

# Some recent results on controllability of coupled parabolic systems: Towards a Kalman condition

F. Ammar Khodja

Clermont-Ferrand, June 2011

## GOAL:

- 1 Show the important differences between scalar and non scalar problems.
- 2 Give necessary and sufficient conditions (*Kalman condition*) which characterize the controllability properties of these systems.

We will only deal with

“Simple” Parabolic Systems: **Coupling Matrices of Constant Coefficients.**

- 1 The parabolic scalar case: The heat equation
- 2 Finite-dimensional systems
- 3 Two simple examples
  - Distributed null controllability of a linear reaction-diffusion system
  - Boundary null controllability of a linear reaction-diffusion system
- 4 The Kalman condition for a class of parabolic systems. Distributed controls
- 5 The Kalman condition for a class of parabolic systems. Boundary controls

# 1. The parabolic scalar case: The heat equation

Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain,  $N \geq 1$ , with boundary  $\partial\Omega$  of class  $C^2$ . Let  $\omega \subseteq \Omega$  be an open subset,  $\gamma \subseteq \partial\Omega$  a relative open subset and let us fix  $T > 0$ .

# 1. The parabolic scalar case: The heat equation

Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain,  $N \geq 1$ , with boundary  $\partial\Omega$  of class  $C^2$ . Let  $\omega \subseteq \Omega$  be an open subset,  $\gamma \subseteq \partial\Omega$  a relative open subset and let us fix  $T > 0$ . We consider the **linear** problems for the **heat equation**:

$$(1) \quad \left\{ \begin{array}{ll} \partial_t y - \Delta y = v 1_\omega & \text{in } Q = \Omega \times (0, T), \\ y = 0 & \text{on } \Sigma = \partial\Omega \times (0, T), \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \end{array} \right.$$

$$(2) \quad \left\{ \begin{array}{ll} \partial_t y - \Delta y = 0 & \text{in } Q, \\ y = v 1_\gamma & \text{on } \Sigma, \quad y(\cdot, 0) = y_0 \quad \text{in } \Omega. \end{array} \right.$$

# 1. The parabolic scalar case: The heat equation

Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain,  $N \geq 1$ , with boundary  $\partial\Omega$  of class  $C^2$ . Let  $\omega \subseteq \Omega$  be an open subset,  $\gamma \subseteq \partial\Omega$  a relative open subset and let us fix  $T > 0$ . We consider the **linear** problems for the **heat equation**:

$$(1) \quad \left\{ \begin{array}{ll} \partial_t y - \Delta y = v 1_\omega & \text{in } Q = \Omega \times (0, T), \\ y = 0 & \text{on } \Sigma = \partial\Omega \times (0, T), \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \end{array} \right.$$

$$(2) \quad \left\{ \begin{array}{ll} \partial_t y - \Delta y = 0 & \text{in } Q, \\ y = v 1_\gamma & \text{on } \Sigma, \\ y(\cdot, 0) = y_0 & \text{in } \Omega. \end{array} \right.$$

In (1) and (2),  $1_\omega$  and  $1_\gamma$  represent resp. the characteristic function of the sets  $\omega$  and  $\gamma$ ,  $y(x, t)$  is the state,  $y_0$  is the **initial datum** and is given in an appropriate space, and  $v$  is the control function (which is localized in  $\omega$  -**distributed control**- or in  $\gamma$  -**boundary control**-).

# 1. The parabolic scalar case: The heat equation

## Theorem (**Distributed Controllability Results**)

Fix  $\omega \subseteq \Omega$  and  $T > 0$ . Then,

- 1 System (1) is **approximately controllable** at time  $T$ , i.e., for any  $\varepsilon > 0$  and  $y_0, y_d \in L^2(\Omega)$  there is  $v \in L^2(Q)$  s.t. the solution  $y$  to (1) satisfies

$$\|y(\cdot, T) - y_d\|_{L^2(\Omega)} \leq \varepsilon.$$

- 2 System (1) is **null controllable** at time  $T$ , i.e., for any  $y_0 \in L^2(\Omega)$  there is  $v \in L^2(Q)$  s.t. the solution  $y$  to (1) satisfies

$$y(\cdot, T) \equiv 0 \text{ in } \Omega. \quad \blacksquare$$

# 1. The parabolic scalar case: The heat equation

## Theorem (**Distributed Controllability Results**)

Fix  $\omega \subseteq \Omega$  and  $T > 0$ . Then,

- 1 System (1) is **approximately controllable** at time  $T$ , i.e., for any  $\varepsilon > 0$  and  $y_0, y_d \in L^2(\Omega)$  there is  $v \in L^2(Q)$  s.t. the solution  $y$  to (1) satisfies

$$\|y(\cdot, T) - y_d\|_{L^2(\Omega)} \leq \varepsilon.$$

- 2 System (1) is **null controllable** at time  $T$ , i.e., for any  $y_0 \in L^2(\Omega)$  there is  $v \in L^2(Q)$  s.t. the solution  $y$  to (1) satisfies

$$y(\cdot, T) \equiv 0 \text{ in } \Omega. \quad \blacksquare$$

## Remark

System (1) is **null controllable** at time  $T$  **if and only if** system (1) is **exactly controllable to the trajectories** at time  $T$ : for every trajectory  $y^*$  of (1) (a solution to (1) associated to  $y_0^* \in L^2(\Omega)$ ) there exists  $v \in L^2(Q)$  such that  $y(\cdot, T) \equiv y^*(\cdot, T)$  in  $\Omega$ . \(\blacksquare\)



# 1. The parabolic scalar case: The heat equation

**Adjoint Problem:** Let us fix  $\varphi_0 \in L^2(\Omega)$  and consider the *adjoint problem*

$$(3) \quad \boxed{\begin{cases} \partial_t \varphi + \Delta \varphi = 0 & \text{in } Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(T) = \varphi_0 & \text{in } \Omega. \end{cases}}$$

# 1. The parabolic scalar case: The heat equation

**Adjoint Problem:** Let us fix  $\varphi_0 \in L^2(\Omega)$  and consider the *adjoint problem*

$$(3) \quad \boxed{\begin{cases} \partial_t \varphi + \Delta \varphi = 0 & \text{in } Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(T) = \varphi_0 & \text{in } \Omega. \end{cases}}$$

It is well known:

## Theorem

System (1) is exactly controllable to trajectories at time  $T$  *if and only if* there exists  $C > 0$  s.t. (*observability inequality*)

$$\|\varphi(0)\|_{L^2(\Omega)}^2 \leq C \iint_{\omega \times (0,T)} |\varphi(x,t)|^2 dx dt,$$

holds for every solution  $\varphi$  to the *adjoint problem* (3) associated to  $\varphi_0 \in L^2(\Omega)$ . ■

