Contrôlabilité d'un problème d'interface diffusive

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collaboration avec Jérôme Le Rousseau and Luc Robbiano
workshop "un peu de contrôle à Clermont"
Clermont-Ferrand le 27 juin 2011

Outline

Introduction: the model

Well-posedness and asymptotic behavior

Controllability

The Carleman estimate
- Statement and preliminaries
- Operators and phase-space regions

Introduction: geometry of the problem

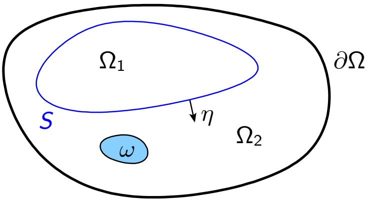

Figure: Geometry of the interface problem

- $\Omega \subset \mathbb{R}^n$,
- $S = (n-1)$-dimensional submanifold of Ω such that $\Omega \setminus S = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$,
- $\omega \subset \Omega_2$ for instance.

Introduction: the problem

Notation $z_{|S_2} := (z_{|\Omega_2})_{|S}$

$$\begin{cases} \partial_t z - \Delta_c z = \mathbb{1}_\omega u & \text{in } (0,T)\times\Omega_1\cup\Omega_2, \\ \partial_t z^s - \Delta_{c^s} z^s = \frac{1}{\delta}\left((c\partial_\eta z)_{|S_2} - (c\partial_\eta z)_{|S_1}\right) & \text{in } (0,T)\times S, \\ z_{|S_1} = z^s = z_{|S_2} & \text{in } (0,T)\times S, \\ z_{|\partial\Omega} = 0; \end{cases} \qquad (\mathcal{P}_\delta)$$

- $+$ initial conditions $(z, z^s)_{|t=0} \in L^2(\Omega_1\cup\Omega_2)\times L^2(S)$
- δ = constant parameter, $0 < \delta \leq \delta_0$
- Diffusion operator on $\Omega_1\cup\Omega_2$: $\Delta_c = \operatorname{div} c(x)\nabla$
- Diffusion operator on S:

$$\Delta_{c^s} = \operatorname{div}^s c^s(x)\nabla^s = \frac{1}{\sqrt{\det(g)}}\sum_{i,j}\partial_{x_i}(c^s g^{ij}\sqrt{\det(g)}\partial_{x_j})$$

(g = metric on S, inherited from the euclidean metric on $\mathbb{R}^n$)

Introduction: the model, derivation in euclidean setting

- Multidimensional structures; useful for efficient numerical methods
- Koch-Zuazua '06: coupled waves in n and $n-1$ dimensions
- Lescarret-Zuazua '10: numerical analysis for waves

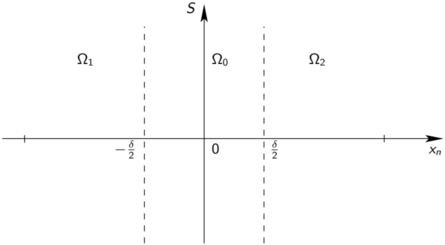

Figure: Local geometry of a three-layer model near the interface $S = \{x_n = 0\}$. The inner layer, Ω_0, shrinks to zero as δ goes to zero.

Introduction: the model, derivation in euclidean setting

Three diffusion coefficients c^j in Ω_j, $j = 1, 0, 2$.
Three diffusion equations

$$\partial_t z^j - \operatorname{div}(c^j \nabla z^j) = 0 \quad \text{in } (0, T) \times \Omega_j, \quad j = 1, 0, 2.$$

Natural transmission conditions at $x_n = \frac{\delta}{2}$ and $x_n = -\frac{\delta}{2}$, i.e.

