Contrôlabilité d’un problème d’interface diffusive

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Outline

Introduction: the model

Well-posedness and asymptotic behavior

Controllability

The Carleman estimate
   Statement and preliminaries
   Operators and phase-space regions
Introduction: geometry of the problem

- $\Omega \subset \mathbb{R}^n$,
- $S = (n - 1)$-dimensional submanifold of $\Omega$ such that $\Omega \setminus S = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$,
- $\omega \subset \Omega_2$ for instance.
Introduction: the problem

Notation $z|_{S_2} := (z|_{\Omega_2})|_S$

\[
\begin{aligned}
\partial_t z - \Delta_c z &= 1_\omega u \quad \text{in } (0, T) \times \Omega_1 \cup \Omega_2, \\
\partial_t z^s - \Delta_{c^s} z^s &= \frac{1}{\delta} \left( (c \partial_\eta z)|_{S_2} - (c \partial_\eta z)|_{S_1} \right) \quad \text{in } (0, T) \times S, \\
z|_{S_1} &= z^s = z|_{S_2} \quad \text{in } (0, T) \times S, \\
z|_{\partial \Omega} &= 0; \\
\end{aligned}
\]

- Initial conditions $(z, z^s)|_{t=0} \in L^2(\Omega_1 \cup \Omega_2) \times L^2(S)$
- $\delta = \text{constant parameter, } 0 < \delta \leq \delta_0$
- Diffusion operator on $\Omega_1 \cup \Omega_2$: $\Delta_c = \text{div } c(x) \nabla$
- Diffusion operator on $S$:
  \[
  \Delta_{c^s} = \text{div}^s c^s(x) \nabla^s = \frac{1}{\sqrt{\text{det}(g)}} \sum_{i,j} \partial_{x_i} (c^s g^{ij} \sqrt{\text{det}(g)} \partial_{x_j})
  \]
  $(g = \text{metric on } S, \text{ inherited from the euclidean metric on } \mathbb{R}^n)$
Introduction: the model, derivation in euclidean setting

- Multidimensional structures; useful for efficient numerical methods
- Koch-Zuazua ’06: coupled waves in $n$ and $n-1$ dimensions
- Lescarret-Zuazua ’10: numerical analysis for waves

Figure: Local geometry of a three-layer model near the interface $S = \{x_n = 0\}$. The inner layer, $\Omega_0$, shrinks to zero as $\delta$ goes to zero.
Introduction: the model, derivation in euclidean setting

Three diffusion coefficients \( c^j \) in \( \Omega_j \), \( j = 1, 0, 2 \).

Three diffusion equations

\[
\partial_t z^j - \text{div}(c^j \nabla z^j) = 0 \quad \text{in} \ (0, T) \times \Omega_j, \quad j = 1, 0, 2.
\]

Natural transmission conditions at \( x_n = \frac{\delta}{2} \) and \( x_n = -\frac{\delta}{2} \), i.e.

- continuity of the solution:

\[
\begin{align*}
 z^1|_{x_n = -\frac{\delta}{2}} &= z^0|_{x_n = -\frac{\delta}{2}}, \\
 z^0|_{x_n = \frac{\delta}{2}} &= z^2|_{x_n = \frac{\delta}{2}}.
\end{align*}
\]

- continuity of the flux:

\[
\begin{align*}
 (c^1 \partial_{x_n} z^1)|_{x_n = -\frac{\delta}{2}} &= (c^0 \partial_{x_n} z^0)|_{x_n = -\frac{\delta}{2}}, \\
 (c^0 \partial_{x_n} z^0)|_{x_n = \frac{\delta}{2}} &= (c^2 \partial_{x_n} z^2)|_{x_n = \frac{\delta}{2}}.
\end{align*}
\]
Introduction: the model, derivation in euclidean setting

- Assumption: $c^0$ does not depend on the normal variable $x_n$. We set $c^s(y) := c^0(y, x_n)$.
- Mean values of $z^0$ in the normal direction $x_n$:

$$z^s(y) := \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} z^0(y, x_n) dx_n$$