# 1. The parabolic scalar case: The heat equation

## FOUR IMPORTANT REFERENCES

- 1 **H.O. FATTORINI, D.L. RUSSELL**, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal. 43 (1971), 272–292.
- 2 **G. LEBEAU, L. ROBBIANO**, *Contrôle exact de l'équation de la chaleur*, Comm. P.D.E. 20 (1995), no. 1-2, 335–356.
- 3 **O. YU. IMANUVILOV**, *Controllability of parabolic equations*, (Russian) Mat. Sb. 186 (1995), no. 6, 109–132; translation in Sb. Math. 186 (1995), no. 6, 879–900.
- 4 **A. FURSIKOV, O. YU. IMANUVILOV**, *Controllability of Evolution Equations*, Lecture Notes Series 34, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

# 1. The parabolic scalar case: The heat equation

## Boundary Controllability Result:

### Theorem

Let  $\gamma \subseteq \partial\Omega$  and  $T > 0$  be given. Then, for any  $y_0 \in H^{-1}(\Omega)$  there exists  $v \in L^2(\Sigma)$  s.t. the solution  $y$  to (2) satisfies

$$y(\cdot, T) \equiv 0 \text{ in } \Omega.$$

**Proof:** It is a consequence of the distributed controllability result. ■

# 1. The parabolic scalar case: The heat equation

## Boundary Controllability Result:

### Theorem

Let  $\gamma \subseteq \partial\Omega$  and  $T > 0$  be given. Then, for any  $y_0 \in H^{-1}(\Omega)$  there exists  $v \in L^2(\Sigma)$  s.t. the solution  $y$  to (2) satisfies

$$y(\cdot, T) \equiv 0 \text{ in } \Omega.$$

**Proof:** It is a consequence of the distributed controllability result. ■

### Important:

Distributed controllability result for system (1) **is equivalent** to the boundary controllability result for system (2).

# 1. The parabolic scalar case: The heat equation

## Boundary Controllability Result:

### Theorem

System (2) is exactly controllable to trajectories at time  $T$  *if and only if* there exists  $C > 0$  s.t. (*observability inequality*)

$$\|\varphi(0)\|_{H_0^1(\Omega)}^2 \leq C \iint_{\gamma \times (0,T)} \left| \frac{\partial \varphi}{\partial n}(x,t) \right|^2,$$

holds for every solution  $\varphi$  to the *adjoint problem* (3) associated to  $\varphi_0 \in H_0^1(\Omega)$  ( $n = n(x)$  is the outward normal unit vector at  $x \in \partial\Omega$ ). ■

# 1. The parabolic scalar case: The heat equation

## Boundary Controllability Result:

### Theorem

System (2) is exactly controllable to trajectories at time  $T$  **if and only if** there exists  $C > 0$  s.t. (**observability inequality**)

$$\|\varphi(0)\|_{H_0^1(\Omega)}^2 \leq C \iint_{\gamma \times (0,T)} \left| \frac{\partial \varphi}{\partial n}(x,t) \right|^2,$$

holds for every solution  $\varphi$  to the **adjoint problem** (3) associated to  $\varphi_0 \in H_0^1(\Omega)$  ( $n = n(x)$  is the outward normal unit vector at  $x \in \partial\Omega$ ). ■

### Summarizing:

- System (1) and system (2) are approximately controllable and exactly controllable to trajectories at time  $T$ .
- The controllability properties of both systems are equivalent. ■

## 2. Finite-dimensional systems

Let us consider the **autonomous linear system**

$$(4) \quad y' = Ay + Bu \quad \text{in } [0, T], \quad y(0) = y_0,$$

where  $A \in \mathcal{L}(\mathbb{C}^n)$  and  $B \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$  are constant matrices,  $y_0 \in \mathbb{C}^n$  and  $u \in L^2(0, T; \mathbb{C}^m)$  is the control.

**Problem:** Given  $y_0, y_d \in \mathbb{R}^n$ , is there a control  $u \in L^2(0, T; \mathbb{R}^m)$  such that the solution  $y$  to the problem satisfies

$$y(T) = y_d?$$



## 2. Finite-dimensional systems

Let us consider the **autonomous linear system**

$$(4) \quad y' = Ay + Bu \quad \text{in } [0, T], \quad y(0) = y_0,$$

where  $A \in \mathcal{L}(\mathbb{C}^n)$  and  $B \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$  are constant matrices,  $y_0 \in \mathbb{C}^n$  and  $u \in L^2(0, T; \mathbb{C}^m)$  is the control.

**Problem:** Given  $y_0, y_d \in \mathbb{R}^n$ , is there a control  $u \in L^2(0, T; \mathbb{R}^m)$  such that the solution  $y$  to the problem satisfies

$$y(T) = y_d?$$

Let us define (*controllability matrix*)

$$\boxed{[A \mid B] = [B \mid AB \mid A^2B \mid \cdots \mid A^{n-1}B] \in \mathcal{L}(\mathbb{C}^{nm}; \mathbb{C}^n).}$$

## 2. Finite-dimensional systems

The following classical result can be found in



R. Kalman, Y.-Ch. Ho, K. Narendra, *Controllability of linear dynamical systems*, 1963

and gives a complete answer to the problem of controllability of finite dimensional autonomous linear systems:

### Theorem

*Under the previous assumptions, the following conditions are equivalent*

- 1 System (4) is **exactly controllable** at time  $T$ , for every  $T > 0$ .
- 2 There exists  $T > 0$  such that system (4) is **exactly controllable** at time  $T$ .
- 3  $\text{rank}[A \mid B] = n$  (**Kalman rank condition**).
- 4  $\ker[A \mid B]^* = \{0\}$ . ■

## 2. Finite-dimensional systems

### Goal

We have a complete characterization of the controllability results for finite-dimensional linear differential systems (a **Kalman condition**). Is it possible to obtain similar results for PDE systems? We will focus on coupled linear **parabolic** systems.

## 2. Finite-dimensional systems

What are the possible generalizations to Systems of Parabolic Equations?