- continuity of the solution:

$$z^1_{|x_n=-\frac{\delta}{2}} = z^0_{|x_n=-\frac{\delta}{2}}, \qquad z^0_{|x_n=\frac{\delta}{2}} = z^2_{|x_n=\frac{\delta}{2}}$$

- continuity of the flux:

$$(c^1 \partial_{x_n} z^1)_{|x_n=-\frac{\delta}{2}} = (c^0 \partial_{x_n} z^0)_{|x_n=-\frac{\delta}{2}}, \quad (c^0 \partial_{x_n} z^0)_{|x_n=\frac{\delta}{2}} = (c^2 \partial_{x_n} z^2)_{|x_n=\frac{\delta}{2}}.$$

Introduction: the model, derivation in euclidean setting

- Assumption: c^0 does not depend on the normal variable x_n. We set $c^s(y) := c^0(y, x_n)$.
- Mean values of z^0 in the normal direction x_n:

$$z^s(y) := \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} z^0(y, x_n) dx_n$$

- Integrating for $x_n \in (-\delta/2, \delta/2)$ the equation in Ω_0 yields

$$\begin{aligned} 0 &= \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} \left(\partial_t z^0 - \operatorname{div}(c^0 \nabla z^0) \right) dx_n \\ &= \partial_t z^s - \operatorname{div}^s(c^s \nabla^s z^s) - \frac{1}{\delta} \left((c^2 \partial_{x_n} z^2)_{|x_n = \frac{\delta}{2}} - (c^1 \partial_{x_n} z^1)_{|x_n = -\frac{\delta}{2}} \right). \end{aligned}$$

Introduction: the limit $\delta \to 0^+$

$$\begin{cases} \partial_t z - \Delta_c z = \mathbb{1}_\omega u & \text{in } (0,T) \times \Omega_1 \cup \Omega_2, \\ \partial_t z^s - \Delta_{c^s} z^s = \frac{1}{\delta}\left((c\partial_\eta z)_{|S_2} - (c\partial_\eta z)_{|S_1}\right) & \text{in } (0,T) \times S, \\ z_{|S_1} = z^s = z_{|S_2} & \text{in } (0,T) \times S, \\ z_{|\partial\Omega} = 0; \end{cases} \qquad (\mathcal{P}_\delta)$$

- Formally, convergence towards the classical transmission problem

$$\begin{cases} \partial_t z - \Delta_c z = \mathbb{1}_\omega u & \text{in } (0,T) \times \Omega_1 \cup \Omega_2, \\ (c\partial_\eta z)_{|S_2} = (c\partial_\eta z)_{|S_1} \text{ and } z_{|S_1} = z_{|S_2} & \text{in } (0,T) \times S, \\ z_{|\partial\Omega} = 0; \end{cases} \qquad (\mathcal{P}_0)$$

- $(\mathcal{P}_0)$ is well-posed
- $(\mathcal{P}_0)$ is null-controllable (Le Rousseau-Robbiano '10)

Introduction: some basic questions

- Well-posedness of $(\mathcal{P}_\delta)$ for fixed δ?
- Convergence of the solutions of $(\mathcal{P}_\delta)$ towards those of $(\mathcal{P}_0)$ when $\delta \to 0^+$?
- Null controllability of $(\mathcal{P}_\delta)$ for fixed δ? i.e. is it possible to drive the solution (z, z^s) to zero in time T?
- Convergence of the control in the limit $\delta \to 0^+$, i.e. uniform controllability?

Reformulation of the problem: function spaces

Hilbert space $\mathcal{H}^0_\delta = L^2(\Omega_1 \cup \Omega_2) \times L^2(S)$ with the inner product

$$\left(Z, \tilde{Z}\right)_{\mathcal{H}^0_\delta} = (z, \tilde{z})_{L^2(\Omega_1 \cup \Omega_2)} + \delta \left(z^s, \tilde{z}^s\right)_{L^2(S)}, \quad Z = (z, z^s), \ \tilde{Z} = (\tilde{z}, \tilde{z}^s),$$

