Lipschitz stability in an inverse problem for non autonomous magnetic Schrödinger equations

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The settings

$\Omega =$ bounded open subset of $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with C^2 boundary Γ.

Time dependent Hamiltonian:

$$H_{\mathbf{a}}(t) = (\mathrm{i}\nabla + \chi(t)\mathbf{a})^2,$$

associated to the magnetic vector potential $\chi(t)\mathbf{a}(x)$, where:

(a) $\chi =$ smooth real valued function on $[0, T]$;

(b) $\mathbf{a} \in \mathrm{H}^1(\Omega)^n$ is bounded, nondivergent and real valued.

Non autonomous magnetic Schrödinger equations

$$\begin{cases} -\mathrm{i}u'(t,x) + H_{\mathbf{a}}(t)u(t,x) = 0, & (t,x) \in Q_T^+ = (0,T) \times \Omega \\ u(t,x) = 0, & (t,x) \in \Sigma_T^+ = (0,T) \times \Gamma \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

for some suitable data u_0.

Γ^+ = open subset of Γ (+ appropriate geometric condition).

Inverse problem: retrieve $\mathbf{a} = (\mathbf{a}_j(x))_{1 \leq j \leq n}$, $x \in \Omega$, in (1.1), from the extra data $\partial_\nu \partial_t^k u \big|_{(0,T)\times\Gamma^+}$, $k = 1, 2$, by n times changing the initial value u_0 suitably.

Magnetic field

- $\Lambda_{\mathbf{a}}(f) = (\partial_\nu + \mathrm{i}\mathbf{a}\cdot\nu)u$, $f \in \mathrm{L}^2(\Sigma_T^+)$, where u is solution to

$$\begin{cases} -\mathrm{i}u'(t,x) - (\mathrm{i}\nabla + \mathbf{a})^2 u(t,x) = 0, & (t,x) \in Q_T^+ \\ u(t,x) = f(t,x), & (t,x) \in \Sigma_T^+ \\ u(0,x) = 0, & x \in \Omega, \end{cases}$$

  is invariant under the gauge transformation of $\mathbf{a}$:

$$\Lambda_{\mathbf{a}+\nabla\psi} = \Lambda_{\mathbf{a}},\ \forall \psi \in \mathrm{C}^1(\overline{\Omega}) \text{ s.t. } \psi_{|\Gamma} = 0.$$

  $\Rightarrow$ at best uniqueness modulo a gauge transform of $\mathbf{a}$ from $\Lambda_{\mathbf{a}}$.

- The magnetic field rot $\mathbf{a}$ is preserved.

The Coulomb gauge class

$$\mathrm{H}_{\mathrm{div}0}(\Omega;\mathbb{R}) := \{\mathbf{a} \in \mathrm{L}^{\infty}(\Omega;\mathbb{R}^n),\ \nabla.\mathbf{a} = 0\}.$$

Admissible potential vectors $\chi(t)\mathbf{a}$:

- $\mathbf{a}_0 \in \mathrm{H}^1(\Omega)^n$ s.t. $\nabla \cdot \mathbf{a}_0 = 0$ and $M > 0$ are fixed:

$$\mathbf{A}(\mathbf{a}_0, M) = \left\{\mathbf{a} \in \mathrm{H}^1(\Omega)^n,\ \nabla \cdot \mathbf{a} = 0, \|\mathbf{a}\|_{\infty} \leq M,\ \mathbf{a}_{|\Gamma} = (\mathbf{a}_0)_{|\Gamma}\right\}.$$

- χ is given and verifies

$$\chi \in \mathrm{C}^3([0,T];\mathbb{R}),\ \chi(0) = 0,\ \chi'(0) \neq 0.$$

Related papers

- Determination of *the magnetic field* from
  - (a) DN-map (full data): Z. Sun '92, G. Nakamura, Z. Sun, G. Uhlmann '95, C. F. Tolmasky '98, M. Salo '04, A. Panchenko '02, G. Eskin '03, M. Bellassoued, M. Choulli '10, G. Eskin '11.
  - (b) Local DN map: D. Dos Santos, C. E. Keinig, J. Sjöstrand, G. Uhlmann '07, L. Tzou '08, H. Ben Joud '09.
- Determination of *the the external electric potential* from a finite number of boundary measurements: L. Baudouin, J.-P. Puel '02, A. Mercado, A. Osses, L. Rosier '08, M. Bellassoued, M. Choulli '09.