### 3. Two simple examples

#### 3.1 Distributed null controllability of a linear reaction-diffusion system

Let us consider the  $2 \times 2$  linear reaction-diffusion system

$$(5) \quad \begin{cases} y_t - D\Delta y = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} y + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v 1_\omega & \text{in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases}$$

Here  $\Omega$ ,  $\omega$  and  $T$  are as before,  $y_0 \in L^2(\Omega; \mathbb{R}^2)$  and

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0 \quad (A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}).$$

## 3. Two simple examples

### 3.1 Distributed null controllability of a linear reaction-diffusion system

Let us consider the  $2 \times 2$  linear reaction-diffusion system

$$(5) \quad \begin{cases} y_t - D\Delta y = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} y + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v \mathbf{1}_\omega & \text{in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases}$$

Here  $\Omega$ ,  $\omega$  and  $T$  are as before,  $y_0 \in L^2(\Omega; \mathbb{R}^2)$  and

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0 \quad (A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}).$$

One has

#### Theorem

System (5) is exactly controllable to trajectories at time  $T$  *if and only if*

$$\det [A \mid B] \neq 0 \iff a_{2,1} \neq 0.$$

## 3. Two simple examples

### 3.1 Distributed null controllability of a linear reaction-diffusion system

**Proof:**  $\Rightarrow$ : If  $a_{2,1} = 0$ , then  $y_2$  is independent of  $v$ .

### 3. Two simple examples

#### 3.1 Distributed null controllability of a linear reaction-diffusion system

**Proof:**  $\Rightarrow$ : If  $a_{2,1} = 0$ , then  $y_2$  is independent of  $v$ .

$\Leftarrow$ : The controllability result for system (5) is equivalent to the **observability inequality**:  $\exists C > 0$  such that

$$\|\varphi_1(\cdot, 0)\|_{L^2}^2 + \|\varphi_2(\cdot, 0)\|_{L^2}^2 \leq C \iint_{\omega \times (0, T)} |\varphi_1(x, t)|^2 dx dt,$$

where  $\varphi$  is the solution associated to  $\varphi_0 \in L^2(\Omega; \mathbb{R}^2)$  of the **adjoint problem**:

$$(6) \quad \begin{cases} -\varphi_t - D\Delta\varphi = A^*\varphi & \text{in } Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(\cdot, T) = \varphi_0 & \text{in } \Omega. \end{cases}$$

It is a consequence of well known **global Carleman estimates** for parabolic equations.



### 3. Two simple examples

#### 3.1 Distributed null controllability of a linear reaction-diffusion system

##### Lemma

There exist a positive regular function,  $\alpha_0$ , and two positive constants  $C_0$  and  $\sigma_0$  (only depending on  $\Omega$  and  $\omega$ ) s.t.

$$\left\{ \begin{array}{l} \mathcal{I}(\phi) \equiv \iint_Q e^{-2s\alpha} [s\rho(t)]^{-1} (|\phi_t|^2 + |\Delta\phi|^2) \\ + \iint_Q e^{-2s\alpha} [s\rho(t)] |\nabla\phi|^2 + \iint_Q e^{-2s\alpha} [s\rho(t)]^3 |\phi|^2 \\ \leq C_0 \left( \iint_{\omega \times (0,T)} e^{-2s\alpha} [s\rho(t)]^3 |\phi|^2 + \iint_Q e^{-2s\alpha} |\phi_t \pm \Delta\phi|^2 \right), \end{array} \right.$$

$\forall s \geq s_0 = \sigma_0(\Omega, \omega)(T + T^2)$  and  $\phi \in L^2(0, T; H_0^1(\Omega))$  s.t.  $\phi_t \pm \Delta\phi \in L^2(Q)$ .  
The functions  $\rho(t)$  and  $\alpha = \alpha(x, t)$  are given by

$$\rho(t) = [t(T - t)]^{-1}, \quad \alpha(x, t) = \alpha_0(x)/t(T - t). \quad \blacksquare$$

## 3. Two simple examples

### 3.1 Distributed null controllability of a linear reaction-diffusion system

Coming back to the **adjoint problem** for system (6), if we apply to  $\phi = \varphi_1$  and  $\phi = \varphi_2$  the previous inequality in  $\omega_0 \subset\subset \omega$ . After some computations we get

$$\mathcal{I}(\varphi_1) + \mathcal{I}(\varphi_2) \leq C_1 s^3 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} [t(T-t)]^{-3} (|\varphi_1|^2 + |\varphi_2|^2),$$

$$\forall s \geq s_1 = \sigma_1(\Omega, \omega_0)(T + T^2).$$

## 3. Two simple examples

### 3.1 Distributed null controllability of a linear reaction-diffusion system

Coming back to the **adjoint problem** for system (6), if we apply to  $\phi = \varphi_1$  and  $\phi = \varphi_2$  the previous inequality in  $\omega_0 \subset\subset \omega$ . After some computations we get

$$\mathcal{I}(\varphi_1) + \mathcal{I}(\varphi_2) \leq C_1 s^3 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} [t(T-t)]^{-3} (|\varphi_1|^2 + |\varphi_2|^2),$$

$$\forall s \geq s_1 = \sigma_1(\Omega, \omega_0)(T + T^2).$$

We now use the first equation in (6),  $a_{2,1}\varphi_2 = -(\varphi_{1,t} + \Delta\varphi_1 + a_{1,1}\varphi_1)$ , to prove ( $\varepsilon > 0$ ):

$$\begin{aligned} s^3 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} [t(T-t)]^{-3} |\varphi_2|^2 &\leq \varepsilon \mathcal{I}(\varphi_2) \\ &+ \frac{C_2}{\varepsilon} s^7 \iint_{\omega \times (0, T)} e^{-2s\alpha} [t(T-t)]^{-7} |\varphi_1|^2. \end{aligned}$$

$$\forall s \geq s_1 = \sigma_1(\Omega, \omega_0)(T + T^2).$$

## 3. Two simple examples

### 3.1 Distributed null controllability of a linear reaction-diffusion system

From the two previous inequalities (**global Carleman estimate**)

$$\mathcal{I}(\varphi_1) + \mathcal{I}(\varphi_2) \leq C_2 s^7 \iint_{\omega \times (0, T)} e^{-2s\alpha} [t(T-t)]^{-7} |\varphi_1|^2,$$

$\forall s \geq s_1 = \sigma_1(\Omega, \omega_0)(T + T^2)$ . Combining this inequality and **energy estimates** for system (6) we deduce the desired **observability inequality**.

## 3. Two simple examples

### 3.1 Distributed null controllability of a linear reaction-diffusion system

#### Remark

- *System (5) is always controllable if we exert a control in each equation (two controls).*
- *The controllability result for system (5) is independent of the diffusion matrix  $D$ . We will see that the situation is more intricate if in the system a general control vector  $B \in \mathbb{R}^2$  is considered.*
- *The same result can be obtained for the approximate controllability at time  $T$ . Therefore, **approximate** and **null controllability** are equivalent concepts.*

## 3. Two simple examples

### 3.1 Distributed null controllability of a linear reaction-diffusion system

#### References

- 1 L. DE TERESA, *Insensitizing controls for a semilinear heat equation*, Comm. Partial Differential Equations 25 (2000), no. 1–2, 39–72.
- 2 F. AK, A. BENABDALLAH, C. DUPAIX ET I. KOSTIN, *Controllability to the trajectories of phase-field models by one control force*, SIAM J. Control Optim. 42 (2003), no. 5, 1661–1689.
- 3 M. GONZÁLEZ-BURGOS, R. PÉREZ-GARCÍA, *Controllability results for some nonlinear coupled parabolic systems by one control force*, Asymptot. Anal. 46 (2006), no. 2, 123–162.
- 4 M. GONZÁLEZ-BURGOS, L. DE TERESA, *Controllability results for cascade systems of  $m$  coupled parabolic PDEs by one control force*, Port. Math. 67 (2010), no. 1, 91–113.