Hilbert space

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with the inner product

$$\left(Z, \tilde{Z}\right)_{\mathcal{H}^1_\delta} = \left(Z, \tilde{Z}\right)_{\mathcal{H}^0_\delta} + \left(c \nabla z, \nabla \tilde{z}\right)_{L^2(\Omega_1 \cup \Omega_2)} + \delta \left(c^s \nabla^s z, \nabla^s \tilde{z}^s\right)_{L^2(S)}$$

Reformulation of the problem

Problem $(\mathcal{P}_\delta)$ can be written as

$$\partial_t Z + A_\delta Z = Bu$$

where $Z = (z, z^s) \in \mathcal{H}^0_\delta$, $B : u \mapsto {}^t(\mathbb{1}_\omega u, 0)$, and

$$A_\delta Z = \begin{pmatrix} -\Delta_c z \\ -\Delta_{c^s} z^s - \frac{1}{\delta}\left((c\partial_\eta z)_{|S_2} - (c\partial_\eta z)_{|S_1}\right) \end{pmatrix}$$

with domain

$$D(A_\delta) = \{(z, z^s) \in \mathcal{H}^1_\delta;\ A_\delta(z, z^s) \in \mathcal{H}^0_\delta\}.$$

Proposition

The operator $(A_\delta, D(A_\delta))$ is a coercive self-adjoint operator on $\mathcal{H}^0_\delta$. System $(\mathcal{P}_\delta)$ is well-posed for an initial condition in $\mathcal{H}^0_\delta$ and a control $u \in L^2((0,T) \times \omega)$.

Convergence $(\mathcal{P}_\delta) \to (\mathcal{P}_0)$ as $\delta \to 0^+$

Notation:

- (z_δ, z_δ^s) solution of $(\mathcal{P}_\delta)$ associated to the control u_δ and the initial data (z_0, z_0^s)
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Proposition

Suppose that $u_\delta \rightharpoonup u$ *in* $L^2((0,T)\times\omega)$*, and that* $(z_0, z_0^s) \in \mathcal{H}^1_\delta(\Omega)$*.*

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$$z_{\delta\,|\Omega_j} \rightharpoonup z_{|\Omega_j}, \quad in \quad L^2(0,T;H^2(\Omega_j)) \cap H^1(0,T;L^2(\Omega_j))$$

for $j = 1, 2$.

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for $j = 1, 2$.

$\Longrightarrow$ Hope: construct for $(\mathcal{P}_\delta)$ a control u_δ uniformly bounded, which yields a particular control for the limit problem.

Controllability of parabolic equations

Methods to prove null controllability for $(\partial_t + A)z = Bu$:

- Fursikov-Imanuvilov ('96): global Carleman estimate for the *parabolic* operator $(-\partial_t + A^*)$, with weight $e^{-\frac{\varphi}{ht(T-t)}}$;

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- Lebeau-Robbiano ('95): Stationary method
  - local Carleman estimates for the *elliptic* operator $-\partial_{x_0}^2 + A^*$,

    $\Downarrow$
  - Interpolation inequalities,

    $\Downarrow$
  - Spectral inequality for sums of eigenfunction of A^* (Lebeau-Zuazua '98, Jerison-Lebeau '99),

    $\Downarrow$
  - iterative construction of the control.

Controllability of parabolic equations with coefficients with jumps

$$\begin{cases} \partial_t z - \Delta_c z = \mathbb{1}_\omega u & \text{in } (0, T) \times \Omega_1 \cup \Omega_2, \\ (c\partial_\eta z)_{|S_2} = (c\partial_\eta z)_{|S_1} \text{ and } z_{|S_1} = z_{|S_2} & \text{in } (0, T) \times S, \\ z_{|\partial\Omega} = 0; \end{cases} \qquad (\mathcal{P}_0)$$

- Doubova-Osses-Puel '02: null controllability of $(\mathcal{P}_0)$ under a technical assumption: "$c_{|S_2} \leq c_{|S_1}$";
- Benabdallah-Dermenjian-Le Rousseau '07: this technical assumption is not necessary in stratified medias;
- Le Rousseau-Robbiano '10: null controllability of $(\mathcal{P}_0)$ in the general case (Lebeau-Robbiano method);
- Le Rousseau-Robbiano '11: null controllability of $(\mathcal{P}_0)$ in the general case (Fursikov-Imanuvilov method);
- Le Rousseau-Lerner '11: null controllability of $(\mathcal{P}_0)$ in the anisotropic case (c is a diffusion matrix).