DtN versus Carleman

- Initial data / No initial data
- geometrical constraints / geometrical constraints
- Finite number of measurements / Infinite number
- Numerical applications / ???

Lipschitz stability inequality

Fix $\varepsilon > 0$ and pick $u_{0,j} \in \mathrm{H}_0^{\max(6,n/2+1+\varepsilon)}(\Omega;\mathbb{R})$, $j = 1, \ldots, n$, s.t.

$$\det DU_0(x) \neq 0,\ x \in \Omega, \text{ where } DU_0(x) := (\partial_{x_j} u_{0,i}(x))_{1\leq i,j\leq n}.$$

Let $\mathbf{a}$ (resp. $\tilde{\mathbf{a}}$) be in $\mathbf{A}(\mathbf{a}_0, M)$, and let u_j (resp. $\tilde{u}_j$) denote the $\mathrm{C}^0([0,T];\mathrm{H}_0^1(\Omega)\cap\mathrm{H}^2(\Omega))$-solution to (1.1) (resp. (1.1) with $\tilde{\mathbf{a}}$) with initial condition $u_{0,j}$, $j = 1, \ldots, n$.

Then there exists $C = C(T, \Omega, \Gamma^+, M, \chi, \{u_{0,j}\}_{j=1}^n) > 0$ satisfying:

$$\begin{aligned} & \|\mathbf{a} - \tilde{\mathbf{a}}\|^2_{\mathrm{L}^2(\Omega)^n} \\ \leq\ & C \sum_{j=1}^{n} \left(\|\partial_\nu \partial_t (u_j - \tilde{u}_j)\|^2_{\mathrm{L}^2(0,T;\Gamma^+)} + \|\partial_\nu \partial_t^2 (u_j - \tilde{u}_j)\|^2_{\mathrm{L}^2(0,T;\Gamma^+)} \right). \end{aligned}$$

Magnetic Schrödinger operators

For every $\mathbf{a} \in \mathrm{C}^0([0,T]; \mathrm{H}_{\mathrm{div}0}(\Omega;\mathbb{R}))$ we have:

$$H(t) = (\mathrm{i}\nabla + \mathbf{a}(t))^2 = -\Delta + 2\mathrm{i}\mathrm{Re}\,(\mathbf{a}(t))\,.\nabla + |\mathbf{a}(t)|^2,\ t \in [0,T].$$

- $H(t) =$ s.a. operator in $\mathcal{H}_0 = \mathrm{L}^2(\Omega)$, associated with the closed, densely defined ≥ 0 sesquilinear form

  $$h(t;u,v) = \langle (\mathrm{i}\nabla + \mathbf{a}(t,x))u, (\mathrm{i}\nabla + \mathbf{a}(t,x))v \rangle_0,$$

  with domain $\mathrm{dom}\ h(t) = \mathrm{dom}\ h(0) = \mathrm{H}_0^1(\Omega) = \mathcal{H}_1$.
- $\mathrm{dom}\ H(t) = \mathrm{H}_0^1(\Omega) \cap \mathrm{H}^2(\Omega) = \mathcal{H}_2$.

The main problems

- Time dependent coefficients (involving the magnetic potential to retrieve) in first and zero orders terms
- The vectorial form of **a**

Solution to non-autonomous Schrödinger equations

Let $f \in \mathrm{L}^2(0, T; \mathcal{H}_{-j})$, $j = 0, 1, 2$.

- A solution to the Schrödinger equation

$$-\mathrm{i}\psi' + H(t)\psi = f \text{ in } \mathrm{L}^2(0, T; \mathcal{H}_{-j}),$$

  is a function $\psi \in \mathrm{L}^2(0, T; \mathcal{H}_{2-j})$ satisfying for every $v \in \mathcal{H}_j$:

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\langle \psi(t), v\rangle_0 + \langle H(t)\psi(t), v\rangle_{-j,j} = \langle f(t), v\rangle_{-j,j} \text{ in } \mathrm{C}_0^\infty(0, T)'.$$

- In this case $\psi \in W(0, T; \mathcal{H}_{2-j}, \mathcal{H}_{-j})$ satisfies

$$-\mathrm{i}\langle \psi'(t), v\rangle_{-j,j} + \langle H(t)\psi(t), v\rangle_{-j,j} = \langle f(t), v\rangle_{-j,j} \text{ in } \mathrm{C}_0^\infty(0, T)',$$

  for every $v \in \mathcal{H}_j$.