### 3. Two simple examples

#### 3.2 Boundary null controllability of a linear reaction-diffusion system

Let us now consider the boundary controllability problem for the one-dimensional linear reaction-diffusion system:

$$(7) \quad \begin{cases} y_t - Dy_{xx} = Ay & \text{in } Q = (0, \pi) \times (0, T), \\ y|_{x=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} v, \quad y|_{x=1} = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

with  $y_0 \in H^{-1}(0, \pi; \mathbb{R}^2)$ ,  $v \in L^2(0, T)$  is the control and

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0 \quad \boxed{(d_1 \neq d_2)}, \quad \text{and } A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

#### Question

Are the controllability properties of system (7) independent of  $d_1$  and  $d_2$ ???

**NO.**

### 3. Two simple examples

#### 3.2 Boundary null controllability of a linear reaction-diffusion system

As before, system (7) is **null controllable** at time  $T$  if and only if the **observability inequality**

$$\|\varphi_1(\cdot, 0)\|_{H_0^1(0,\pi)}^2 + \|\varphi_2(\cdot, 0)\|_{H_0^1(0,\pi)}^2 \leq C \int_0^T |\varphi_{1,x}(0, t)|^2 dt,$$

holds. Again  $\varphi$  is the solution associated with  $\varphi_0 \in H_0^1(0, \pi; \mathbb{R}^2)$  of the **adjoint problem**:

$$(8) \quad \begin{cases} -\varphi_t - D\varphi_{xx} = A^*\varphi & \text{in } Q, \\ \varphi|_{x=0} = \varphi|_{x=1} = 0 & \text{on } (0, T), \\ \varphi(\cdot, T) = \varphi_0 & \text{in } (0, \pi). \end{cases}$$

Let us see that, in general, this inequality fails (**even if**  $a_{2,1} = 1 \neq 0!$ ).



### 3. Two simple examples

#### 3.2 Boundary null controllability of a linear reaction-diffusion system

#### A necessary condition:

##### Proposition

Assume that system (7) is null controllable at time  $T$ . Then  $(\lambda_k = k^2)$ ,

$$d_1 \lambda_k \neq d_2 \lambda_j, \quad \forall k, j \geq 1 \quad (\iff \sqrt{d_1/d_2} \notin \mathbb{Q}).$$

**Proof:** By contradiction, assume that  $d_1 \lambda_k = d_2 \lambda_j$  for some  $k, j$  and take  $K = \max\{k, j\}$ . The idea is transforming system (8) into an o.d. system.

### 3. Two simple examples

#### 3.2 Boundary null controllability of a linear reaction-diffusion system

##### A necessary condition:

###### Proposition

Assume that system (7) is null controllable at time  $T$ . Then  $(\lambda_k = k^2)$ ,

$$d_1 \lambda_k \neq d_2 \lambda_j, \quad \forall k, j \geq 1 \quad (\iff \sqrt{d_1/d_2} \notin \mathbb{Q}).$$

**Proof:** By contradiction, assume that  $d_1 \lambda_k = d_2 \lambda_j$  for some  $k, j$  and take  $K = \max\{k, j\}$ . The idea is transforming system (8) into an o.d. system. Let us consider the sequence of eigenvalues and normalized eigenfunctions of  $-\partial_{xx}$  on  $(0, \pi)$  with homogenous Dirichlet boundary conditions:

$$\lambda_k = k^2, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, \quad k \geq 1, \quad x \in (0, \pi).$$

**Idea:** Take  $\varphi_0 \in X_K = \{\varphi_0 = \sum_{\ell=1}^K a_\ell \phi_\ell : a_\ell \in \mathbb{R}^2\} \subset H_0^1(0, \pi; \mathbb{R}^2)$ .

### 3. Two simple examples

#### 3.2 Boundary null controllability of a linear reaction-diffusion system

Consider also

$$B_K = \begin{pmatrix} B \\ \vdots \\ B \end{pmatrix} \in \mathbb{R}^{2K}, \quad (B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}) \quad \text{and}$$

$$\mathcal{L}_K^* = \text{diag} (-\lambda_1 D + A^*, -\lambda_2 D + A^*, \dots, -\lambda_K D + A^*) \in \mathcal{L}(\mathbb{R}^{2K}).$$

Taking in (8) arbitrary initial data  $\varphi_{0,K} = \sum_{\ell=1}^K a_\ell \phi_\ell \in H_0^1(0, \pi; \mathbb{R}^2)$  where  $a_\ell \in \mathbb{R}^2$ , it is not difficult to see that system (8) is equivalent to the o.d. system

$$(9) \quad -Z' = \mathcal{L}_K^* Z \quad \text{on } [0, T], \quad Z(0) = Z_0 \in \mathbb{R}^{2K}.$$

From the **observability inequality** for system (8) we deduce the **unique continuation property** for the solutions to (9):

$$\boxed{B_K^* Z(\cdot) = 0 \quad \text{in } (0, T) \implies Z \equiv 0.}$$

### 3. Two simple examples

#### 3.2 Boundary null controllability of a linear reaction-diffusion system

In particular system

$$Y' = \mathcal{L}_K Y + B_K v \quad \text{on } [0, T], \quad Y(0) = Y_0 \in \mathbb{R}^{2K}.$$

is exactly controllable at time  $T$ . Then

$$\text{rank} [\mathcal{L}_K \mid B_K] = 2K.$$

We deduce that  $\mathcal{L}_K^*$  cannot have eigenvalues with **geometric multiplicity** 2 or greater.

But  $\theta = -d_1 \lambda_k = -d_2 \lambda_j$  is an eigenvalue of  $\mathcal{L}_K^*$  with two linearly independent eigenvectors  $V_1, V_2 \in \mathbb{R}^{2K}$  given by:

$$\begin{cases} V_1 = (V_{1,\ell})_{1 \leq \ell \leq K}, & V_{1,k} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } V_{1,\ell} = 0 \quad \forall \ell \neq k, \\ V_2 = (V_{2,\ell})_{1 \leq \ell \leq K}, & V_{2,j} = \begin{pmatrix} \frac{1}{\lambda_j(d_1 - d_2)} \\ 0 \end{pmatrix} \text{ and } V_{2,\ell} = 0 \quad \forall \ell \neq j. \blacksquare \end{cases}$$

### 3. Two simple examples

#### 3.2 Boundary null controllability of a linear reaction-diffusion system

The result has been proved in

- E. FERNÁNDEZ-CARA, M. GONZÁLEZ-BURGOS, L. DE TERESA, *Boundary controllability of parabolic coupled equations*, J. Funct. Anal. 259 (2010), no. 7, 1720–1758.