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... uniformly w.r.t. $0 < \delta \leq \delta_0$.

Consequences

Notation: $\mathscr{E}_{\delta,j} = (e_{\delta,j}, e^s_{\delta,j})$, $j \in \mathbb{N}$, eigenfunctions of A_δ associated with the positive eigenvalues $\mu_{\delta,j} \in \mathbb{R}$, $j \in \mathbb{N}$.
$\mathscr{E}_{\delta,j}$: Hilbert basis of $\mathcal{H}^0_\delta$

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Theorem (Spectral inequality)

For $\delta_0 > 0$, there exists $C > 0$ such that for all $0 < \delta \leq \delta_0$ and $\mu \in \mathbb{R}$, we have

$$\|Z\|_{\mathcal{H}^0_\delta} \leq Ce^{C\sqrt{\mu}}\|z\|_{L^2(\omega)}, \qquad Z = (z, z^s) \in \operatorname{span}\{\mathscr{E}_{\delta,j}; \mu_{\delta,j} \leq \mu\}.$$

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Theorem (Null controllability)

Let $\delta_0 > 0$. For all $T > 0$ and $\omega \subset \Omega$, $\exists C > 0$ s.t.: for all $Z_0 = (z_0, z_0^s) \in \mathcal{H}^0_\delta$ and $0 < \delta \leq \delta_0$, there exists $u_\delta \in L^2((0,T) \times \omega)$ s.t. the solution (z, z^s) of $(\mathcal{P}_\delta)$ satisfies $(z(T), z^s(T)) = (0,0)$ and moreover

$$\|u_\delta\|_{L^2((0,T)\times\omega)} \leq C\|Z_0\|_{\mathcal{H}^0_\delta}.$$

Some remarks

- Uniform controllability as $\delta \to 0^+$; extract a subsequence u_δ weakly convergent in $L^2((0,T)\times\omega) \Longrightarrow$ the associated solution of Problem $(\mathcal{P}_\delta)$ converges towards a controlled solution of Problem $(\mathcal{P}_0)$.
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  For $(\mathcal{P}_0)$, we hence construct a control function robust w.r.t. small viscous perturbations in the interface.
- A boundary control from a subset of $\partial\Omega$ also holds.
- A control result from the interface S is false (in general) $\to$ counterexamples on the flat cylinder.

The (semi-global) Carleman estimate

Notation: V a small neighborhood of S in Ω
ζ a cut-off function $\text{supp}(\zeta) \subset V$, $\zeta = 1$ on S
Consider the system

$$\begin{cases} -\partial_{x_0}^2 w - \Delta_c w = f & \text{in } (0, X_0) \times V \\ -\partial_{x_0}^2 w^s - \Delta_{c^s} w^s & \\ \qquad = \frac{1}{\delta}\left((c\partial_\eta w)_{|(0,X_0)\times S_2} - (c\partial_\eta w)_{|(0,X_0)\times S_1} + \theta^s\right) & \text{in } (0, X_0) \times S, \\ w_{|(0,X_0)\times S_1} = w^s = w_{|(0,X_0)\times S_2} & \text{in } (0, X_0) \times S. \end{cases} \tag{1}$$

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For a well-chosen weight function φ, $\forall \delta_0 > 0$, $\exists C > 0$, and $h_0 > 0$ such that

$$h\|e^{\varphi/h} w\|_0^2 + h^3\|e^{\varphi/h}\nabla_{x_0,x} w\|_0^2 + h|e^{\varphi/h} w_{|S}|_0^2 + h^3 \sum_{j=1,2} |e^{\varphi/h}\nabla_{x_0,x} w_{|S_j}|_0^2$$
$$\leq C\left(h^4\|e^{\varphi/h} f\|_0^2 + h^2\delta^2\|\zeta e^{\varphi/h} f_{|V_2}\|_0^2 + h^3|e^{\varphi/h}\theta^s|_0^2 \right),$$

for all $0 < \delta < \delta_0$, $0 < h \leq h_0$, for (w, θ^s, f) satisfying (1).