Existence + uniqueness results

Let $\mathbf{a} \in \mathrm{C}^1([0, T]; \mathrm{H}_{\mathrm{div}0}(\Omega; \mathbb{R}))$.
For $j = 0, 1, 2$, we consider the Cauchy problem

$$\begin{cases} -\mathrm{i}\psi' + H(t)\psi = f \text{ in } \mathrm{L}^2(0, T; \mathcal{H}_{-j}) \\ \psi(0) = \psi_0, \end{cases} \tag{2.2}$$

where $\psi_0 \in \mathcal{H}_{2-j}$ and $f \in \mathrm{L}^2(0, T; \mathcal{H}_{-j})$.

- For $j = 0, 1$, $\exists! \psi \in \mathrm{C}^0([0, T]; \mathcal{H}_{2-j}) \cap \mathrm{C}^1([0, T]; \mathcal{H}_{-j})$, solution to (2.2), provided $f \in W(0, T; \mathcal{H}_0, \mathcal{H}_{-j})$.
- For $j = 2$ there exists at most one solution to (2.2).

Sketch of the proof (case $j = 2$)

- For all $\tau \in (0, T)$, the first line of (2.2) with $f = 0$ states

$$\langle \psi'(\tau), v \rangle_{-2,2} = -\mathrm{i} \langle \psi(\tau), H(\tau) v \rangle_0, \ \ v \in \mathcal{H}_2.$$

- Choose $v = R(\tau)\psi(\tau)$ with $R(\tau) := (1 + H(\tau))^{-1}$:

$$\langle \psi'(\tau), R(\tau)\psi(\tau) \rangle_{-2,2} = \mathrm{i}(\| R(\tau)^{1/2} \psi(\tau) \|_0^2 - \| \psi(\tau) \|_0^2). \quad (2.3)$$

- As $\frac{\mathrm{d}}{\mathrm{d}\tau} R(\tau)\psi(\tau) = R(\tau)\psi'(\tau) + R(\tau)H'(\tau)R(\tau)\psi(\tau)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \| R(\tau)^{1/2} \psi(\tau) \|_0^2 = \langle \psi(\tau), R(\tau)H'(\tau)R(\tau)\psi(\tau) \rangle_0 \quad (2.4)$$
$$+2 \underbrace{\mathrm{Re} \left(\langle \psi'(\tau), R(\tau)\psi(\tau) \rangle_{-2,2} \right)}_{=0 \text{ from } (2.3)}.$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}\tau}\| \overbrace{\tilde{\psi}(\tau)}^{R(\tau)^{1/2}\psi(\tau)} \|_0^2 = \langle \tilde{\psi}(\tau), R(\tau)^{1/2}H'(\tau)R(\tau)^{1/2}\tilde{\psi}(\tau)\rangle_0$$
$$\leq \|H'(\tau)R(\tau)^{1/2}\|\|\tilde{\psi}(\tau)\|_0^2.$$

Since

- $\|H'(\tau)R(\tau)^{1/2}\| = 2\|\mathbf{a}'(\tau).(\mathrm{i}\nabla + \mathbf{a}(\tau))R(\tau)^{1/2}\| \leq C,$
- $\psi(0) = 0,$

we obtain

$$\|\tilde{\psi}(t)\|_0^2 \leq C\int_0^t \|\tilde{\psi}(\tau)\|_0^2 d\tau,\ \ t \in [0, T],$$

$\stackrel{\text{Gronwall}}{\Longrightarrow} \tilde{\psi}(t) = 0 \Rightarrow \psi(t) = 0.$

Time differentiation of (2.2)

Let $f \in W(0, T; \mathcal{H}_0)$ and $\psi_0 \in \mathcal{H}_2$.

The $\mathrm{C}^0([0, T]; \mathcal{H}_2) \cap \mathrm{C}^1([0, T]; \mathcal{H}_0)$-solution ψ to (2.2) satisfies:

- $\psi'' \in \mathrm{L}^2(0, T; \mathcal{H}_{-2})$.
- ψ' is solution to the system:

$$\begin{cases} -\mathrm{i}\psi'' + H(t)\psi' = f' - H'(t)\psi \text{ in } \mathrm{L}^2(0, T; \mathcal{H}_{-2}) \\ \psi'(0) = -\mathrm{i}(H(0)\psi_0 - f(0)), \end{cases}$$

where

$$H'(t) = 2\mathbf{a}'(t).(\mathrm{i}\nabla + \mathbf{a}(t)).$$

Differentiating (1.1) (case where $\mathbf{a}(t) = \chi(t)\mathbf{a}$ is C^3)

Let u_0 be such that $\Delta^k u_0 \in \mathcal{H}_2,\ k = 0, 1, 2$.

