terminology:

\[
E(u, f) := \frac{1}{2} \| u_t - \nabla \cdot (a(x) \nabla u) + du - f 1_{\omega} \|_{H^{-1}(Q_T)}^2
\]
A numerical application in 1D (inner controllability)

\[ N = 1, \Omega = (0, 1), \omega = (0.2, 0.5), u_0(x) = \sin(\pi x), c(x) = c_0 = 0.25, T = 1/2, \]
d := 0
Starting point of the algorithm: \((u, f) = (u_0(x)(1 - t/T)^2, 0) \in \mathcal{A}\)

\[ u_0(x) = \sin(\pi x) - \text{Control acting on } \omega = (0.2, 0.5) - \varepsilon = 10^{-6} - \log_{10}(E_h(u_h^n)) \text{ (dashed line)} \text{ and } \log_{10}(|g_h^n|_A) \text{ (full line)} \text{ vs. the iteration } n \text{ of the CG algorithm.} \]
A numerical application in 1D (inner controllability)

\[(u, f) \in \mathcal{A} \text{ along } Q_T \text{ at convergence}\]
A numerical application in 1D (inner controllability)

Isovalues along $Q_T$ of the corresponding corrector $v$: $\|v\|_{H^1(Q_T)} \approx 10^{-4}$
PART V
Inverse problem - Reconstruction of $y$ from $y_{qT}$

**Inverse Problem - Reconstruction of \( y \) from \( y_{q_T} \)**

\( \Omega \subset \mathbb{R}^N (N \geq 1) - T > 0, \ c \in C^1(\overline{\Omega}, \mathbb{R}), \ d \in L^\infty(Q_T), \ y_0 \in H \)

\[
\begin{cases}
L y := y_t - \nabla \cdot (c \nabla y) + dy = f, & Q_T := \Omega \times (0, T) \\
y = 0, & \Sigma_T := \partial \Omega \times (0, T) \\
y(\cdot, 0) = y_0, & \Omega.
\end{cases}
\]

\[(53)\]

**Inverse Problem**: Distributed observation on \( q_T = \omega \times (0, T), \omega \subset \Omega \)

\[
\begin{cases}
X = L^2(q_T), \\
\text{Given } (y_{obs}, f) \in (L^2(q_T), X), \text{ find } y \text{ s.t. } (53) \text{ and } y - y_{obs} = 0 \text{ on } q_T
\end{cases}
\]

**Well-known Difficulty**:

\[
\left( Ly \in L^2(Q_T), y \in L^2(q_T), y_{|\Sigma_T} = 0 \right) \implies y \in C([\delta, T], H^1_0(\Omega)), \quad \forall \delta > 0
\]
Carleman ineq.

\[
\left\{ \begin{array}{l}
\iint_{Q_T} \rho_0^{-2} |q|^2 \, dx \, dt \\
\leq C \left( \iint_{Q_T} \rho^{-2} |L^* q|^2 \, dx \, dt + \iint_{q_T} \rho_0^{-2} |q|^2 \, dx \, dt \right) , \forall q \in P.
\end{array} \right.
\]

(54)

t \mapsto T - t

\[\tilde{\rho}(x, t) = \rho(x, T - t), \quad \tilde{\rho}_0(x, t) = \rho_0(x, T - t)\]

\[
\left\{ \begin{array}{l}
\iint_{Q_T} \tilde{\rho}_0^{-2} |y|^2 \, dx \, dt \\
\leq C \left( \iint_{Q_T} \tilde{\rho}^{-2} |Ly|^2 \, dx \, dt + \iint_{q_T} \tilde{\rho}_0^{-2} |y|^2 \, dx \, dt \right) , \forall y \in Y.
\end{array} \right.
\]

(55)
Carleman ineq.

\[
\left\{ \begin{align*}
\int_{Q_T} \rho_0^{-2} |q|^2 \, dx \, dt \\
\int_{q_T} \rho_0^{-2} |q|^2 \, dx \, dt
\end{align*} \right. 
\leq C \left( \int_{Q_T} \rho^{-2} |L^* q|^2 \, dx \, dt + \int_{q_T} \rho_0^{-2} |q|^2 \, dx \, dt \right), \forall q \in P.
\]