#### Remark

- *Again, the system is always null controllable at time  $T$  if we exert **two controls**.*
- *In fact, system (7) is approximately controllable at time  $T$   $\iff$*

$$\sqrt{d_1/d_2} \notin \mathbb{Q}.$$

## 4. The Kalman condition for a class of parabolic systems. Distributed controls

Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain,  $N \geq 1$ , with boundary  $\partial\Omega$  of class  $C^2$ . Let  $\omega \subseteq \Omega$  be an open subset and let us fix  $T > 0$ .

For  $n, m \in \mathbb{N}$  we consider the following  $n \times n$  parabolic system

$$(10) \quad \begin{cases} \partial_t y - D\Delta y = Ay + Bv \mathbf{1}_\omega & \text{in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot, 0) = y_0(\cdot) & \text{in } \Omega, \end{cases}$$

with  $A \in \mathcal{L}(\mathbb{R}^n)$  and  $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$  are **constant** matrices  $y_0 \in L^2(\Omega; \mathbb{R}^n)$  and

$$D = \text{diag}(d_1, d_2, \dots, d_n) \in \mathcal{L}(\mathbb{R}^n), \quad (d_i > 0, \forall i).$$

$v \in L^2(Q; \mathbb{R}^m)$  is the control ( $m$  components).

### Remark

This problem is **well posed**: For any  $y_0 \in L^2(\Omega; \mathbb{R}^n)$  and  $v \in L^2(Q; \mathbb{R}^m)$ , problem (10) has a **unique solution**  $y \in L^2(0, T; H_0^1) \cap C^0([0, T]; L^2)$ . ■

## 4. The Kalman condition for a class of parabolic systems. Distributed controls

$$(10) \quad \begin{cases} \partial_t y - D\Delta y = Ay + Bv \mathbf{1}_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \quad y(\cdot, 0) = y_0(\cdot) & \text{in } \Omega. \end{cases}$$

### Remark

We want to control the whole system ( $n$  *equations*) with  $m$  *controls*. The most interesting case is  $m < n$  or even  $m = 1$ .

### *Difficulties:*

- 1 In general  $m < n$ .
- 2  $D$  is not the identity matrix.

## 4. The Kalman condition for a class of parabolic systems. Distributed controls

### The adjoint problem:

$$(11) \quad \begin{cases} -\partial_t \varphi = (D\Delta + A^*)\varphi & \text{in } Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(\cdot, T) = \varphi_0 & \text{in } \Omega, \end{cases}$$

where  $\varphi_0 \in L^2(\Omega; \mathbb{R}^n)$ . Then, the **exact controllability to the trajectories** of system (10) **is equivalent** to the existence of  $C > 0$  such that, for every  $\varphi_0 \in L^2(\Omega; \mathbb{R}^n)$ , the solution  $\varphi \in C^0([0, T]; L^2(\Omega; \mathbb{R}^n))$  to the adjoint system (11) satisfies the **observability inequality**:

$$(12) \quad \|\varphi(\cdot, 0)\|_{L^2(\Omega)}^2 \leq C \iint_{\omega \times (0, T)} |B^* \varphi(x, t)|^2,$$



## 4. The Kalman condition for a class of parabolic systems. Distributed controls

Let us consider  $\{\lambda_k\}_{k \geq 1}$  the sequence of eigenvalues for  $-\Delta$  with homogeneous Dirichlet boundary conditions and  $\{\phi_k\}_{k \geq 0}$  the corresponding normalized eigenfunctions.

### Theorem (A Necessary Condition)

If system (10) is **null controllable** at time  $T$  **then**

$$(13) \quad \text{rank} [-\lambda_k D + A \mid B] = n, \quad \forall k \geq 1.$$

where

$$[-\lambda_k D + A \mid B] = [B \mid (-\lambda_k D + A)B \mid (-\lambda_k D + A)^2 B \mid \cdots \mid (-\lambda_k D + A)^{n-1} B].$$

## 4. The Kalman condition for a class of parabolic systems. Distributed controls

Let us consider  $\{\lambda_k\}_{k \geq 1}$  the sequence of eigenvalues for  $-\Delta$  with homogenous Dirichlet boundary conditions and  $\{\phi_k\}_{k \geq 0}$  the corresponding normalized eigenfunctions.

### Theorem (A Necessary Condition)

If system (10) is **null controllable** at time  $T$  **then**

$$(13) \quad \text{rank} [-\lambda_k D + A \mid B] = n, \quad \forall k \geq 1.$$

where

$$[-\lambda_k D + A \mid B] = [B \mid (-\lambda_k D + A)B \mid (-\lambda_k D + A)^2 B \mid \cdots \mid (-\lambda_k D + A)^{n-1} B].$$

**Proof:** Reasoning by contradiction:  $\exists k \geq 1$  such that  $\text{rank} [-\lambda_k D + A \mid B] < n$ . Then the o.d.s.

$$-Z' = (-\lambda_k D + A^*)Z \quad \text{in } (0, T),$$

is not  $B^*$ -observable at time  $T$ .

## 4. The Kalman condition for a class of parabolic systems. Distributed controls

There exists  $Z_0 \in \mathbb{R}^n$ ,  $Z_0 \neq 0$ , such that the solution  $Z$  to the previous system satisfies  $Z(\cdot) = 0$  on  $(0, T)$ . But

$$\varphi(x, t) = Z(t)\phi_k(x)$$

is the solution to **adjoint problem** (11) associated to  $\varphi_0(x) = Z_0\phi_k$  and

$$B^*\varphi(x, t) = 0, \quad \forall (x, t) \in \Omega \times (0, T).$$

Then, the **observability inequality** (12) fails and system (10) is not **null controllable** at time  $T$ . ■

### Remark

Observe that, if condition (13) is not satisfied, then system (10) is neither **approximately controllable** nor **null controllable** at time  $T$  (for any  $T > 0$ ) even if  $\omega = \Omega$ . ■

## 4. The Kalman condition for a class of parabolic systems. Distributed controls

### Question:

Is condition (13) a **sufficient condition** for the **null controllability** of system (10)? ■

## 4. The Kalman condition for a class of parabolic systems. Distributed controls

### Question:

Is condition (13) a **sufficient condition** for the **null controllability** of system (10)? ■