Then $\exists! u \in \mathrm{C}^2([0, T]; \mathcal{H}_2) \cap \mathrm{C}^3([0, T]; \mathcal{H}_0)$ solution to

$$\begin{cases} -\mathrm{i}u^{(k+1)}(t,x) + H_{\mathbf{a}}(t)u^{(k)}(t,x) = f_k(t,x), & (t,x) \in Q_T^+ \\ u^{(k)}(t,x) = 0, & (t,x) \in \Sigma_T^+ \\ u^{(k)}(0,x) = \mathrm{i}^k \Delta^k u_0(x), & x \in \Omega, \end{cases}$$

for $k = 0, 1, 2$, where

$$f_k = \begin{cases} 0 & \text{if } k = 0 \\ -H'_{\mathbf{a}}(t)u & \text{if } k = 1 \\ -H''_{\mathbf{a}}(t)u - 2H'_{\mathbf{a}}(t)u' & \text{if } k = 2, \end{cases}$$

Linearized system

- Let u (resp. $\tilde{u}$) denote the $\mathrm{C}^2([0,T];\mathcal{H}_2) \cap \mathrm{C}^3([0,T];\mathcal{H}_0)$ solution to (1.1) with $H(t) = H_{\mathbf{a}}(t)$ (resp. (1.1) with $H(t) = H_{\tilde{\mathbf{a}}}(t)$).
- $v = u - \tilde{u}$ is solution in $\mathrm{L}^2(0,T;\mathcal{H}_0)$ to

$$\begin{cases} -\mathrm{i}v'(t,x) + H_{\mathbf{a}}(t)v(t,x) = f(t,x), & (t,x) \in Q_T^+ \\ v(t,x) = 0, & (t,x) \in \Sigma_T^+ \\ v(0,x) = 0, & x \in \Omega, \end{cases}$$

where $f = \chi(\tilde{\mathbf{a}} - \mathbf{a}).\,(2\mathrm{i}\nabla + \chi(\mathbf{a} + \tilde{\mathbf{a}}))\,\tilde{u}$.

- $w = v'$ is solution in $\mathrm{L}^2(0, T; \mathcal{H}_0)$ to

$$\begin{cases} -\mathrm{i}w'(t,x) + H_{\mathbf{a}}(t)w(t,x) = g(t,x) & (t,x) \in Q_T^+ \\ w(t,x) = 0, & (t,x) \in \Sigma_T^+ \\ w(0,x) = 0, & x \in \Omega, \end{cases}$$

where $g = f' - H_{\mathbf{a}}'(t)v$.

- $y = w'$ is solution in $\mathrm{L}^2(0, T; \mathcal{H}_0)$ to

$$\begin{cases} -\mathrm{i}y'(t,x) + H_{\mathbf{a}}(t)y(t,x) = q(t,x), & (t,x) \in Q_T^+ \\ y(t,x) = 0, & (t,x) \in \Sigma_T^+ \\ y(0,x) = -2\chi'(0)(\tilde{\mathbf{a}}(x) - \mathbf{a}(x)).\nabla u_0(x), & x \in \Omega, \end{cases}$$

where $q := g' - H_{\mathbf{a}}'(t)w$.

- $w, y \in \mathrm{C}^0([0, T]; \mathcal{H}_2) \cap \mathrm{C}^1([0, T]; \mathcal{H}_0)$.

Time symmetrization

- The method we use requires that a Carleman estimate for y be established in $Q_T = (-T, T) \times \Omega$ (in order to center the problem around the initial condition u_0).
- Extend u on $Q_T^- = (-T, 0) \times \Omega$ by setting
  (i) $\chi(t) = -\chi(-t),\ t \in (-T, 0)$,
  (ii) $u(t, x) = \overline{u(-t, x)},\ (t, x) \in Q_T^-$.
- Condition: u_0 is real valued.