\( t \mapsto T - t \)

\[
\tilde{\rho}(x, t) = \rho(x, T - t), \quad \tilde{\rho}_0(x, t) = \rho_0(x, T - t)
\]

\[
\left\{ \begin{align*}
\int_{Q_T} \tilde{\rho}_0^{-2} |y|^2 \, dx \, dt \\
\int_{q_T} \tilde{\rho}_0^{-2} |y|^2 \, dx \, dt
\end{align*} \right. 
\leq C \left( \int_{Q_T} \tilde{\rho}^{-2} |L y|^2 \, dx \, dt + \int_{q_T} \tilde{\rho}_0^{-2} |y|^2 \, dx \, dt \right), \forall y \in Y.
\]

(54)
Second order mixed formulation .... as in the previous part

We then define the following extremal problem :

\[
\begin{align*}
\begin{cases}
\text{Minimize } & J(y) := \frac{1}{2} \int \int_{Q_T} \rho_0^{-2} |y(x, t) - y_{obs}(x, t)|^2 dx \, dt + r \int \int_{Q_T} (\rho^{-1} L y)^2 dx \, dt \\
\text{Subject to } & y \in \mathcal{W} := \left\{ y \in \mathcal{Y} : \rho^{-1} L y = 0 \text{ in } L^2(Q_T) \right\}
\end{cases}
\end{align*}
\]

with \( \rho_0, \rho \in \mathcal{R} \) where \( (\rho_\star \in \mathbb{R}_+^*) \)

\[
\mathcal{R} := \{ w : w \in C(Q_T); w \geq \rho_\star > 0 \text{ in } Q_T; \ w \in L^\infty(\Omega \times (\delta, T)) \ \forall \delta > 0 \}
\]

Let \( \mathcal{Y}_0 := \left\{ y \in C^2(\bar{Q}_T) : y = 0 \text{ on } \Sigma_T \right\} \) and for \( \eta > 0, \rho \in \mathcal{R} \), the bilinear form by

\[
(y, \bar{y})_{\mathcal{Y}_0} := \int \int_{Q_T} \rho_0^{-2} y \bar{y} dx \, dt + \eta \int \int_{Q_T} \rho^{-2} L y L \bar{y} dx \, dt, \quad \forall y, \bar{y} \in \mathcal{Y}_0.
\]

Let \( \mathcal{Y} \) be the completion of \( \mathcal{Y}_0 \) for this scalar product endowed with the norm

\[
\|y\|_{\mathcal{Y}}^2 := \|\rho_0^{-1} y\|_{L^2(q_T)}^2 + \eta\|\rho^{-1} L y\|_{L^2(Q_T)}^2, \quad \forall y \in \mathcal{Y}.
\]
Second order mixed formulation .... as in the previous part

We then define the following extremal problem :

\[
\begin{align*}
\text{Minimize } J(y) := & \frac{1}{2} \int_0^T \int_{\Omega} \rho_0^{-2} |y(x, t) - y_{\text{obs}}(x, t)|^2 \, dx \, dt \\
& + r \int_0^T \int_{Q_T} (\rho^{-1} L y)^2 \, dx \, dt \\
\text{Subject to } y \in \mathcal{W} := \left\{ y \in \mathcal{Y} : \rho^{-1} L y = 0 \text{ in } L^2(Q_T) \right\}
\end{align*}
\]

with \( \rho_0, \rho \in \mathcal{R} \) where \( (\rho^* \in \mathbb{R}_+^+) \)

\[
\mathcal{R} := \left\{ w : w \in C(Q_T); w \geq \rho^* > 0 \text{ in } Q_T; w \in L^\infty(\Omega \times (\delta, T)) \forall \delta > 0 \right\}
\]