Let us now introduce the **unbounded matrix operator**

$$\mathcal{K} = [D\Delta + A \mid B] = [B \mid (D\Delta + A)B \mid \cdots \mid (D\Delta + A)^{n-1}B],$$

$$\left\{ \begin{array}{l} \mathcal{K} : D(\mathcal{K}) \subset L^2(\Omega; \mathbb{R}^{nm}) \rightarrow L^2(\Omega; \mathbb{R}^n), \text{ with} \\ D(\mathcal{K}) := \{y \in L^2(\Omega; \mathbb{R}^{nm}) : \mathcal{K}y \in L^2(\Omega; \mathbb{R}^n)\}. \end{array} \right.$$

Then,

### Proposition

$\ker \mathcal{K}^* = \{0\}$  **if and only if** condition (13) holds. ■

## 4. The Kalman condition for a class of parabolic systems. Distributed controls

### Theorem (Kalman condition)

System (10) is **exactly controllable to trajectories** (resp., **approximately controllable**) at time  $T$  **if and only if**

$$\ker \mathcal{K}^* = \{0\} \iff \text{rank} [-\lambda_k D + A \mid B] = n, \quad \forall k \geq 1.$$

### Remark

One can prove, either there exists  $k_0 \geq 1$  such that

$$\text{rank} [-\lambda_k D + A \mid B] = n, \quad \forall k \geq k_0$$

or

$$\text{rank} [-\lambda_k D + A \mid B] < n, \quad \forall k \geq 1.$$

## 4. The Kalman condition for a class of parabolic systems. Distributed controls

Controllability (outside a finite dimensional space) **if and only if** the algebraic Kalman condition  $\boxed{\text{rank} [-\lambda_k D + A \mid B] = n}$  is satisfied for one frequency  $k \geq 1$ .

### Remark

System (10) can be *exactly controlled to the trajectories* with one control force ( $m = 1$  and  $B \in \mathbb{R}^n$ ) even if  $\boxed{A \equiv 0}$ . Indeed, let us assume that  $B = (b_i)_{1 \leq i \leq n} \in \mathbb{R}^n$ . Then,

$$\left[ (-\lambda_k D + A) \mid B \right] = \begin{bmatrix} b_1 & (-\lambda_k d_1) b_1 & \cdots & (-\lambda_k d_1)^{n-1} b_1 \\ b_2 & (-\lambda_k d_2) b_2 & \cdots & (-\lambda_k d_2)^{n-1} b_2 \\ \vdots & \vdots & \ddots & \vdots \\ b_n & (-\lambda_k d_n) b_n & \cdots & (-\lambda_k d_n)^{n-1} b_n \end{bmatrix} \in \mathcal{L}(\mathbb{R}^n),$$

and (13) holds **if and only if**  $b_i \neq 0$  for every  $i$  and  $d_i$  are **distinct**. ■

## 4. The Kalman condition for a class of parabolic systems. Distributed controls

**Idea of the proof:** The objective is to prove the **observability inequality** (12):

$$(12) \quad \|\varphi(\cdot, 0)\|_{L^2(\Omega)}^2 \leq C \iint_{\omega \times (0, T)} |B^* \varphi(x, t)|^2.$$

To this end we use two arguments:

- Prove a Carleman type observability estimate for a **scalar equation of order  $n$  in time**,
- Prove a **coercivity** property for the Kalman operator  $\mathcal{K}$ .



## 4. The Kalman condition for a class of parabolic systems. Distributed controls

Let us consider  $\varphi$  a regular solution of the **adjoint system** (11) and take

$$\Phi = \sum_{i=1}^n \alpha_i \varphi_i, \quad \text{with } \alpha_i \in \mathbb{R} \quad \forall i : 1 \leq i \leq n.$$

Then,  $\Phi$  is a regular solution to the **linear scalar equation of order  $n$**  in time

$$\begin{cases} \det(I_d \partial_t + D\Delta + A^*) \Phi = 0 & \text{in } Q, \\ \Delta^j \Phi = 0 & \text{on } \Sigma, \quad \forall j \geq 1. \end{cases}$$

## 4. The Kalman condition for a class of parabolic systems. Distributed controls

The **key point** is to prove a Carleman inequality for the solutions to the previous problem:

### Theorem

Let  $n, k_1, k_2 \in \mathbb{N}$ . There exist two constants  $r_0$  and  $C$  (only depending on  $\Omega, \omega, n, D, A, k_1$  and  $k_2$ ) such that

$$\sum_{i=0}^{k_1} \sum_{j=0}^{k_2} \mathcal{J}(3 - 4(i + j), \Delta^i \partial_t^j \Phi) \leq C \iint_{\omega \times (0, T)} e^{-2s\alpha} [s\rho(t)]^{3+r_0} |\Phi|^2, \quad ,$$

$\forall s \geq s_0 = \sigma_0(\Omega, \omega)(T + T^2)$  (see Lemma 6) and  $\Phi$  solution to the previous problem. ■

## 4. The Kalman condition for a class of parabolic systems. Distributed controls

From this result and after some operations, one deduces

$$\int_0^T e^{\frac{-2sM_0}{i(T-i)}} [s\rho(t)]^3 \|\Delta^k \mathcal{K}^* \varphi\|_{L^2(\Omega)^{nm}}^2 \leq C \iint_{\omega \times (0,T)} e^{-2s\alpha} [s\rho(t)]^{3+r} |B^* \varphi|^2$$

for every  $s \geq \sigma_0 (T + T^2)$ . In this inequality,  $\rho$  and  $\alpha$  are as in Lemma 6,  $M_0 = \max_{\bar{\Omega}} \alpha_0$  and  $r \geq 0$  is an integer only depending on  $n$ .

### Remark

*The previous inequality is a **partial observability estimate**. It is valid even if the Kalman condition does not hold, i.e., even if  $\ker \mathcal{K}^* \neq \{0\}$ . ■*

## 4. The Kalman condition for a class of parabolic systems. Distributed controls

The **coercivity** property of  $\mathcal{K}$ :

### Theorem

Assume that  $\ker \mathcal{K}^* = \{0\}$  and consider  $k = (n-1)(2n-1)$ . Then there exists  $C > 0$  such that if  $z \in L^2(\Omega)^n$  satisfies  $\mathcal{K}^* z \in D(\Delta^k)^{nm}$ , one has

$$\|z\|_{L^2(\Omega)^n}^2 \leq C \|\Delta^k \mathcal{K}^* z\|_{L^2(\Omega)^{nm}}^2.$$

So, from the previous inequality we get

$$\int_0^T e^{\frac{-2sM_0}{i(T-i)}} [s\rho(t)]^3 \|\varphi\|_{L^2(\Omega)^{nm}}^2 \leq C \iint_{\omega \times (0,T)} e^{-2s\alpha} [s\rho(t)]^{3+r} |B^* \varphi|^2$$

and the **observability inequality** (12):

$$(12) \quad \|\varphi(\cdot, 0)\|_{L^2(\Omega)}^2 \leq C \iint_{\omega \times (0,T)} |B^* \varphi(x, t)|^2.$$

## 4. The Kalman condition for a class of parabolic systems. Distributed controls

### Summarizing

- 1 We have established a **Kalman condition**

$$\ker \mathcal{K}^* = \{0\}$$

which characterizes the controllability properties of system (10).

