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Postdoctoral Position

Constructive approximation of controls for nonlinear partial differential equations and applications

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Location: Laboratoire de Mathématiques Blaise Pascal (http://recherche.math.univ-bpclermont.fr/)
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Salary: The net salary is approximatively 2100 euros. Funds from the Challenge 2 of the I-Site project.
Starting date: between September 1st, 2021 and January 1st, 2022.
Duration: 12 months.
Required qualification: Applicants should have a PhD in Applied Mathematics (automatic control theory, dynamical systems, partial differential equations) or in Scientific computing.
General topics: Partial differential equations, Control theory, Numerical approximation.
Mobility: The position includes travel facilities.

Scientific context. The controllability of distributed parameter systems has attracted the interest of many researchers and important progress have been made during the last decades. This topic can be briefly summarized as follows: let X be a Banach space, t the time variable and $\mathbf{x} = \mathbf{x}(t) : [0, T] \to X$ a solution of the following dynamical system :

$$\begin{cases} \dot{\mathbf{x}}(t;\xi) = A_\xi(\mathbf{x}(t;\xi)) + C_\xi \mathbf{u}(t) + f_\xi(t), \quad t > 0, \\ \mathbf{x}(0;\xi) = \mathbf{x}_{0,\xi}, \end{cases} \tag{1}$$

starting at the point $\mathbf{x}_{0,\xi}$. The main issue in control theory is to act on the system through an additional control function $\mathbf{u}(t)$ such that the state $\mathbf{x}(t)$ reaches a given target $\mathbf{x_T}$ at a final controllability time T; that is $\|\mathbf{x}(T,\xi) - \mathbf{x_T}\|_X = 0$. This scheme is illustrated in Figure 1. ξ denotes a random variable that takes into account the uncertainties on the various data of the system. We refer to [1] in the deterministic case for an overview of nonlinear controllability problems involving partial differential equations and systems.

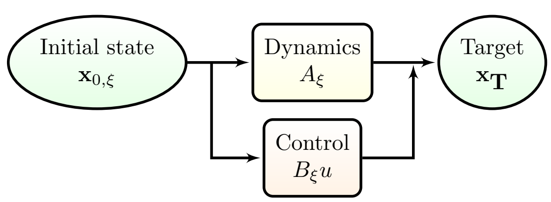

Figure 1: Schematic view of a controlled system.

Given a controllable nonlinear system of the form (1) uniformly with respect to the initial and terminal conditions, the primary goal of the position is to construct a sequence $(\mathbf{x}_k, u_k)_{k\in\mathbb{N}}$ of functions converging strongly to a controlled pair for (1).

For partial differential equations, this issue remains largely open since in many situations controllability results are obtained using nonconstructive fixed point arguments. This is notably the case for the semilinear wave equation

$$
\begin{cases} \partial_{tt} y - \Delta y + g(y) = f 1_\omega, & \text{in } \Omega \times (0,T), \\ y = 0, & \text{on } \partial\Omega \times (0,T), \\ (y(\cdot,0), y_t(\cdot,0)) = (u_0, u_1), & \text{in } \Omega, \end{cases} \tag{2}
$$

posed over $\Omega \times (0,T)$ where Ω is bounded and smooth domain of $\mathbb{R}^d$, $d \in \mathbb{N}$. Assuming a growth condition at infinity on the nonlinear function g, precisely that $\limsup_{|r|\to\infty} \frac{|g(r)|}{|r| \ln^{1/2} |r|} = 0$, the Leray-Schauder theorem applied to a linearization of (2) leads to the existence of control functions f, acting on the subset ω such that the corresponding solution is driven to rest at a time T large enough, i.e $(y(\cdot,T), \partial_t y(\cdot,T)) = (0,0)$. We refer to [2] for $d = 1$ and to [3] for $d > 1$. In the multidimensional case, the subset ω acts on a small neighborhood of $\partial\Omega$. Recent advances have been however obtained in the direction. In [4], [5], a convergent sequence to a controlled pair for (2) is explicitly constructed via a Least-squares method: the algorithm is based on the extremal problem

$$
\min_{(y,f)\in\mathcal{A}} E(y,f) = \|\partial_{tt} y - \Delta y + g(y) - f\, 1_\omega\|^2_{L^2(Q_T)}
$$

over a convex space $\mathcal{A}$ which incorporates the initial and final constraints on the state. A minimizing sequence $(y_k, f_k)_{k\geq 0}$ for E is shown to strongly converge with a super-linear rate toward a controlled pair (y, f) for (2) with the explicit structure

$$
(y,f) = (y_0, f_0) + \sum_{k\geq 1} \lambda_k (Y_k^1, F_k^1)
$$

where (y_0, f_0) and (Y_k^1, F_k^1) are controlled pairs for linear wave equations. Interestingly, the method is general and can be applied for instance to the parabolic case: we refer to [6,7] devoted to semi-linear heat equations and to [8,9] devoted to the direct problem for the Navier-Stokes equations.

Research project. The main objective is, both from a theoretical and numerical viewpoint, to

- extend these recents results to other equations and systems involving notably nonlinearities depending on the gradient of the state;
- compare the underlying linearization strategy (equivalent to the one obtained from the damped Newton method) with other simpler linearizations;
- combine a fixed point method and continuation techniques for the nonlinear terms;
- analyze the influence of uncertainties on the efficiency of the various methods;
- apply the least-square approach to real life applications coming from traffic flows and fluid flows.

This program will be adapted according to the experiences and preferences of the candidate.

Interested person may contact and send by email a CV to Arnaud Münch. The position will remain open until filled.

Bibliography

[1] J.M. Coron. *Control and nonlinearity.* Mathematical Surveys and Monographs. 136. 2007.

[2] E. Zuazua. *Exact controllability for semilinear wave equations in one space dimension.* Ann. Inst. H. Poincaré Anal. Non Linéaire. 1993

[3] X. Fu, J. Yong, X. Zhang. *Exact controllability for multidimensional semilinear hyperbolic equations*, SICON 2007.

[4] A. Münch, E. Trelat, *Constructive exact control of semilinear 1D wave equations by a least-squares approach*, arXiv:2011.08462

[5] A. Bottois, J. Lemoine, A. Münch. *Constructive exact controls for semilinear wave equations.* arXiv:2101.06446.

[6] I. Gayte-Marin, J.Lemoine, A. Münch. *Approximation of null controls for semilinear heat equations using a least-squares approach.* ESAIM:COCV.

[7] J. Lemoine, A. Münch, *Constructive exact control of semilinear 1D heat equations.* arXiv:2103.09640.

[8] J. Lemoine, A. Münch, *Resolution of the implicit Euler scheme for the Navier-Stokes equation through a least-squares method.* Numerische Mathematik. 2021

[9] J. Lemoine, A. Münch, *A fully space-time least-squares method for the unsteady Navier-Stokes system.* J. Mathematical Fluid Mechanics.