Geometric condition on Γ^+

We introduce an open set $\Gamma^+ \subset \Gamma$ and $\tilde{\beta} \in \mathrm{C}^4(\overline{\Omega}; \mathbb{R}_+)$, satisfying:

(a) $|\nabla \tilde{\beta}(x)| \geq C_0 > 0$ for all $x \in \Omega$;

(b) $\partial_\nu \widetilde{\beta}(\sigma) = \nabla \tilde{\beta}(\sigma).\nu(\sigma) \leq 0$ for all $\sigma \in \Gamma^- = \Gamma \backslash \Gamma^+$;

(c) $\exists \Lambda_1 > 0$, $\exists \epsilon > 0$ such that

$$\lambda |\nabla \tilde{\beta}(x).\zeta|^2 + \langle D^2 \tilde{\beta} \zeta, \zeta \rangle_{\mathbb{C}^n} \geq \epsilon |\zeta|^2, \ \forall \zeta \in \mathbb{C}^n, \ \forall \lambda > \Lambda_1,$$

where $D^2 \tilde{\beta} = \left(\frac{\partial^2 \tilde{\beta}}{\partial x_i \partial x_j} \right)_{1 \leq i,j \leq n}$.

(d) Classical example : $x_0 \in \mathbb{R}^n \backslash \bar{\Omega}$, $\tilde{\beta}(x) = |x - x_0|^2$ and

$$\{x \in \Gamma, \ (x - x_0).\nu(x) \geq 0\} \subset \Gamma^+.$$

Definitions

- **Weight functions:**

$$\varphi(t,x) = \frac{e^{\lambda\beta(x)}}{(T+t)(T-t)}, \ \eta(t,x) = \frac{e^{2\lambda K} - e^{\lambda\beta(x)}}{(T+t)(T-t)},$$

for $(t,x) \in Q_T$ and $\lambda > 0$, where:

$$\beta := \widetilde{\beta} + K \text{ and } K := m\|\tilde{\beta}\|_\infty \text{ for some } m > 1.$$

- **Differential operators:**

$$M_1 := i\partial_t + \Delta + s^2|\nabla\eta|^2, \ M_2 := s(i\eta' + 2\nabla\eta.\nabla + (\Delta\eta)),$$

in such a way that

$$M_1 + M_2 = e^{-s\eta} L e^{s\eta}, \ L := i\partial_t + \Delta.$$

Global Carleman estimate

There exists $\lambda_0 > 0$, $s_0 > 0$ and $C_0 = C_0(T, \Omega, \Gamma^+, \lambda_0, s_0) > 0$ s.t.

$$
\begin{aligned}
I(z) &= s^3\lambda^4 \|\mathrm{e}^{-s\eta}\varphi^{3/2} z\|^2_{\mathrm{L}^2(Q_T)} + s\lambda\|\mathrm{e}^{-s\eta}\varphi^{1/2}|\nabla z|\|^2_{\mathrm{L}^2(Q_T)} \\
&\quad + \sum_{j=1,2} \|e^{-s\eta} M_j z\|^2_{\mathrm{L}^2(Q_T)} \\
&\leq C_0 \left(s\lambda \int_{-T}^{T}\int_{\Gamma^+} \mathrm{e}^{-2s\eta(t,\sigma)}\varphi(t,\sigma)\partial_\nu\beta(\sigma)|\partial_\nu z(t,\sigma)|^2 \mathrm{d}\sigma \mathrm{d}t \right. \\
&\quad \left. + \|\mathrm{e}^{-s\eta} L z\|^2_{\mathrm{L}^2(Q_T)} \right),
\end{aligned}
$$

for all $\lambda \geq \lambda_0$, all $s \geq s_0$, and all $z \in \mathrm{L}^2(-T, T; \mathcal{H}_1)$ satisfying

$$
Lz \in \mathrm{L}^2(Q_T) \text{ and } \partial_\nu z \in \mathrm{L}^2(-T, T; \mathrm{L}^2(\Gamma)).
$$

Carleman inequality for w

Recall that $-\mathrm{i}w' + H_{\mathbf{a}}(t)w = g = f' - H'_{\mathbf{a}}(t)v$ in $\mathrm{L}^2(-T,T;\mathcal{H}_0)$.