Let \( \mathcal{Y}_0 := \left\{ y \in C^2(\overline{Q_T}) : y = 0 \text{ on } \Sigma_T \right\} \) and for \( \eta > 0, \rho \in \mathcal{R} \), the bilinear form by

\[
(y, \bar{y})_{\mathcal{Y}_0} := \int_0^T \int_{\Omega} \rho_0^{-2} y \bar{y} \, dx \, dt + \eta \int_0^T \int_{Q_T} \rho^{-2} L y L \bar{y} \, dx \, dt, \quad \forall y, \bar{y} \in \mathcal{Y}_0.
\]

Let \( \mathcal{Y} \) be the completion of \( \mathcal{Y}_0 \) for this scalar product endowed with the norm

\[
\| y \|_{\mathcal{Y}}^2 := \| \rho_0^{-1} y \|_{L^2(Q_T)}^2 + \eta \| \rho^{-1} L y \|_{L^2(Q_T)}^2, \quad \forall y \in \mathcal{Y}.
\]
Mixed formulation

Find \((y, \lambda) \in \mathcal{Y} \times L^2(Q_T)\) solution of

\[
\begin{equation}
\begin{cases}
  a_r(y, y) + b(y, \lambda) &= l(y) \quad \forall y \in \mathcal{Y}, \\
  b(y, \lambda) &= 0 \quad \forall \lambda \in L^2(Q_T),
\end{cases}
\end{equation}
\]

(56)

where

\[
a_r : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}, \quad a(y, y) := \iint_{q_T} \rho_0^{-2} y \, y \, dx \, dt + r \iint_{Q_T} \rho^{-2} L y \, L y \, dx \, dt
\]

\[
b : \mathcal{Y} \times L^2(Q_T) \to \mathbb{R}, \quad b(y, \lambda) := \iint_{Q_T} \rho^{-1} L y \, \lambda \, dx \, dt
\]

\[
l : \mathcal{Y} \to \mathbb{R}, \quad l(y) := \iint_{q_T} \rho_0^{-2} y \, y_{obs} \, dx \, dt.
\]
Mixed formulation

Corollary

Let \( \rho_0 \in \mathcal{R}, \rho \in \mathcal{R} \cap L^\infty(Q_T) \) and assume \( \exists K \text{ s.t.} \)

\[
\rho_0 \leq K \rho_{c,0}, \quad \rho \leq K \rho_c \quad \text{in} \quad Q_T.
\]

If \((y, \lambda)\) is the solution of the mixed formulation (56), then \( \exists C > 0 \) such that

\[
\| \rho_{c,0}^{-1} y \|_{L^2(Q_T)} \leq C \| y \|_Y.
\]
First order formulation involving $y$ and the flux $p = c(x) \nabla y$.

\[
\begin{cases}
    \mathcal{I}(y, p) := y_t - \nabla \cdot p + d y = f, & \mathcal{J}(y, p) := c(x) \nabla y - p = 0 \quad \text{in} \quad Q_T, \\
    y = 0 \quad \text{on} \quad \Sigma_T, \\
    y(x, 0) = y_0(x) \quad \text{in} \quad \Omega.
\end{cases}
\]

($y_0, f) \in L^2(\Omega) \times L^2(Q_T) \implies p \in L^2(Q_T), y \in L^2(0, T, H^1_0(\Omega)), y_t \in L^2(0, T, H^{-1}(\Omega))$

- **Inverse Problem**: Distributed observation on $q_T = \omega \times (0, T), \omega \subset \Omega$

\[
\begin{cases}
    X = L^2(q_T), \\
    \text{Given} \ (y_{obs}, f) \in (L^2(q_T), X), \text{find} \ (y, p) \ s.t. \ \{(57) \text{ and } y - y_{obs} = 0 \ \text{on} \ q_T\}
\end{cases}
\]
Parabolic case: $H^1_0 - L^2$ first order formulation of the parabolic

The extremal problem is then:

$$\begin{align*}
\text{Minimize} & \quad J(y, p) := \frac{1}{2} \int_0^T \int_{Q_T} \rho_0^{-2} |y(x, t) - y_{obs}(x, t)|^2 \, dx \, dt + r. \\
(y, p) \in \mathcal{V} & := \left\{ (y, p) \in \mathcal{U} : \rho_1^{-1} J(y, p) = 0 \text{ in } L^2(\Omega_T), \quad \rho_1^{-1} \mathcal{I}(y, p) = 0 \text{ in } L^2(\Omega_T) \right\}
\end{align*}$$

