Control of a retarded parabolic equation

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Clermont-Ferrand - Septembre 2014

The control issue

Let $T > 0$, $\omega \subset \Omega$ and consider the following control problem:

$$\begin{cases} y_t - (\Delta + a)\, y = by\,(t-h) + u 1_\omega & \text{in } \Omega_T := (0, T) \times \Omega, \\ y = 0 & \text{on } \Gamma_T = (0, T) \times \partial\Omega, \\ y(0) = y^0 & \text{in } \Omega. \\ y = \theta & \text{in } \Omega_{-h} = (-h, 0) \times \Omega \end{cases}$$

- $a, b \in L^\infty(Q_T)$,
- $h > 0$ is a delay parameter
- $y^0 \in L^2(\Omega)$, $\theta \in L^2(\Omega_{-h})$.
- $u \in L^2(Q_T)$: control function.

Two control issues

- Null-controllability (NC) issue:

$T > 0, \; y^0 \in L^2(\Omega), \theta \in L^2(\Omega_{-h}) \; :$

$$\exists ? u, \; : \; y\left(T, \cdot ; y^0, \theta, u\right) = 0 \text{ on } (0, \pi).$$

- Approximate controllability (AC) issue:

$T > 0, \; \left(y^0, \theta, y^1\right) \in L^2(\Omega) \times L^2(\Omega_{-h}) \times L^2(\Omega)$ given:

$$\forall \varepsilon > 0, \; \exists ? u : \; \left\| y\left(T, \cdot ; y^0, \theta, u\right) - y^1 \right\| < \varepsilon$$

Existence and regularity results

From Artola's paper (1967): If $(y_0, \theta, u) \in L^2(\Omega) \times L^2(\Omega_{-h_N}) \times L^2\left(\Omega_T\right)$,

there exists a unique solution $y \in L^2((-h, T) \times \Omega)$ such that:

$$y \in L^2\left(0, T; H_0^1\left(\Omega\right)\right) \cap C\left([0, T]\,; L^2\left(\Omega\right)\right), \;\; y' \in L^2\left(0, T; H^{-1}\left(\Omega\right)\right),$$

and there exists $C > 0$ which does not depend on (y_0, f, u) such that:

$$\begin{aligned} & \sup_{t\in[0,T]} \|y(t)\|^2_{L^2(\Omega)} + \int_{\Omega_T} |\nabla y|^2 + \int_0^T \|y'\|^2_{H^{-1}(\Omega)} \\ \leq \;\; & C\left(\|(y_0, \theta)\|^2 + \|u\|^2_{L^2(\Omega_T)}\right) \end{aligned}$$

Approximate controllability

Theorem

For any $T > 0$ and $a, b \in L^\infty(Q_T)$, the system is approximately controllable (AC).

Proof. It is a consequence of the (AC) of:

$$\begin{cases} y' = \Delta y + a\, y + f + \chi_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0, & \text{in } \Omega. \end{cases}$$

for any $f \in L^2(Q_T)$. The (AC) property of this last system is itself a consequence of the (AC) of

$$\begin{cases} y' = \Delta y + a\, y + \chi_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y_0, & \text{in } \Omega. \end{cases}$$

Null controllability

Theorem

Assume that $a, b \in L^{\infty}(\Omega_T)$ *and:*

$$\lim_{t \to T} (T - t) \ln \|b(t)\|_{L^{\infty}(\Omega \setminus \overline{\omega})} = -\infty.$$

Then the system is null controllable. Moreover, u can be chosen such that:

$$\|u\|_{L^2(\Omega_T)} \leq C_{T,h} \|(y_0, \theta)\|_{M_2}$$

Comments

- If $\mathrm{supp}(b) \subset (0, T) \times \omega$, the assumption on b is trivially satisfied and the (NC) property is a consequence of the known (NC) property of

$$\begin{cases} y' = \Delta y + a\, y + 1_\omega v & \text{in } Q_T \\ y = 0 & \text{on } \Sigma_T \\ y(0) = y_0, \; y(T) = 0 & \text{in } \Omega \end{cases}$$