Global Carleman estimate applied to $Lw = 2\chi\mathbf{a}.(\mathrm{i}\nabla + \chi\mathbf{a})w - g$:

$$I(w) \leq C_0 \left(s\lambda \int_{-T}^{T} \int_{\Gamma^+} \mathrm{e}^{-2s\eta(t,\sigma)} \varphi(t,\sigma)\partial_\nu\beta(t,\sigma) \, |\partial_\nu w(t,\sigma)|^2 \, \mathrm{d}\sigma \mathrm{d}t \right.$$
$$\left. + \|\mathrm{e}^{-s\eta} f'\|^2_{\mathrm{L}^2(Q_T)} + \sum_{\rho=v,w} \|\mathrm{e}^{-s\eta}(|\rho|^2 + |\nabla\rho|^2)^{1/2}\|^2_{\mathrm{L}^2(Q_T)} \right).$$

- $\nabla v(x,t) = \int_0^t \nabla w(\xi,x)\mathrm{d}\xi$ so $\exists \kappa = \kappa(T,\lambda_0) > 0$ s.t.
  $$\|\mathrm{e}^{-s\eta}(|v|^2 + |\nabla v|^2)^{1/2}\|^2_{\mathrm{L}^2(Q_T)} \leq \frac{\kappa}{s}\|\mathrm{e}^{-s\eta}(|w|^2 + |\nabla w|^2)^{1/2}\|^2_{\mathrm{L}^2(Q_T)}.$$
- $\|\mathrm{e}^{-s\eta}(|w|^2 + |\nabla w|^2)^{1/2}\|^2_{\mathrm{L}^2(Q_T)} \ll I(w)$, for $s \gg 1$,

Hence w may remove $\sum_{\rho=v,w} \dots$ from the r.h.s. provided $s \gg 1$.

Carleman estimate for y

Similarly $Ly = 2\chi\mathbf{a}.(\mathrm{i}\nabla + \chi\mathbf{a})y - q$ with $q = g' - H_{\mathbf{a}}'(t)w$ entails:

$$I(y) \leq C_0 \Bigg(s\lambda \int_{-T}^{T} \int_{\Gamma^+} \mathrm{e}^{-2s\eta(t,\sigma)} \varphi(t,\sigma)\, \partial_\nu\beta(t,\sigma)\, |\partial_\nu y(t,\sigma)|^2 \mathrm{d}\sigma \mathrm{d}t$$

$$+ \|\mathrm{e}^{-s\eta} f''\|^2_{\mathrm{L}^2(Q_T)} + \sum_{\rho=v,w,y} \|\mathrm{e}^{-s\eta}(|\rho|^2 + |\nabla\rho|^2)^{1/2}\|^2_{\mathrm{L}^2(Q_T)} \Bigg).$$

Since

$$\|\mathrm{e}^{-s\eta}\,(|v|^2 + |\nabla v|^2)^{1/2}\|^2_{\mathrm{L}^2(Q_T)} \leq \frac{\kappa}{s} \|\mathrm{e}^{-s\eta}\,(|w|^2 + |\nabla w|^2)^{1/2}\|^2_{\mathrm{L}^2(Q_T)},$$

and

$$\|\mathrm{e}^{-s\eta}(|\rho|^2 + |\nabla\rho|^2)^{1/2}\|^2_{\mathrm{L}^2(Q_T)} \ll I(\rho), \text{ for } s \gg 1, \ \rho = w, y,$$

We find (upon taking $s \gg 1$) that :

$$I(y) \leq C_0 \left(s\lambda \int_{-T}^{T} \int_{\Gamma^+} \mathrm{e}^{-2s\eta(t,\sigma)} \varphi(t,\sigma) \partial_\nu \beta(t,\sigma) \, |\partial_\nu y(t,\sigma)|^2 \, \mathrm{d}\sigma \mathrm{d}t + \| \mathrm{e}^{-s\eta} f'' \|^2_{\mathrm{L}^2(Q_T)} + I(w) \right).$$

Recall that

$$I(w) \leq C_0 \left(s\lambda \int_{-T}^{T} \int_{\Gamma^+} \mathrm{e}^{-2s\eta(t,\sigma)} \varphi(t,\sigma) \partial_\nu \beta(t,\sigma) \, |\partial_\nu w(t,\sigma)|^2 \, \mathrm{d}\sigma \mathrm{d}t + \| \mathrm{e}^{-s\eta} f' \|^2_{\mathrm{L}^2(Q_T)} \right),$$

$$\begin{aligned} f' &= (\tilde{\mathbf{a}} - \mathbf{a}).[\chi(\tilde{\mathbf{a}} + \mathbf{a})(2\chi' \tilde{u} + \chi \tilde{u}') + 2\mathrm{i}(\chi' \nabla \tilde{u} + \chi \nabla \tilde{u}')], \\ f'' &= (\tilde{\mathbf{a}} - \mathbf{a}).[2\mathrm{i}(\chi'' \nabla \tilde{u} + 2\chi' \nabla \tilde{u}' + \chi \nabla \tilde{u}'') \\ &\quad + (\tilde{\mathbf{a}} + \mathbf{a})(2\chi'^2 + \chi\chi'')\tilde{u} + 4\chi\chi' \tilde{u}' + \chi^2 \tilde{u}''], \end{aligned}$$

+ the "conservation" of charge + energy...