- Integrating for $x_n \in (-\delta/2, \delta/2)$ the equation in $\Omega_0$ yields

$$0 = \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} \left( \partial_t z^0 - \text{div}(c^0 \nabla z^0) \right) dx_n$$

$$= \partial_t z^s - \text{div}^s (c^s \nabla^s z^s) - \frac{1}{\delta} \left( (c^2 \partial_{x_n} z^2)_{|x_n = \delta/2} - (c^1 \partial_{x_n} z^1)_{|x_n = -\delta/2} \right).$$
Introduction: the limit $\delta \to 0^+$

\[
\begin{cases}
\partial_t z - \Delta_c z = \mathbbm{1}_\omega u & \text{in } (0, T) \times \Omega_1 \cup \Omega_2, \\
\partial_t z^s - \Delta_c z^s = \frac{1}{\delta} \left( (c \partial_\eta z)|_{S_2} - (c \partial_\eta z)|_{S_1} \right) & \text{in } (0, T) \times S, \\
z|_{S_1} = z^s = z|_{S_2} & \text{in } (0, T) \times S, \\
z|_{\partial \Omega} = 0;
\end{cases} \tag{P_\delta}
\]

- Formally, convergence towards the classical transmission problem

\[
\begin{cases}
\partial_t z - \Delta_c z = \mathbbm{1}_\omega u & \text{in } (0, T) \times \Omega_1 \cup \Omega_2, \\
(c \partial_\eta z)|_{S_2} = (c \partial_\eta z)|_{S_1} \quad \text{and } z|_{S_1} = z|_{S_2} & \text{in } (0, T) \times S, \\
z|_{\partial \Omega} = 0;
\end{cases} \tag{P_0}
\]

- $(P_0)$ is well-posed
- $(P_0)$ is null-controllable (Le Rousseau-Robbiano '10)
### Introduction: some basic questions

- Well-posedness of \( (P_{\delta}) \) for fixed \( \delta \)?
- Convergence of the solutions of \( (P_{\delta}) \) towards those of \( (P_0) \) when \( \delta \to 0^+ \)?
- Null controllability of \( (P_{\delta}) \) for fixed \( \delta \)? i.e. is it possible to drive the solution \( (z, z^s) \) to zero in time \( T \)?
- Convergence of the control in the limit \( \delta \to 0^+ \), i.e. uniform controllability?
Reformulation of the problem: function spaces

Hilbert space \( H_\delta^0 = L^2(\Omega_1 \cup \Omega_2) \times L^2(S) \) with the inner product
\[
(Z, \tilde{Z})_{H_\delta^0} = (z, \tilde{z})_{L^2(\Omega_1 \cup \Omega_2)} + \delta (z^s, \tilde{z}^s)_{L^2(S)}, \quad Z = (z, z^s), \quad \tilde{Z} = (\tilde{z}, \tilde{z}^s),
\]

Hilbert space
\[
H_\delta^1 = \{ Z = (z, z^s) \in H^1(\Omega_1 \cup \Omega_2) \times H^1(S) ; z|_{\partial \Omega} = 0 ; z|_{S_1} = z^s = z|_{S_2} \},
\]

with the inner product
\[
(Z, \tilde{Z})_{H_\delta^1} = (Z, \tilde{Z})_{H_\delta^0} + (c \nabla z, \nabla \tilde{z})_{L^2(\Omega_1 \cup \Omega_2)} + \delta (c^s \nabla^s z, \nabla^s \tilde{z}^s)_{L^2(S)}.
\]
Reformulation of the problem

Problem \((\mathcal{P}_\delta)\) can be written as

\[
\partial_t Z + A_\delta Z = Bu
\]

where \(Z = (z, z^s) \in \mathcal{H}^0_\delta\), \(B : u \mapsto t(1_\omega u, 0)\), and

\[
A_\delta Z = \begin{pmatrix}
-\Delta_c z \\
-\Delta_c z^s - \frac{1}{\delta} ((c\partial_\eta z)|_{S_2} - (c\partial_\eta z)|_{S_1})
\end{pmatrix}
\]

with domain

\[
D(A_\delta) = \{(z, z^s) \in \mathcal{H}^1_\delta; \ A_\delta(z, z^s) \in \mathcal{H}^0_\delta\}.
\]

Proposition

The operator \((A_\delta, D(A_\delta))\) is a coercive self-adjoint operator on \(\mathcal{H}^0_\delta\).