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- If b is constant on Q_T, then the system has not the (NC) property.
- The found controls do not work at time $T + \tau$ for any $\tau > 0$

Reduction to an observability inequality

Theorem

Let $T > 0$. The system is null controllable at time T if and only if there exists a constant $C > 0$ such that for any $\varphi_0 \in L^2(\Omega)$, the solution of the backward linear system

$$\begin{cases} -\varphi'(t) = (\Delta + a)\,\varphi(t) + b\varphi(t+h) & \text{in } \Omega_T \\ \varphi = 0 & \text{on } \Gamma_T \\ \varphi(T) = \varphi_0, & \text{in } \Omega \\ \varphi = 0, & \text{in } (T, T+h) \times \Omega \end{cases}$$

satisfies the observability inequality:

$$\int_\Omega \varphi^2(0) + \int_{-h}^{0} \int_\Omega \left| \left(\chi_{[0,\min(h,T)]} b\varphi \right)(s+h) \right|^2 \leq C \int_{\omega_T} \varphi^2.$$

Reduction to an observability inequality

The observability inequality to be proved writes:

- $T \leq h$

$$\int_{\Omega} \varphi^2(0) + \int_0^T \int_{\Omega} |b\varphi|^2 \leq C \int_{\omega_T} \varphi^2$$

Reduction to an observability inequality

The observability inequality to be proved writes:

- $T \leq h$
$$\int_\Omega \varphi^2(0) + \int_0^T \int_\Omega |b\varphi|^2 \leq C \int_{\omega_T} \varphi^2$$
- $T > h$
$$\int_\Omega \varphi^2(0) + \int_0^h \int_\Omega |b\varphi|^2 \leq C \int_{\omega_T} \varphi^2$$

.

Proof of the observability inequality: First step

Let $T > 0$ and $T_h = \max(0, T - h)$. On $(T_h, T) \times \Omega$, the solution φ of the adjoint problem satisfies

$$\begin{cases} \varphi_t + \Delta\varphi = -a\,\varphi & \text{in} \quad \Omega_h = (T_h, T) \times \Omega \\ \varphi = 0 & \text{on} \quad \Sigma_h = (T_h, T) \times \partial\Omega \end{cases}$$

From the global Carleman inequality, it follows:

$$\int_{T_h}^{T} \int_{\Omega} e^{-2\frac{\tau}{(t-T_h)(T-t)}} \varphi^2 \leq C \int_{\omega_T} \varphi^2.$$

Proof of the observability inequality: Second step

Lemma

There exists $K = K(a, b) > 0$ *such that for any solution* φ *of the adjoint problem, the function*

$$E(t) := e^{Kt} \left(\int_\Omega \varphi^2(t, x) + \int_\Omega \int_t^{t+\min(h,T)} \left(\chi_{[0,T]} b\varphi \right)^2 (s)\,ds\,dx \right),$$

is non decreasing on $[0, T]$.

Remark. $E(0) = \int_\Omega \varphi^2(0, x) + \int_\Omega \int_0^{\min(h,T)} \left(\chi_{[0,T]} b\varphi \right)^2$ is the left hand-side in the observability inequality.

Proof of the observability inequality: Third step

For simplicity, assume $T > h$.
Using the energy E, we can write

$$E(0) \preceq \int_{T-h+\nu}^{T} e^{-2\frac{\tau}{(t-T+h)(T-t)}-Kt} E(t)\, dt$$

$$= \int_{T-h+\nu}^{T} e^{-2\frac{\tau}{(t-T+h)(T-t)}} \left(\int_\Omega \varphi^2(t,x) + \int_\Omega \int_t^{t+h} \left(\chi_{[0,T]} b\varphi\right)^2 (s) \right)$$

$$\preceq \underbrace{\int_{T-h+\nu}^{T} e^{-2\frac{\tau}{(t-T+h)(T-t)}} \int_\Omega \left(\int_t^{t+h} \left(\chi_{[0,T]} b_i \varphi\right)^2 (\tau) d\tau \right)}_{=I} + \int_{\omega_T} \varphi^2$$