$\mathcal{U}$ - completion of $\mathcal{U}_0 := \left\{ (y, p) \in C^1(\overline{\Omega_T}) \times C^1(\overline{\Omega_T}) : y = 0 \text{ on } \Sigma_T \right\}$ for

$$
((y, p), (\overline{y}, \overline{p}))_{\mathcal{U}_0} = \int_0^T \int_{Q_T} \rho_0^{-2} y \overline{y} \, dx \, dt + \eta_1 \int_0^T \int_{Q_T} \rho_1^{-1} J(y, p) \cdot \mathcal{J}(\overline{y}, \overline{p}) \, dx \, dt + \eta_2 \int_0^T \int_{Q_T} \rho_1^{-2} \mathcal{I}(y, p) \mathcal{I}(\overline{y}, \overline{p}) \, dx \, dt \quad \forall (y, p), (\overline{y}, \overline{p}) \in \mathcal{U}_0.
$$

for any $\eta_1, \eta_2 > 0$ and any $\rho, \rho_0, \rho_1 \in \mathcal{R}$

$$
\|(y, p)\|^2_{\mathcal{U}} := \|\rho_0^{-1} y\|^2_{L^2(\Omega_T)} + \eta_1 \|\rho_1^{-1} J(y, p)\|^2_{L^2(\Omega_T)} + \eta_2 \|\rho_1^{-1} \mathcal{I}(y, p)\|^2_{L^2(\Omega_T)}.
$$
Parabolic case: $H^1_0 - L^2$ first order formulation of the parabolic

The extremal problem is then:

$$
\begin{align*}
\text{Minimize } & J(y, p) := \frac{1}{2} \iint_{Q_T} \rho_0^{-2} |y(x, t) - y_{obs}(x, t)|^2 \, dx \, dt + r. \\
\text{subject to } & (y, p) \in \nu := \left\{ (y, p) \in \mathcal{U} : \rho_1^{-1} \mathcal{J}(y, p) = 0 \text{ in } L^2(Q_T), \quad \rho_1^{-1} \mathcal{I}(y, p) = 0 \text{ in } L^2(Q_T) \right\}
\end{align*}
$$

$\mathcal{U}$ - completion of $\mathcal{U}_0 := \left\{ (y, p) \in C^1(\overline{Q}_T) \times C^1(\overline{Q}_T) : y = 0 \text{ on } \Sigma_T \right\}$ for

$$
((y, p), (\bar{y}, \bar{p}))_{\mathcal{U}_0} = \iint_{Q_T} \rho_0^{-2} y \bar{y} \, dx \, dt + \eta_1 \iint_{Q_T} \rho_1^{-2} \mathcal{J}(y, p) \cdot \mathcal{J}(\bar{y}, \bar{p}) \, dx \, dt \\
+ \eta_2 \iint_{Q_T} \rho^{-2} \mathcal{I}(y, p) \mathcal{I}(\bar{y}, \bar{p}) \, dx \, dt \quad \forall (y, p), (\bar{y}, \bar{p}) \in \mathcal{U}_0.
$$

for any $\eta_1, \eta_2 > 0$ and any $\rho, \rho_0, \rho_1 \in \mathcal{R}$

$$
\| (y, p) \|^2_{\mathcal{U}} := \| \rho_0^{-1} y \|^2_{L^2(q_T)} + \eta_1 \| \rho_1^{-1} \mathcal{J}(y, p) \|^2_{L^2(Q_T)} + \eta_2 \| \rho^{-1} \mathcal{I}(y, p) \|^2_{L^2(Q_T)}.
$$
Parabolic case: $H^1_0 - L^2$ first order formulation - Mixed formulation