System \((\mathcal{P}_\delta)\) is well-posed for an initial condition in \(\mathcal{H}^0_\delta\) and a control \(u \in L^2((0, T) \times \omega)\).
Convergence \((P_\delta) \to (P_0)\) as \(\delta \to 0^+\)

Notation:

- \((z_\delta, z_\delta^s)\) solution of \((P_\delta)\) associated to the control \(u_\delta\) and the initial data \((z_0, z_0^s)\)
- \(z\) solution of \((P_0)\) associated to the control \(u\) and the initial data \(z_0\)
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Proposition

Suppose that \(u_\delta \rightharpoonup u\) in \(L^2((0, T) \times \omega)\), and that \((z_0, z_0^s) \in \mathcal{H}^1_\delta(\Omega)\).
Convergence $(\mathcal{P}_\delta) \to (\mathcal{P}_0)$ as $\delta \to 0^+$

Notation:

- $(z_\delta, z^s_\delta)$ solution of $(\mathcal{P}_\delta)$ associated to the control $u_\delta$ and the initial data $(z_0, z^s_0)$
- $z$ solution of $(\mathcal{P}_0)$ associated to the control $u$ and the initial data $z_0$

Proposition

Suppose that $u_\delta \rightharpoonup u$ in $L^2((0, T) \times \omega)$, and that $(z_0, z^s_0) \in \mathcal{H}^{1}_{\delta}(\Omega)$. Then $z_\delta$ satisfies

$$z_\delta|_{\Omega_j} \rightharpoonup z|_{\Omega_j}, \quad \text{in} \quad L^2(0, T; H^2(\Omega_j)) \cap H^1(0, T; L^2(\Omega_j))$$

for $j = 1, 2$. 
Convergence \((\mathcal{P}_\delta)\rightarrow(\mathcal{P}_0)\) as \(\delta \rightarrow 0^+\)

Notation:

- \((z_\delta, z^s_\delta)\) solution of \((\mathcal{P}_\delta)\) associated to the control \(u_\delta\) and the initial data \((z_0, z^s_0)\)
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Suppose that \(u_\delta \rightharpoonup u\) in \(L^2((0, T) \times \omega)\), and that \((z_0, z^s_0) \in \mathcal{H}^1_\delta(\Omega)\).

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\[ z_\delta|_{\Omega_j} \rightharpoonup z|_{\Omega_j}, \quad \text{in} \quad L^2(0, T; H^2(\Omega_j)) \cap H^1(0, T; L^2(\Omega_j)) \]

for \(j = 1, 2\).

\(\implies\) Hope: construct for \((\mathcal{P}_\delta)\) a control \(u_\delta\) uniformly bounded, which yields a particular control for the limit problem.
Controllability of parabolic equations

Methods to prove null controllability for \((\partial_t + A)z = Bu\):

- Fursikov-Imanuvilov ('96): global Carleman estimate for the \textit{parabolic} operator \((-\partial_t + A^*)\), with weight \(e^{-\frac{\varphi}{ht(T-t)}}\);
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- Lebeau-Robbiano ('95): Stationary method
  - local Carleman estimates for the \textit{elliptic} operator \(-\partial_{x_0}^2 + A^*\),
  \[\Downarrow\]
  - Interpolation inequalities,
  \[\Downarrow\]
  - Spectral inequality for sums of eigenfunction of \(A^*\) (Lebeau-Zuazua '98, Jerison-Lebeau '99),
  \[\Downarrow\]
  - iterative construction of the control.
Controllability of parabolic equations with coefficients with jumps

\[
\begin{cases}
\partial_t z - \Delta_c z = 1_\omega u & \text{in } (0, T) \times \Omega_1 \cup \Omega_2, \\
(c \partial_\eta z)|_{S_2} = (c \partial_\eta z)|_{S_1} \text{ and } z|_{S_1} = z|_{S_2} & \text{in } (0, T) \times S, \\
z|_{\partial \Omega} = 0;
\end{cases}
\]  

(P_0)

- Doubova-Osses-Puel '02: null controllability of (P_0) under a technical assumption: “c|_{S_2} \leq c|_{S_1}’’;
- Benabdallah-Dermenjian-Le Rousseau '07: this technical assumption is not necessary in stratified media;
- Le Rousseau-Robbiano '10: null controllability of (P_0) in the general case (Lebeau-Robbiano method);
- Le Rousseau-Robbiano '11: null controllability of (P_0) in the general case (Fursikov-Imanuvilov method);
- Le Rousseau-Lerner '11: null controllability of (P_0) in the anisotropic case (c is a diffusion matrix).
Main result: a Carleman estimate
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Main result

Theorem (Carleman estimate)

A Carleman estimate for the operator $-\partial_{x_0}^2 + A_\delta$ holds...
Main result: a Carleman estimate

Main result

**Theorem (Carleman estimate)**

A Carleman estimate for the operator \(-\partial_{x_0}^2 + A_\delta\) holds... ... uniformly w.r.t. \(0 < \delta \leq \delta_0\).
Consequences

Notation: $\mathcal{E}_{\delta,j} = (e_{\delta,j}, e_{\delta,j}^s)$, $j \in \mathbb{N}$, eigenfunctions of $A_{\delta}$ associated with the positive eigenvalues $\mu_{\delta,j} \in \mathbb{R}$, $j \in \mathbb{N}$.