- 2 The **Kalman condition** for system (10)  $\ker \mathcal{K}^* = \{0\}$  generalizes the **algebraic Kalman condition**  $\ker[A \mid B]^* = \{0\}$  for o.d.s.
- 3 This **Kalman condition** is also equivalent to the **approximate controllability** of system (10) at time  $T$ . Again, **approximate** and **null controllability** are equivalent concepts for system (10).

## 4. The Kalman condition for a class of parabolic systems. Distributed controls

A special case:  $D = Id$ .

It is possible to get better results when  $D = Id$ . In this case system (10) is given by

$$(14) \quad \begin{cases} \partial_t y - \Delta y = Ay + Bv \mathbf{1}_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \quad y(\cdot, 0) = y_0(\cdot) & \text{in } \Omega, \end{cases}$$

where again  $A \in \mathcal{L}(\mathbb{R}^n)$  and  $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$  are **constant** matrices and  $y_0 \in L^2(\Omega; \mathbb{R}^n)$  is given. In this case,  $\ker \mathcal{K}^* = \{0\}$  is **equivalent** to the **algebraic Kalman condition**

$$\text{rank} [A \mid B] = \text{rank} [B \mid AB \mid A^2 B \mid \dots \mid A^{n-1} B] = n.$$

In this case we can obtain a better Carleman inequality for the adjoint system

$$(15) \quad \begin{cases} -\partial_t \varphi - \Delta \varphi = A^* \varphi & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \quad \varphi(\cdot, T) = \varphi_0(\cdot) & \text{in } \Omega. \end{cases}$$

## 4. The Kalman condition for a class of parabolic systems. Distributed controls

### Theorem

There exist a positive function  $\alpha_0 \in C^2(\bar{\Omega})$  (only depending on  $\Omega$  and  $\omega$ ), positive constants  $C_0$  and  $\sigma_0$  (only depending on  $\Omega$ ,  $\omega$ ,  $n$ ,  $m$ ,  $A$  and  $B$ ) and a positive integer  $\ell \geq 3$  (only depending on  $n$  and  $m$ ) such that, if  $\text{rank}[A \mid B] = n$ , for every  $\varphi_0 \in L^2(Q; \mathbb{R}^n)$ , the solution  $\varphi$  to (15) satisfies

$$\mathcal{I}(\varphi) \leq C_0 \left( s^\ell \iint_{\omega \times (0, T)} e^{-2s\alpha} \rho(t)^\ell |B^* \varphi|^2 \right),$$

$\forall s \geq s_0 = \sigma_0 (T + T^2)$ . In this inequality,  $\alpha(x, t)$ ,  $\rho(t)$  and  $\mathcal{I}(z)$  are as in Lemma 6. ■

## 4. The Kalman condition for a class of parabolic systems. Distributed controls

### References

- 1 F. AK, A. BENABDALLAH, C. DUPAIX, M. GONZÁLEZ-BURGOS., *A generalization of the Kalman rank condition for time-dependent coupled linear parabolic systems*, Differ. Equ. Appl. 1 (2009), no. 3, 139–151.

$$D = I_d, A = A(t) \text{ and } B = B(t).$$

- 2 F. AK, A. BENABDALLAH, C. DUPAIX, M. GONZÁLEZ-BURGOS., *Kalman rank condition for the localized distributed controllability of a class of linear parabolic systems*, J. Evol. Equ. 9 (2009), no. 2, 267–291.

$D$  diagonal matrix,  $A$  and  $B$  constant matrices.



## 5. The Kalman condition for a class of parabolic systems. Boundary controls

Let us consider the **boundary controllability problem**:

$$(16) \quad \begin{cases} y_t = y_{xx} + Ay & \text{in } Q = (0, \pi) \times (0, T), \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

where  $A \in \mathcal{L}(\mathbb{C}^n)$  and  $B \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^n)$  are two given matrices and  $y_0 \in H^{-1}(0, \pi; \mathbb{C}^n)$  is the initial datum. In system (16),  $v \in L^2(0, T; \mathbb{C}^m)$  is the control function (to be determined).

**Simpler problem:** One-dimensional case and  $D = Id$ .

This problem has been studied in the case  $n = 2$ :

- E. FERNÁNDEZ-CARA, M. GONZÁLEZ-BURGOS., L. DE TERESA, *Boundary controllability of parabolic coupled equations*, J. Funct. Anal. 259 (2010), no. 7, 1720–1758.

## 5. The Kalman condition for a class of parabolic systems. Boundary controls

### Theorem

$n = 2, m = 1$ . Let  $A \in \mathcal{L}(\mathbb{C}^2)$  and  $B \in \mathbb{C}^2$  be given and let us denote by  $\mu_1$  and  $\mu_2$  the eigenvalues of  $A^*$ . Then (16) is **exactly controllable to the trajectories** at any time  $T > 0$  **if and only if**  $\text{rank} [A \mid B] = 2$  and

$$\lambda_k - \lambda_j \neq \mu_1 - \mu_2 \quad \forall k, j \in \mathbb{N} \text{ with } k \neq j. \quad \blacksquare$$

### Remark ( $n = 2, m = 1$ )

For the previous **boundary controllability problem**, one has

- 1 A complete characterization of the **exact controllability to trajectories** at time  $T$ .
- 2 **Boundary controllability** and **distributed controllability** are not equivalent
- 3 **Approximate controllability**  $\iff$  **null controllability**.