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For a well-chosen weight function φ, $\forall \delta_0 > 0$, $\exists C > 0$, *and* $h_0 > 0$ *such that*

$$h\|e^{\varphi/h} w\|_0^2 + h^3\|e^{\varphi/h}\nabla_{x_0,x} w\|_0^2 + h|e^{\varphi/h} w_{|S}|_0^2 + h^3 \sum_{j=1,2} |e^{\varphi/h}\nabla_{x_0,x} w_{|S_j}|_0^2$$
$$\leq C\left(h^4\|e^{\varphi/h} f\|_0^2 + h^2\delta^2\|\zeta e^{\varphi/h} f_{|V_2}\|_0^2 + h^3|e^{\varphi/h}\theta^s|_0^2\right),$$

for all $0 < \delta < \delta_0$, $0 < h \leq h_0$, *for* (w, θ^s, f) *satisfying* (1).

The (semi-global) Carleman estimate: comments

- in the r.h.s., the term
$$h^2\delta^2\|\zeta e^{\varphi/h} f_{|V_2}\|_0^2$$
prevents from absorbing first order terms (δ being fixed)
$\Longrightarrow$ prevents from patching together local estimates
$\Longrightarrow$ we cannot work in local coordinates only
$\Longrightarrow$ we have to work on whole V
- $\delta = 0$ we recover the Carleman estimate of Le Rousseau-Robbiano '10.

The (semi-global) Carleman estimate: comments

- Conditions on φ (keep in mind that ω is in Ω_2):
  - φ continuous on $\overline{V}$ (in particular at the interface);
  - $\nabla\varphi \neq 0$ on $\overline{V}$;
  - φ satisfies a Hörmander subellipticity condition on $\overline{V}$;
  - φ is increasing in the normal direction from V_1 to V_2 ;
  - $\partial_\eta \varphi|_{S_2} - \partial_\eta \varphi|_{S_1}$ sufficiently large;
  - $\nabla_{x_0,x'}\varphi$ sufficiently small.

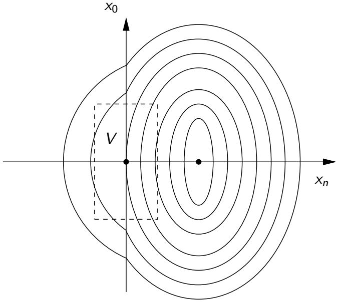


Figure: Level sets of an example of an admissible weight functions φ in flat coordinates.

Sketch of the proof: simplifications

- Omit the additional x_0-direction
- Local (flat) coordinates in a neighborhood of S: $S = \{x_n = 0\}$
- $V = \{x = (x', x_n), \text{ s.t. } -\varepsilon < x_n < \varepsilon\}$
- $V_2 = \{0 < x_n < \varepsilon\}$ = right side
- $V_1 = \{-\varepsilon < x_n < 0\}$ = left side
- In the local setting: $w_2 \to w^r$ (right) and $w_1 \to w^l$ (left)
- Keep in mind that ω is in Ω_2, i.e. on the right side
- The principal symbol of $-\Delta_{c^{r/l}}$ is $c^{r/l}(\xi_n^2 + |\xi'|^2)$, with $\xi' = (\xi_1, \cdots, \xi_{n-1})$
- The principal symbol of $-\Delta_{c^s}$ is $c^s|\xi'|^2$,