$\mathcal{E}_{\delta,j}$: Hilbert basis of $\mathcal{H}_\delta^0$
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$\mathcal{E}_{\delta,j}$: Hilbert basis of $\mathcal{H}_\delta^0$

**Theorem (Spectral inequality)**

*For $\delta_0 > 0$, there exists $C > 0$ such that for all $0 < \delta \leq \delta_0$ and $\mu \in \mathbb{R}$, we have*

$$\|Z\|_{\mathcal{H}_\delta^0} \leq C e^{C\sqrt{\mu}} \|z\|_{L^2(\omega)}, \quad Z = (z, z^s) \in \text{span}\{\mathcal{E}_{\delta,j}; \mu_{\delta,j} \leq \mu\}.$$
Consequences

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$\mathcal{E}_{\delta,j}$: Hilbert basis of $\mathcal{H}_\delta^0$

**Theorem (Spectral inequality)**

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$$\|Z\|_{\mathcal{H}_\delta^0} \leq C e^{C \sqrt{\mu}} \|z\|_{L^2(\omega)}, \quad Z = (z, z^s) \in \text{span}\{\mathcal{E}_{\delta,j}; \mu_{\delta,j} \leq \mu\}.$$

**Theorem (Null controllability)**

Let $\delta_0 > 0$. For all $T > 0$ and $\omega \subset \Omega$, $\exists C > 0$ s.t.: for all $Z_0 = (z_0, z_0^s) \in \mathcal{H}_\delta^0$ and $0 < \delta \leq \delta_0$, there exists $u_\delta \in L^2((0, T) \times \omega)$ s.t. the solution $(z, z^s)$ of $(\mathcal{P}_\delta)$ satisfies $(z(T), z^s(T)) = (0, 0)$ and moreover

$$\|u_\delta\|_{L^2((0, T) \times \omega)} \leq C \|Z_0\|_{\mathcal{H}_\delta^0}.$$
Some remarks

- Uniform controllability as $\delta \to 0^+$; extract a subsequence $u_\delta$ weakly convergent in $L^2((0, T) \times \omega)$ $\implies$ the associated solution of Problem $(\mathcal{P}_\delta)$ converges towards a controlled solution of Problem $(\mathcal{P}_0)$.
  For $(\mathcal{P}_0)$, we hence construct a control function robust w.r.t. small viscous perturbations in the interface.
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- A boundary control from a subset of $\partial \Omega$ also holds.
Some remarks

- Uniform controllability as $\delta \to 0^+$; extract a subsequence $u_\delta$ weakly convergent in $L^2((0, T) \times \omega) \implies$ the associated solution of Problem ($\mathcal{P}_\delta$) converges towards a controlled solution of Problem ($\mathcal{P}_0$).
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- A boundary control from a subset of $\partial \Omega$ also holds.

- A control result from the interface $S$ is false (in general) $\rightarrow$ counterexamples on the flat cylinder.
The (semi-global) Carleman estimate

Notation: $V$ a small neighborhood of $S$ in $\Omega$
$\zeta$ a cut-off function $\text{supp}(\zeta) \subset V$, $\zeta = 1$ on $S$
Consider the system

\[
\begin{cases}
-\partial^2_{x_0} w - \Delta_c w = f & \text{in } (0, X_0) \times V \\
-\partial^2_{x_0} w^s - \Delta_{c^s} w^s \\
\quad = \frac{1}{\delta} \left( (c \partial_\eta w)_{|(0, x_0) \times S_2} - (c \partial_\eta w)_{|(0, x_0) \times S_1} + \theta^s \right) & \text{in } (0, X_0) \times S, \quad (1) \\
w_{|(0, x_0) \times S_1} = w^s = w_{|(0, x_0) \times S_2} & \text{in } (0, X_0) \times S.
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w|_{(0,x_0) \times S_1} = w^s = w|_{(0,x_0) \times S_2} & \text{in } (0, X_0) \times S.
\end{cases}
$$