## 5. The Kalman condition for a class of parabolic systems. Boundary controls

### What does happen if $n > 2$ ?

We consider again  $\{\lambda_k\}_{k \geq 1}$  the sequence of eigenvalues for  $-\partial_{xx}$  in  $(0, \pi)$  with homogenous Dirichlet boundary conditions and  $\{\phi_k\}_{k \geq 0}$  the corresponding normalized eigenfunctions:

$$\lambda_k = k^2, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, \quad k \geq 1, \quad x \in (0, \pi).$$

## 5. The Kalman condition for a class of parabolic systems. Boundary controls

### Notation

For  $k \geq 1$ , we introduce  $L_k = -\lambda_k I_d + A \in \mathcal{L}(\mathbb{C}^n)$  and the matrices

$$B_k = \begin{pmatrix} B \\ \vdots \\ B \end{pmatrix} \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^{nk}), \quad \mathcal{L}_k = \begin{pmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & L_k \end{pmatrix} \in \mathcal{L}(\mathbb{C}^{nk}),$$

and let us write the Kalman matrix associated with the pair  $(\mathcal{L}_k, B_k)$ :

$$\mathcal{K}_k = [\mathcal{L}_k \mid B_k] = [B_k \mid \mathcal{L}_k B_k \mid \mathcal{L}_k^2 B_k \mid \cdots \mid \mathcal{L}_k^{nk-1} B_k] \in \mathcal{L}(\mathbb{C}^{mnk}, \mathbb{C}^{nk}).$$

## 5. The Kalman condition for a class of parabolic systems. Boundary controls

### Theorem

Let us fix  $A \in \mathcal{L}(\mathbb{C}^n)$  and  $B \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^n)$ . Then, system (16) is **exactly controllable to trajectories** at time  $T$  if and only if

$$(17) \quad \text{rank } \mathcal{K}_k = nk, \quad \forall k \geq 1.$$

### Remark

- 1 This result gives a complete characterization of the **exact controllability to trajectories** at time  $T$ : **Kalman condition**.
- 2 If for  $k \geq 1$  one has  $\text{rank } \mathcal{K}_k = nk$ , then  $\text{rank } [A \mid B] = n$  and system (14) is **exactly controllable to trajectories** at time  $T$ . But  $\text{rank } [A \mid B] = n$  does not imply condition (17). So **boundary controllability** and **distributed controllability** are not equivalent.

## 5. The Kalman condition for a class of parabolic systems. Boundary controls

### Remark

Condition (17) is also a **necessary** and **sufficient condition** for the **boundary approximate controllability** of system (16). Then

**Approximate controllability**  $\iff$  **null controllability**.

**Adjoint Problem:**

$$(18) \quad \begin{cases} -\varphi_t = \varphi_{xx} + A^* \varphi & \text{in } Q, \\ \varphi(0, \cdot) = \varphi(\pi, \cdot) = 0 & \text{on } (0, T), \\ \varphi(\cdot, T) = \varphi_0 & \text{in } (0, \pi), \end{cases}$$

with  $\varphi_0 \in H_0^1(0, \pi; \mathbb{C}^n)$ . Then, system (16) is **exactly controllable to trajectories** at time  $T \iff$  for a constant  $C > 0$  one has

$$\|\varphi(\cdot, 0)\|_{H_0^1(0, \pi; \mathbb{C}^n)}^2 \leq C \int_0^T |B^* \varphi_x(0, t)|^2 dt.$$

## 5. The Kalman condition for a class of parabolic systems. Boundary controls

**Necessary implication.** We reason as before: if  $\text{rank } \mathcal{K}_k < nk$ , for some  $k \geq 1$ , then the o.d.s.

$$-Z' = \mathcal{L}_k^* Z \quad \text{on } (0, T), \quad Z(T) = Z_0 \in \mathbb{C}^{nk}$$

is not  $B_k^*$ -observable on  $(0, T)$ , i.e., there exists  $Z_0 \neq 0$  s.t.  $B_k^* Z(t) = 0$  for every  $t \in (0, T)$ . From  $Z_0$  it is possible to construct  $\varphi_0 \in H_0^1(0, \pi; \mathbb{C}^n)$  with  $\varphi_0 \neq 0$  such that the corresponding solution to the **adjoint problem** (17) satisfies

$$B_x^* \varphi_x(0, t) = 0 \quad \forall t \in (0, T).$$

**As a consequence:** The **unique continuation property** and the previous **observability inequality** for the **adjoint problem** fail:

Neither **approximate** nor **null controllability** at any  $T$  for system (14).

## 5. The Kalman condition for a class of parabolic systems. Boundary controls

**Sufficient implication.** For the proof we follow the ideas from

- H.O. FATTORINI, D.L. RUSSELL, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal. 43 (1971), 272–292.

Two “big” steps:

- 1 We reformulate the null controllability problem for system (16) as a **vector moment problem**.
- 2 Existence and bounds of a family **biorthogonal** to appropriate complex matrix exponentials.



## 5. The Kalman condition for a class of parabolic systems. Boundary controls

**Sufficient implication.** For the proof we follow the ideas from

- H.O. FATTORINI, D.L. RUSSELL, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal. 43 (1971), 272–292.

Two “big” steps:

- 1 We reformulate the null controllability problem for system (16) as a **vector moment problem**.
- 2 Existence and bounds of a family **biorthogonal** to appropriate complex matrix exponentials.

Let us fix  $\eta \geq 1$ , an integer,  $T \in (0, \infty]$  and  $\{\Lambda_k\}_{k \geq 1} \subset \mathbb{C}$  a sequence. Let us recall that the family  $\{\varphi_{k,j}\}_{k \geq 1, 0 \leq j \leq \eta-1} \subset L^2(0, T; \mathbb{C})$  is **biorthogonal** to  $\{t^j e^{-\Lambda_k t}\}_{k \geq 1, 0 \leq j \leq \eta-1}$  if one has

$$\int_0^T t^j e^{-\Lambda_k t} \varphi_{l,i}^*(t) dt = \delta_{kl} \delta_{ij}, \quad \forall (k,j), (l,i) : k, l \geq 1, 0 \leq i, j \leq \eta - 1.$$

## 5. The Kalman condition for a class of parabolic systems. Boundary controls

### Theorem

Assume that for two positive constants  $\delta$  and  $\rho$  one has

$$\left\{ \begin{array}{l} \Re \Lambda_k \geq \delta |\Lambda_k|, \quad |\Lambda_k - \Lambda_l| \geq \rho |k - l|, \quad \forall k, l \geq 1, \\ \sum_{k \geq 1} \frac{1}{|\Lambda_k|} < \infty. \end{array} \right.$$

Then,  $\exists \{\varphi_{k,j}\}_{k \geq 1, 0 \leq j \leq \eta-1}$  **biorthogonal** to  $\{t^j e^{-\Lambda_k t}\}_{k \geq 1, 0 \leq j \leq \eta-1}$  such that, for every  $\varepsilon > 0$ , there exists  $C(\varepsilon, T) > 0$  satisfying

$$\|\varphi_{k,j}\|_{L^2(0,T;\mathbb{C})} \leq C(\varepsilon, T) e^{\varepsilon \Re \Lambda_k}, \quad \forall (k,j) : k \geq 1, 0 \leq j \leq \eta - 1.$$

## 5. The Kalman condition for a class of parabolic systems. Boundary controls

### Reference

F. AMMAR-KHODJA, A. BENABDALLAH, M. G.-B., L. DE TERESA, *The Kalman condition for the boundary controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials*, submitted.