Precisely, we set $\mathcal{X} := L^2(Q_T) \times L^2(Q_T)$ and then we consider the following mixed formulation: find $((y, p), (\lambda, \mu)) \in \mathcal{U} \times \mathcal{X}$ solution of

$$
\begin{align*}
\left\{ \begin{array}{l}
ar_r((y, p), (\overline{y}, \overline{p})) + b((\overline{y}, \overline{p}), (\lambda, \mu)) = l(\overline{y}, \overline{p}) & \forall (\overline{y}, \overline{p}) \in \mathcal{U}, \\
b((y, p), (\overline{\lambda}, \overline{\mu})) = 0 & \forall (\overline{\lambda}, \overline{\mu}) \in \mathcal{X},
\end{array} \right.
\end{align*}
$$

(58)

where

$$
\begin{align*}
ar_r : \mathcal{U} \times \mathcal{U} &\rightarrow \mathbb{R}, \quad ar_r((y, p), (\overline{y}, \overline{p})) := \iint_{Q_T} \rho_0^{-2} y \overline{y} \, dx \, dt \\
+ r_1 \iint_{Q_T} \rho_1^{-2} \mathcal{J}(y, p) \cdot \mathcal{J}(\overline{y}, \overline{p}) \, dx \, dt + r_2 \iint_{Q_T} \rho^{-2} \mathcal{I}(y, p) \mathcal{I}(\overline{y}, \overline{p}) \, dx \, dt
\end{align*}
$$

$$
\begin{align*}
b : \mathcal{U} \times \mathcal{X} &\rightarrow \mathbb{R}, \quad b((y, p), (\lambda, \mu)) := \iint_{Q_T} \rho_1^{-1} \mathcal{J}(y, p) \cdot \mu \, dx \, dt + \iint_{Q_T} \rho^{-1} \mathcal{I}(y, p) \lambda \, dx \, dt
\end{align*}
$$

$$
\begin{align*}
l : \mathcal{U} &\rightarrow \mathbb{R}, \quad l(y, p) := \iint_{Q_T} \rho_0^{-2} y y_{obs} \, dx \, dt.
\end{align*}
$$

$\forall \mathbf{r} = (r_1, r_2) \in (\mathbb{R}^+)^2$
Proposition (Imanuvilov-Puel-Yamamoto, 2010)

\[ \rho_p(x, t) := \exp\left(\frac{\beta(x)}{t^2}\right), \quad \beta(x) := K_1 \left(e^{K_2} - e^{\beta_0(x)}\right), \]
\[ \rho_{p,0}(x, t) := t\rho_p(x, t), \quad \rho_{p,1}(x, t) := t^{-1}\rho_p(x, t), \quad \rho_{p,2}(x, t) := t^{-2}\rho_p(x, t) \]

\[ \exists C = C(\omega, T) > 0 \text{ s.t.} \]
\[ \|\rho_{p,0}^{-1}y\|_{L^2(Q_T)}^2 + \|\rho_{p,1}^{-1}\nabla y\|_{L^2(Q_T)}^2 \leq C \left(\|\rho_{p}^{-1}G\|_{L^2(Q_T)}^2 + \|\rho_{p,2}^{-1}g\|_{L^2(Q_T)}^2 + \|\rho_{p,0}^{-1}y\|_{L^2(q_T)}^2\right), \]

for any

\[ \begin{cases} y \in \mathcal{K} := \left\{ y \in L^2(0, T; H^1_0(\Omega)) : y_t \in L^2(0, T; H^{-1}(\Omega)) \right\}, \\
Ly = g + \nabla \cdot G \text{ in } Q_T, \quad (g, G) \in L^2(Q_T) \times L^2(Q_T). \end{cases} \]

\[ \begin{cases} Ly = \mathcal{I}(y, p) - \nabla \cdot \mathcal{J}(y, p), \\
\mathcal{J}(y, p) := c(x)\nabla y - p, \quad \mathcal{I}(y, p) := y_t - \nabla \cdot p + dy \end{cases} \]
$N = 1$ - Heat eq. Comparison with the standard method