Here is used the assumption on b which implies: for any number $r > 0$, there exists $\delta > 0$ such that

$$\|b(t)\|_{L^\infty(\Omega\setminus\overline{\omega})} \leq e^{-\frac{r}{T-t}}, \ t \in (T-\delta, T).$$

So that

$$\begin{aligned} I &\preceq \left(\int_{T-h+\nu}^{T} e^{-2\tau\gamma} dt\right) \int_{T-h+\nu}^{T} \left(\int_{\Omega\setminus\overline{\omega}} (b\varphi)^2(s) + \int_{\omega} b_i^2 \varphi^2(s)\right) dxds \\ &\preceq \int_{T-h+\nu}^{T} e^{-\frac{2r}{(T-t)}} \int_{\Omega\setminus\overline{\omega}} \varphi^2(t) dxdt + \|b_i\|_\infty^2 \int_{\omega_T} \varphi^2(s) \\ &\preceq \int_{\omega_T} \varphi^2 \end{aligned}$$

where the following inequality has been used:

$$e^{-\frac{2r}{(T-t)}} \leq e^{-\frac{2\tau}{\left(t-T_{h_1}\right)(T-t)}}, \ t \in (T-h+\nu, T)$$

This concludes the proof.

Open problems I

The same techniques give the same results for

$$\begin{cases} y' = \Delta y + \int_{-h}^{0} \, y\,(t+s)\, d\mu(s) + \chi_\omega u & \text{in } \Omega_T \\ y = 0 & \text{on } \Sigma_T \\ y(0,\cdot) = y_0 & \text{in } \Omega \\ y = \theta & \text{in } (-h,0)\times\Omega \end{cases}$$

where μ is the Stieljes measure

$$\mu(t,s) = -\sum_{j=1}^{m} 1_{(-\infty,h_j]} b_j(t,s) - \int_s^0 b(t,\sigma)d\sigma,\ \ s \in (-h,0)\,,$$

where $0 < h_1 < \cdots < h_m = h$ and $b, b_j \in L^1\left(-h,0;L^\infty\left(\Omega_T\right)\right)$.

- If $b = 0$, the previous techniques solve the null-controllability problem.
- If $b \neq 0$, the problem is widely open.

Open problems II

Systems of parabolic retarded equations:

$$\begin{cases} y' = D\Delta y + Ay + B\ y\left(t-h\right) + \chi_\omega Cu & \text{in } \Omega_T \\ y = 0 & \text{on } \Sigma_T \\ y(0,\cdot) = y_0 & \text{in } \Omega \\ y = f & \text{in } \left(-h,0\right)\times\Omega \end{cases}$$

where $A, B \in L^\infty\left(\Omega_T, \mathcal{L}\left(\mathbb{R}^n\right)\right)$ and $C \in L^\infty\left(\Omega_T, \mathcal{L}\left(\mathbb{R}^m, \mathbb{R}^n\right)\right)$ with $\left(m,n\right) \in \mathbb{N}^2 \backslash \left(0,0\right)$ and data $u \in L^2\left(\Omega_T, \mathbb{R}^m\right)$, $\left(y_0, f\right) \in L^2\left(\Omega, \mathbb{R}^n\right) \times L^2\left(\left(-h,0\right)\times\Omega, \mathbb{R}^n\right)$.
To our knowledge, there does not exist null controllability results similar to the case $B = 0$.

Open problems III

The semilinear delayed equation.

$$\begin{cases} y'(t) = \Delta y(t) + f\,(y(t)) + b(t)\; y(t-h) + \chi_\omega u(t) & \text{in}\,\Omega_T\,, \\ y = 0, & \text{on}\;\Gamma_T\,, \\ y(0,\cdot) = y_0\,, & \text{in}\;\Omega, \\ y = \theta & \text{in}\;\Omega_{-h}, \end{cases} \tag{1}$$

where the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is regular and superlinear.

- It is possible to deduce the *local controllability to trajectories* at time T.
  But a global controllability result *à la* Fernandez-Cara and Zuazua is an open problem.
- In the literature: results for the approximate controllability for lipschitz nonlinearities.

Merci pour votre attention!