Uniform time boundedness of charge and energy

- There exists $c_0 = c_0(T, \|\mathbf{a}\|_{\mathrm{C}^1([0,T])}, \Omega) > 0$ s.t. the $\mathrm{C}^0([0,T];\mathcal{H}_1) \cap \mathrm{C}^1([0,T];\mathcal{H}_{-1})$-solution ψ to (2.2), satisfies:

$$\|\psi(t)\|_1 \leq c_0(\|\psi_0\|_1 + \|f\|_{W(0,T;\mathcal{H}_0,\mathcal{H}_{-1})}),\ t \in [0,T],$$

  for every $\psi_0 \in \mathcal{H}_1$ and $f \in W(0,T;\mathcal{H}_0,\mathcal{H}_{-1})$.
- **Consequence:** $\exists c = c(T, \|\chi\|_{\mathrm{C}^3([0,T])}, \Omega, M) > 0$ such that $\tilde{u}$ satisfies:

$$\left\|(\partial^j \tilde{u}/\partial t^j)(t)\right\|_1 \leq c \sum_{k=0}^{j} \|\Delta^k u_0\|_1,\ t \in [0,T],\ j = 0,1,2.$$

...thus we end up getting:

Global Carleman estimate for y

$\exists \lambda_0 > 0$, $s_0 > 0$ and $C_1 = C_1(T, \Omega, \Gamma^+, \lambda_0, s_0, u_0) > 0$ s.t.

$$I(y) \leq C_1 \left(s\lambda \sum_{\rho=y,w} \int_{-T}^{T} \int_{\Gamma^+} \mathrm{e}^{-2s\eta} \varphi \partial_\nu \beta |\partial_\nu \rho|^2 \mathrm{d}\sigma \mathrm{d}t + \|\mathrm{e}^{-s\eta}(\mathbf{a} - \tilde{\mathbf{a}})\|^2_{\mathrm{L}^2(Q_T)^n} \right), \ \lambda \geq \lambda_0, \ s \geq s_0,$$

where

$$I(y) = s^3\lambda^4 \|\mathrm{e}^{-s\eta}\varphi^{3/2} y\|^2_{\mathrm{L}^2(Q_T)} + s\lambda \|\mathrm{e}^{-s\eta}\varphi^{1/2}|\nabla y|\|^2_{\mathrm{L}^2(Q_T)} + \sum_{j=1,2} \|M_j e^{-s\eta} y\|^2_{\mathrm{L}^2(Q_T)},$$

$$M_1 = \mathrm{i}\partial_t + \Delta + s^2|\nabla \eta|^2.$$

Stability inequality

Set $\psi = \mathrm{e}^{-s\eta} y$ so that $\psi(-T,.) = 0$, and

$$\begin{aligned} \mathcal{I} &= \|\mathrm{e}^{-s\eta(0,.)} y(0,.)\|^2_{\mathrm{L}^2(\Omega)} = \int_{-T}^{0}\int_\Omega \partial_t |\psi(t,x)|^2 \mathrm{d}x\mathrm{d}t \\ &= 2\mathrm{Re}\left(\int_{-T}^{0}\int_\Omega \psi'(t,x)\overline{\psi(t,x)}\mathrm{d}x\mathrm{d}t\right) \\ &= 2\mathrm{Im}\left(\int_{-T}^{0}\int_\Omega M_1\psi(t,x)\overline{\psi(t,x)}\mathrm{d}t\mathrm{d}x\right). \end{aligned}$$

Hence for all $s \geq s_0$, $\lambda \geq \lambda_0$:

$$|\mathcal{I}| \leq Cs^{-3/2}\lambda^{-2}\left(s^3\lambda^4\|\mathrm{e}^{-s\eta}\varphi^{3/2}y\|^2_{\mathrm{L}^2(Q_T)} + \|M_1(\mathrm{e}^{-s\eta}y)\|^2_{\mathrm{L}^2(Q_T)}\right).$$