$y_0(x) = \sin(\pi x)^{20}, \quad Q_T = (0, 1) \times (0, T), \quad q_T = (0.7, 0.8) \times (0, T), \quad T = 1/2$

$$\min_{y_0h} \left( J_h(y_0h) + \frac{h^2}{2} \|y_0h\|_{L^2(\Omega)}^2 \right) \quad \text{vs.} \quad \min_{\lambda_h} J^{**}(\lambda_h) \quad \text{over} \quad \Lambda_h \quad (59)$$

$$\frac{\|y - y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}} \approx 5.86 \times 10^{-2}, \quad \frac{\|y - y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}} \approx 7.70 \times 10^{-2}$$
$N = 1$ - Comparison with the standard method

\[ y_0(x) = \sin(\pi x)^{20}, \quad Q_T = (0, 1) \times (0, T), \quad q_T = (0.7, 0.8) \times (0, T), \quad T = 1/2 \]

Restriction at \((0, 1) \times \{0\}\)
\( N = 1 \) - Comparison with the standard method

![Graph showing the evolution of the relative residual \( \| g^k \| / \| g^0 \| \) w.r.t. iterate \( k \).]
THE VARIATIONAL APPROACH CAN BE USED IN THE CONTEXT OF MANY OTHER CONTROLLABLE SYSTEMS FOR WHICH APPROPRIATE CARLEMAN ESTIMATES ARE AVAILABLE.

THE APPROXIMATION IS ROBUST (WE HAVE TO INVERSE SYMMETRIC DEFINITE POSITIVE AND VERY SPARSE MATRICE WITH DIRECT LU AND CHOLESKY SOLVERS)

WITH CONFORMAL TIME-SPACE FINITE ELEMENTS APPROXIMATIONS, WE OBTAIN STRONG CONVERGENCE RESULTS WITH RESPECT TO $h = (\Delta x, \Delta t)$.

THE PRICE TO PAY IS TO USED $C^1$ FINITE ELEMENTS (AT LEAST IN SPACE) ..... BUT WE MAY INTRODUCE LOWER ORDER SYSTEM.


THIS APPROACH MAY BE APPLIED FOR INVERSE PROBLEMS, OBSERVATION PROBLEMS, RECONSTRUCTION OF DATA, ....
Final comments - Application

▶ Approximation of observability constant

\[ C_{obs}(T, \omega) = \sup_{\varphi_T \in H_0^1(\Omega)} \frac{\| \varphi(\cdot, 0) \|_{L^2(\Omega)}^2}{\| \varphi \|_{L^2(\omega \times (0, T))}^2}. \] (60)

Example: \( y_t - \epsilon y_{xx} + y_x = 0, \ y(0, t) = \varphi_\epsilon, \ \epsilon > 0 \)

▶ Optimization of the support of the control

\[ \inf_{\omega \subset \Omega, |\omega| = L|\Omega|} \| \varphi_\omega \|_{L^2(\omega \times (0, T))} \leq L \in (0, 1) \] (61)
Final comments - Application

- Approximation of observability constant

\[ C_{obs}(T, \omega) = \sup_{\varphi \in H^1_0(\Omega)} \frac{\|\varphi(\cdot, 0)\|_{L^2(\omega \times (0, T))}^2}{\|\varphi\|_{L^2(\omega)}^2}. \]  

Example: \( y_t - \epsilon y_{xx} + y_x = 0, \ y(0, t) = \nu_\epsilon, \ \epsilon > 0 \)

- Optimization of the support of the control

\[ \inf_{\omega \subset \Omega, |\omega| = L|\Omega|} \|\nu_\omega\|_{L^2(\omega \times (0, T))}, \quad L \in (0, 1) \]
Approximation of observability constant
\[
C_{\text{obs}}(T, \omega) = \sup_{\varphi_T \in H_0^1(\Omega)} \frac{\|\varphi(\cdot, 0)\|^2_{L^2(\Omega)}}{\|\varphi\|^2_{L^2(\omega \times (0, T))}}. \tag{60}
\]

Example: \(y_t - \epsilon y_{xx} + y_x = 0, \quad y(0, t) = v_\epsilon, \quad \epsilon > 0\)

Optimization of the support of the control
\[
\inf_{\omega \subset \Omega, |\omega| = L|\Omega|} \|v_\omega\|_{L^2(\omega \times (0, T))}, \quad L \in (0, 1) \tag{61}
\]