(1)

Theorem (Carleman estimate, with J. Le Rousseau and L. Robbiano)

For a well-chosen weight function $\varphi$, $\forall \delta_0 > 0$, $\exists C > 0$, and $h_0 > 0$ such that

$$
h \| e^{\varphi/h} w \|_0^2 + h^3 \| e^{\varphi/h} \nabla_{x_0,x} w \|_0^2 + h \| e^{\varphi/h} w \|_{S_0}^2 + h^3 \sum_{j=1,2} | e^{\varphi/h} \nabla_{x_0,x} w |_{S_j}^2 \\
\leq C \left( h^4 \| e^{\varphi/h} f \|_0^2 + h^2 \delta^2 \| \zeta e^{\varphi/h} f \|_{V_2}^2 + h^3 \| e^{\varphi/h} \theta^s \|_0^2 \right),
$$

for all $0 < \delta < \delta_0$, $0 < h \leq h_0$, for $(w, \theta^s, f)$ satisfying (1).
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-\partial_{x_0}^2 w - \Delta_c w &= f \quad \text{in } (0, X_0) \times V \\
-\partial_{x_0}^2 w^s - \Delta_c^s w^s &= \frac{1}{\delta} \left( (c \partial_\eta w)_{|(0,X_0) \times S_2} - (c \partial_\eta w)_{|(0,X_0) \times S_1} + \theta^s \right) \quad \text{in } (0, X_0) \times S, \\
w_{|(0,X_0) \times S_1} &= w^s = w_{|(0,X_0) \times S_2} 
\end{align*}
\]

(1)

Theorem (Carleman estimate, with J. Le Rousseau and L. Robbiano)

For a well-chosen weight function \( \varphi \), \( \forall \delta_0 > 0 \), \( \exists C > 0 \), and \( h_0 > 0 \) such that

\[
h \| e^{\varphi/h} w \|_0^2 + h^3 \| e^{\varphi/h} \nabla_{x_0,x} w \|_0^2 + h \| e^{\varphi/h} w \|_S^2 + h^3 \sum_{j=1,2} | e^{\varphi/h} \nabla_{x_0,x} w |_{S_j}^2 \]
\[
\leq C \left( h^4 \| e^{\varphi/h} f \|_0^2 + h^2 \delta^2 \| \zeta e^{\varphi/h} f \|_{V_2}^2 + h^3 | e^{\varphi/h} \theta^s |_0^2 \right),
\]

for all \( 0 < \delta < \delta_0 \), \( 0 < h \leq h_0 \), for \((w, \theta^s, f)\) satisfying (1).
The (semi-global) Carleman estimate: comments

- In the r.h.s., the term
  \[ h^2 \delta^2 \| \zeta e^{\varphi/h} f |_{V_2} \|_0^2 \]
  prevents from absorbing first order terms (\( \delta \) being fixed)
  \( \implies \) prevents from patching together local estimates
  \( \implies \) we cannot work in local coordinates only
  \( \implies \) we have to work on whole \( V \)

- \( \delta = 0 \) we recover the Carleman estimate of Le Rousseau-Robbiano '10.
The (semi-global) Carleman estimate: comments

- Conditions on \( \varphi \) (keep in mind that \( \omega \) is in \( \Omega_2 \)):
  - \( \varphi \) continuous on \( \overline{V} \) (in particular at the interface);
  - \( \nabla \varphi \neq 0 \) on \( \overline{V} \);
  - \( \varphi \) satisfies a Hörmander subellipticity condition on \( \overline{V} \);
  - \( \varphi \) is increasing in the normal direction from \( V_1 \) to \( V_2 \);
  - \( \partial_\eta \varphi|_{S_2} - \partial_\eta \varphi|_{S_1} \) sufficiently large;
  - \( \nabla_{x_0,x'} \varphi \) sufficiently small.