Use Global Carleman estimate for y:

$$\mathcal{I} \leq Cs^{-1/2}\lambda^{-1}\left(\sum_{\rho=w,y}\int_{-T}^{T}\int_{\Gamma^+} \mathrm{e}^{-2s\eta}\varphi\partial_\nu\beta|\partial_\nu\rho|^2\mathrm{d}\sigma\mathrm{d}t + s^{-1}\lambda^{-1}\|\mathrm{e}^{-s\eta}(\tilde{\mathbf{a}}-\mathbf{a})\|^2_{\mathrm{L}^2(Q_T)^n}\right), \ \lambda \geq \lambda_0, s \geq s_0.$$

Add the two following ingredients:

- $\mathcal{I} = 4\chi'(0)^2\|\mathrm{e}^{-s\eta(0,.)}(\tilde{\mathbf{a}}-\mathbf{a}).\nabla u_0\|_0^2$
- $\eta(t,x) \geq \eta(0,x), \ (t,x) \in Q_T$

$+$ recall that $w = (u-\tilde{u})'$, $y = (u-\tilde{u})''$...

... and find out for all $j = 1, \ldots, n$, that:

$$C\left(\|\mathrm{e}^{-s\eta(0,.)}(\tilde{\mathbf{a}}-\mathbf{a}).\nabla u_{0,j}\|^2_{\mathrm{L}^2(\Omega)} - s^{-3/2}\lambda^{-2}\|\mathrm{e}^{-s\eta(0,.)}(\tilde{\mathbf{a}}-\mathbf{a})\|^2_{\mathrm{L}^2(\Omega)^n}\right)$$

$$\leq s^{-1/2}\lambda^{-1}\int_{-T}^{T}\int_{\Gamma^+}\mathrm{e}^{-2s\eta}\varphi\partial_\nu\beta\times\left(\sum_{k=1,2}|\partial_\nu\partial_t^k(\tilde{u}_j-u_j)|^2\right)\mathrm{d}\sigma\mathrm{d}t.$$

Summing up over $j = 1, \ldots, n$, we end up getting:

$$C\left(\|\mathrm{e}^{-s\eta(0,.)}DU_0(\tilde{\mathbf{a}}-\mathbf{a})\|^2_{\mathrm{L}^2(\Omega)^n} - ns^{-3/2}\lambda^{-2}\|\mathrm{e}^{-s\eta(0,.)}(\tilde{\mathbf{a}}-\mathbf{a})\|^2_{\mathrm{L}^2(\Omega)^n}\right)$$

$$\leq s^{-1/2}\lambda^{-1}\int_{-T}^{T}\int_{\Gamma^+}\mathrm{e}^{-2s\eta}\varphi\partial_\nu\beta\left(\sum_{\substack{j=1,\ldots,n\\ k=1,2}}|\partial_\nu\partial_t^k(\tilde{u}_j-u_j)|^2\right)\mathrm{d}\sigma\mathrm{d}t.$$

$\{\mu_j(x)\}_{j=1}^n \subset \mathbb{R}_+^n$

$=$ non decreasing sequence of the singular values of $DU_0(x)$

$$\|DU_0(x)v\|_{\mathbb{C}^n} \geq \mu_1(x)\|v\|_{\mathbb{C}^n},\ x \in \Omega,\ v \in \mathbb{C}^n.$$

- $u_{0,j} \in \mathrm{H}^p(\Omega),\ p > n/2+1 \Rightarrow u_{0,j} \in \mathrm{C}^1(\overline{\Omega}),\ j = 1, \ldots, n$, hence:
$$\mu_1 \in \mathrm{C}^0(\overline{\Omega}; \mathbb{R}_+).$$
- $\det\ DU_0(x) \neq 0,\ x \in \Omega \Rightarrow \mu_1(x) > 0,\ x \in \Omega.$

$$\Longrightarrow \mu = \mu(\Omega, \{u_{0,j}\}_{j=1}^n) := \inf_{x \in \Omega} \mu_1(x) > 0.$$

As a consequence:

$$\|\mathrm{e}^{-s\eta(0,.)} DU_0(\tilde{\mathbf{a}} - \mathbf{a})\|_{\mathrm{L}^2(\Omega)^n} \geq \mu \|\mathrm{e}^{-s\eta(0,.)}(\tilde{\mathbf{a}} - \mathbf{a})\|_{\mathrm{L}^2(\Omega)^n}.$$

Perspectives

(a) Avoid the geometric hypothesis (FBI method)
(b) Improve the logarithmic inequality obtained in this case
(c) Treat the case of Yang-Mills potentials