Sketch of the proof: 1 conjugation with the weight function

- conjugated differential operators

$$P_\varphi^{r/l} = h^2 e^{\varphi^{r/l}/h} \frac{1}{c^{r/l}} (-\Delta_{c^{r/l}}) e^{-\varphi^{r/l}/h}, \quad P_\varphi^s = h^2 e^{\varphi_{|S}/h} \frac{1}{c^s} (-\Delta_{c^s}) e^{-\varphi_{|S}/h},$$

- conjugated functions

$$v^{r/l} = e^{\varphi^{r/l}/h} w^{r/l}, \quad v^s = e^{\varphi_{|S}/h} w^s,$$

- the elliptic system can be rewritten as

$$\begin{cases} P_\varphi^{r/l} v^{r/l} = F_\varphi^{r/l} & \text{in } V, \\ P_\varphi^s v^s = \frac{hi}{c^s \delta} \left(c^r (D_{x_n} + i\partial_{x_n}\varphi^r) v^r_{|x_n=0} + c^l (D_{x_n} + i\partial_{x_n}\varphi^l) v^l_{|x_n=0} + \Theta_\varphi^s \right) & \text{in } S, \\ v^{r/l}_{|x_n=0} = v^s & \text{in } S, \end{cases}$$

  with $D_{x_n} = h\partial_{x_n}/i$.

- consider $P_\varphi^{r/l}$ and P_φ^s as semi-classical differential operators.

Sketch of the proof: 1 analysis of the operators

Calculus rule: operator $\longleftrightarrow$ symbol

$$D = \frac{h}{i}\partial \longleftrightarrow \xi$$

Sketch of the proof: 1 analysis of the operators

Calculus rule: operator $\longleftrightarrow$ symbol

$$D = \frac{h}{i}\partial \longleftrightarrow \xi$$

semiclassical principal symbols

$$\begin{aligned}
p_\varphi^s(x', \xi') = \sigma(P_\varphi^s) &= |\xi'|^2 - |\nabla_{x'}\varphi_{|S}|^2 + 2i\xi' \cdot \nabla_{x'}\varphi_{|S}, \\
p_\varphi^{r/l}(x, \xi) = \sigma(P_\varphi^{r/l}) &= |\xi|^2 - |\nabla\varphi|^2 + 2i\xi \cdot \nabla\varphi^{r/l} \\
&= \xi_n^2 + 2i\xi_n\partial_{x_n}\varphi^{r/l} + \underbrace{|\xi'|^2 - |\nabla\varphi|^2}_{=q_2(x,\xi')} + 2i\underbrace{\xi' \cdot \nabla_{x'}\varphi^{r/l}}_{=q_1(x,\xi')} \\
&= \xi_n^2 + 2i\xi_n\partial_{x_n}\varphi^{r/l} + q_2 + 2iq_1(x, \xi') \\
&= \left(\xi_n - \rho^{r/l,+}(x, \xi')\right)\left(\xi_n - \rho^{r/l,-}(x, \xi')\right)
\end{aligned}$$

Sketch of the proof: 1 analysis of the operators

Calculus rule: operator $\longleftrightarrow$ symbol

$$D = \frac{h}{i}\partial \longleftrightarrow \xi$$

semiclassical principal symbols

$$\begin{aligned}
p^s_\varphi(x',\xi') = \sigma(P^s_\varphi) &= |\xi'|^2 - |\nabla_{x'}\varphi_{|S}|^2 + 2i\xi'\cdot\nabla_{x'}\varphi_{|S},\\
p^{r/l}_\varphi(x,\xi) = \sigma(P^{r/l}_\varphi) &= |\xi|^2 - |\nabla\varphi|^2 + 2i\xi\cdot\nabla\varphi^{r/l}\\
&= \xi_n^2 + 2i\xi_n\partial_{x_n}\varphi^{r/l} + \underbrace{|\xi'|^2 - |\nabla\varphi|^2}_{=q_2(x,\xi')} + 2i\underbrace{\xi'\cdot\nabla_{x'}\varphi^{r/l}}_{=q_1(x,\xi')}\\
&= \xi_n^2 + 2i\xi_n\partial_{x_n}\varphi^{r/l} + q_2 + 2iq_1(x,\xi')\\
&= \left(\xi_n - \rho^{r/l,+}(x,\xi')\right)\left(\xi_n - \rho^{r/l,-}(x,\xi')\right)
\end{aligned}$$