Figure: Level sets of an example of an admissible weight functions \( \varphi \) in flat coordinates.
Sketch of the proof: simplifications

- Omit the additional $x_0$-direction
- Local (flat) coordinates in a neighborhood of $S$: $S = \{x_n = 0\}$
- $V = \{x = (x', x_n), \text{ s.t. } -\varepsilon < x_n < \varepsilon\}$
- $V_2 = \{0 < x_n < \varepsilon\} = \text{right side}$
- $V_1 = \{-\varepsilon < x_n < 0\} = \text{left side}$
- In the local setting: $w_2 \to w^r$ (right) and $w_1 \to w^l$ (left)
- Keep in mind that $\omega$ is in $\Omega_2$, i.e. on the right side
- The principal symbol of $-\Delta_{c^{r/l}}$ is $c^{r/l}(\xi_n^2 + |\xi'|^2)$, with $\xi' = (\xi_1, \cdots, \xi_{n-1})$
- The principal symbol of $-\Delta_{c^s}$ is $c^s|\xi'|^2$,
Sketch of the proof: 1 conjugation with the weight function

- conjugated differential operators

\[ P_{p}^{r/l} = h^2 e^{\varphi^{r/l}/h} \frac{1}{c^{r/l}} (-\Delta c^{r/l}) e^{-\varphi^{r/l}/h}, \quad P_{p}^{s} = h^2 e^{\varphi|s/h} \frac{1}{c^{s}} (-\Delta c^{s}) e^{-\varphi|s/h}, \]

- conjugated functions

\[ v^{r/l} = e^{\varphi^{r/l}/h} w^{r/l}, \quad v^{s} = e^{\varphi|s/h} w^{s}, \]

- the elliptic system can be rewritten as

\[
\begin{align*}
P_{p}^{r/l} v^{r/l} &= F^{r/l} & \text{in } V, \\
P_{p}^{s} v^{s} &= \frac{hi}{c^{s} \delta} (c^{r}(D_{x_{n}} + i\partial_{x_{n}} \varphi^{r}) v^{r}_{|x_{n}=0} + c^{l}(D_{x_{n}} + i\partial_{x_{n}} \varphi^{l}) v^{l}_{|x_{n}=0} + \Theta^{s}_{p}) & \text{in } S, \\
v^{r/l}_{|x_{n}=0} &= v^{s} & \text{in } S,
\end{align*}
\]

with \( D_{x_{n}} = h\partial_{x_{n}}/i. \)

- consider \( P_{p}^{r/l} \) and \( P_{p}^{s} \) as semi-classical differential operators.
Sketch of the proof: 1 analysis of the operators

Calculus rule: \( \text{operator} \leftrightarrow \text{symbol} \)

\[
D = \frac{h}{i} \partial \leftrightarrow \xi
\]
Sketch of the proof: 1 analysis of the operators

Calculus rule: \[ \text{operator} \longleftrightarrow \text{symbol} \]

\[ D = \frac{h}{i} \partial \longleftrightarrow \xi \]

semiclassical principal symbols

\[ p^s_\varphi(x', \xi') = \sigma(P^s_\varphi) = |\xi'|^2 - |\nabla_{x'} \varphi|_S^2 + 2i\xi' \cdot \nabla_{x'} \varphi|_S, \]

\[ p^{r/l}_\varphi(x, \xi) = \sigma(P^{r/l}_\varphi) = |\xi|^2 - |\nabla \varphi|^2 + 2i\xi \cdot \nabla \varphi^{r/l} \]

\[ = \xi_n^2 + 2i\xi_n \partial_{x_n} \varphi^{r/l} + |\xi'|^2 - |\nabla \varphi|^2 + 2i\xi' \cdot \nabla_{x'} \varphi^{r/l} = q_2(x, \xi') = q_1(x, \xi') \]

\[ = \xi_n^2 + 2i\xi_n \partial_{x_n} \varphi^{r/l} + q_2 + 2i q_1(x, \xi') \]

\[ = (\xi_n - \rho^{r/l,+}(x, \xi'))(\xi_n - \rho^{r/l,-}(x, \xi')) \]

Important remarks:

• the ellipticity properties of \( P^{r/l}_\varphi \) depend on the localization of the roots \( \rho^{r,l} \) in the complex plane!

• the ellipticity properties of \( P^s_\varphi \) depend only on \( \text{Char}(P^s_\varphi) = \{(x', \xi') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, p^s_\varphi(x', \xi') = 0 \} \).
Sketch of the proof: 1 analysis of the operators

Calculus rule: operator $\leftrightarrow$ symbol

$$D = \frac{h}{i}\partial \leftrightarrow \xi$$

Semiclassical principal symbols

$$p^s_\varphi(x',\xi') = \sigma(P^s_\varphi) = |\xi'|^2 - |\nabla_{x'}\varphi|_S^2 + 2i\xi' \cdot \nabla_{x'}\varphi|_S,$$

$$p^{r/l}_\varphi(x,\xi) = \sigma(P^{r/l}_\varphi) = |\xi|^2 - |\nabla\varphi|^2 + 2i\xi \cdot \nabla\varphi^{r/l}$$

$$= \xi_n^2 + 2i\xi_n\partial_{x_n}\varphi^{r/l} + |\xi'|^2 - |\nabla\varphi|^2 + 2i \underbrace{\xi' \cdot \nabla_{x'}\varphi^{r/l}}_{=q_2(x,\xi')} + \underbrace{|\xi|_2^2}_{=q_1(x,\xi')},$$

$$= \xi_n^2 + 2i\xi_n\partial_{x_n}\varphi^{r/l} + q_2 + 2iq_1(x,\xi')$$

$$= (\xi_n - \rho^{r/l,+}(x,\xi'))(\xi_n - \rho^{r/l,-}(x,\xi'))$$

Important remarks:

- the ellipticity properties of $P^{r/l}_\varphi$ depend on the localization of the roots $\rho^+, \rho^-$ in the complex plane!

- the ellipticity properties of $P^s_\varphi$ depend only on $\text{Char}(P^s_\varphi) = \{(x',\xi') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, p^s_\varphi(x',\xi') = 0\}$. 
Sketch of the proof: localization of the roots

At the interface,

\[
\begin{cases}
P^s_{\varphi} v^s = \frac{hi}{c^s\delta} (c^r(D_{x_n} + i\partial_{x_n}\varphi^r)v^r_{|x_n=0} + c^l(D_{x_n} + i\partial_{x_n}\varphi^l)v^l_{|x_n=0} + \Theta^s) & \text{in } S, \\
v^l_{|x_n=0} = v^r_{|x_n=0} = v^s & \text{in } S,
\end{cases}
\]

- THREE unknowns: $v^s$, $D_{x_n} v^r_{|x_n=0}$,
- ONE equation at the interface
- We have to find two more relations to “solve” the system
- $v^r_{|x_n=0}$ solves an “elliptic” problem on each side $\implies$ yield other relations...
- ... depending on the localization of the roots $\rho^+, \rho^-$ in the complex plane!
Sketch of the proof: localization of the roots

Focus on ONE SIDE ONLY (i.e. either $r$ or $l$): analysis of the roots $\rho^\pm(x, \xi')$:

1. **VERY GOOD configuration:** TWO relations between the two traces $v|_{x_n=0}$ and $D_{x_n}v|_{x_n=0}$. i.e. both traces are determined!

2. **GOOD configuration:** $\approx$ ONE relation between the two traces $v|_{x_n=0}$ and $D_{x_n}v|_{x_n=0}$

3. **GOOD configuration:** ONE relation between the two traces $v|_{x_n=0}$ and $D_{x_n}v|_{x_n=0}$

4. **BAD configuration:** NO relation between the two traces $v|_{x_n=0}$ and $D_{x_n}v|_{x_n=0}$
Sketch of the proof: localization of the roots

The previous can be quantified

\[ q_2(x, \xi') = |\xi'|^2 - |\nabla \varphi|^2, \quad q_1(x, \xi') = \xi' \cdot \nabla_x \varphi^{r/l} \]

Lemma (Lebeau-Robbiano '97, Le Rousseau-Robbiano '10)

Let

\[ \mu^{r/l}(x, \xi') = q^{r/l}_2(x, \xi') + \frac{(q^{r/l}_1(x, \xi'))^2}{(\partial_{x_n} \varphi^{r/l})^2}, \]

We have

- \( \mu^r > 0 \iff \text{GOOD for } P^r_{\varphi} \) (high frequencies)
- \( \mu^r \leq 0 \iff \text{VERY GOOD for } P^r_{\varphi} \) (low frequencies)
- \( \mu^l > 0 \iff \text{GOOD for } P^l_{\varphi} \) (high frequencies)
- \( \mu^l \leq 0 \iff \text{BAD for } P^l_{\varphi} \) (low frequencies)
Sketch of the proof: localization of the roots

Lemma

The properties of the weight function \( \varphi \) imply that

- \( \mu^l \leq 0 \implies \mu^r \leq 0 \), i.e. BAD situation \( P^l_\varphi \implies \) VERY GOOD situation \( P^r_\varphi \) (low frequencies)
- \( p^s_\varphi = 0 \implies \mu^l \leq 0 \)
Sketch of the proof: localization of the roots

Lemma

The properties of the weight function $\varphi$ imply that

- $\mu^l \leq 0 \iff \mu^r \leq 0$, i.e. BAD situation $P^l_\varphi \iff$ VERY GOOD situation $P^r_\varphi$ (low frequencies)
- $p^s_\varphi = 0 \implies \mu^l \leq 0$

Figure: Microlocal ellipticity regions
Sketch of the proof: microlocal decomposition

- Decompose the tangent phase-space into four regions $\mathcal{E}$, $\mathcal{X}$, $\mathcal{F}$, $\mathcal{G}$;
- In each region $\mathcal{E}$, $\mathcal{X}$, $\mathcal{F}$, $\mathcal{G}$: produce a microlocal Carleman estimate;
- Finally: patch together these 4 microlocal Carleman estimates.

Figure: Microlocal ellipticity regions
Sketch of the proof: microlocal decomposition

- Loss of a power $\frac{\delta^2}{\hbar^2}$ on $\text{Char}(P^s_\varphi)$; i.e. in region $\mathcal{G}$. Why?
- $P^l_\varphi$ BAD $\implies$ NO relation between $v^l_{|x_n=0}$ and $D_{x_n}v^l_{|x_n=0}$
- $P^r_\varphi$ VERY GOOD $\implies$ the two traces $v^r_{|x_n=0}$ and $D_{x_n}v^r_{|x_n=0}$ are determined.
- Hence, using

\[
\begin{cases}
    P^s_\varphi v^s = \frac{\hbar i}{c^s \delta} (c^r (D_{x_n} + i\partial_{x_n} \varphi^r) v^r_{|x_n=0} + c^l (D_{x_n} + i\partial_{x_n} \varphi^l) v^l_{|x_n=0} + \Theta^s_\varphi) & \text{in } S, \\
    v^l_{|x_n=0} = v^r_{|x_n=0} = v^s & \text{in } S,
\end{cases}
\]

we can deduce $D_{x_n}v^l_{|x_n=0}$!
Sketch of the proof: loss of $h^2$

- loss of a power $\frac{\delta^2}{h^2}$ on $\text{Char}(P^s_\varphi)$; i.e. in region $G$. Why?
- $P^r_\varphi$ VERY GOOD $\implies$ the two traces $v^r|_{x_n=0}$ and $D_{x_n}v^r|_{x_n=0}$ are determined.
- Hence, using

\[
\begin{cases}
P^s_\varphi v^s = \frac{hi}{c^s\delta} (c^r(D_{x_n} + i\partial_{x_n}\varphi^r)v^r|_{x_n=0} + c^l(D_{x_n} + i\partial_{x_n}\varphi^l)v^l|_{x_n=0} + \Theta^s) & \text{in } S, \\
v^l|_{x_n=0} = v^r|_{x_n=0} = v^s & \text{in } S,
\end{cases}
\]

we can deduce $D_{x_n}v^l|_{x_n=0}$!

\[
c^lD_{x_n}v^l|_{x_n=0} = \frac{c^s\delta}{hi} P^s_\varphi v^s - c^r D_{x_n}v^r|_{x_n=0} - i\partial_{x_n}\varphi^r v^r|_{x_n=0} - i\partial_{x_n}\varphi^l v^l|_{x_n=0} + \Theta^s
\]

\[
|D_{x_n}v^l|_{x_n=0}|_0 \lesssim \frac{\delta}{h} \left|v^r|_{x_n=0}|_0 + |D_{x_n}v^r|_{x_n=0}|_0 + |v^r|_{x_n=0}|_0 + |\Theta^s|_0\right|
\]

estimated on the right by $\|P^r_\varphi v^r\|_0$

- Recall that everything has to be done globally on $T^*S$
Other geometries

- Semiglobal estimates can be patched together $\implies$ a global Carleman estimate (see Le Rousseau-Robbiano '11)
- Several interfaces can be considered
- Other geometries can be addressed $\implies$ semiglobal nature of the Carleman estimate

Figure: Other geometrical situations: (a) $\Omega$ is an bounded open subset of $\mathbb{R}^n$; (b) and (c) $\Omega$ is a compact manifold with boundary.
Some open problems

- Optimality of the powers of $h$?
- Produce a parabolic Carleman estimate (in the spirit of Fursikov-Imanuilov/Le Rousseau-Robbiano ’11)?
- Case of three non-conformal metrics at the interface (in the spirit of Le Rousseau-Lerner ’11)?
- If $S$ touches the boundary?