Important remarks:

- the ellipticity properties of $P^{r/l}_\varphi$ depend on the localization of the roots ρ^+,ρ^- in the complex plane!
- the ellipticity properties of P^s_φ depend only on $\mathrm{Char}(P^s_\varphi) = \{(x',\xi')\in\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}, p^s_\varphi(x',\xi') = 0\}$.

Sketch of the proof: localization of the roots

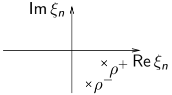
At the interface,

$$\begin{cases} P^s_\varphi v^s = \frac{hi}{c^s \delta}\left(c^r(D_{x_n} + i\partial_{x_n}\varphi^r)v^r_{|x_n=0} + c^l(D_{x_n} + i\partial_{x_n}\varphi^l)v^l_{|x_n=0} + \Theta^s_\varphi\right) & \text{in } S, \\ v^l_{|x_n=0} = v^r_{|x_n=0} = v^s & \text{in } S, \end{cases}$$

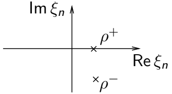
- THREE unknowns: v^s, $D_{x_n} v^{r/l}_{|x_n=0}$,
- ONE equation at the interface
- We have to find two more relations to "solve" the system
- $v^{r/l}$ solves an "elliptic" problem on each side $\Longrightarrow$ yield other relations...
- ... depending on the localization of the roots ρ^+, ρ^- in the complex plane!

Sketch of the proof: localization of the roots

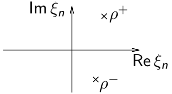
Focus on ONE SIDE ONLY (i.e. either r or l): analysis of the roots $\rho^{\pm}(x,\xi')$:



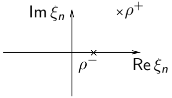
VERY GOOD configuration: TWO relations between the two traces $v_{|x_n=0}$ and $D_{x_n}v_{|x_n=0}$. i.e. both traces are determined!



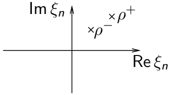
GOOD configuration: $\approx$ ONE relation between the two traces $v_{|x_n=0}$ and $D_{x_n}v_{|x_n=0}$



GOOD configuration: ONE relation between the two traces $v_{|x_n=0}$ and $D_{x_n}v_{|x_n=0}$



BAD configuration: NO relation between the two traces $v_{|x_n=0}$ and $D_{x_n}v_{|x_n=0}$



BAD configuration: NO relation between the two traces $v_{|x_n=0}$ and $D_{x_n}v_{|x_n=0}$

Sketch of the proof: localization of the roots

The previous can be quantified

$$q_2(x,\xi') = |\xi'|^2 - |\nabla\varphi|^2, \qquad q_1(x,\xi') = \xi' \cdot \nabla_{x'}\varphi^{r/l}$$

Lemma (Lebeau-Robbiano '97, Le Rousseau-Robbiano '10)

Let

$$\mu^{r/l}(x,\xi') = q_2^{r/l}(x,\xi') + \frac{\left(q_1^{r/l}(x,\xi')\right)^2}{\left(\partial_{x_n}\varphi^{r/l}\right)^2},$$

We have

- $\mu^r > 0 \iff$ *GOOD for* P_φ^r *(high frequencies)*
- $\mu^r \leq 0 \iff$ *VERY GOOD for* P_φ^r *(low frequencies)*
- $\mu^l > 0 \iff$ *GOOD for* P_φ^l *(high frequencies)*
- $\mu^l \leq 0 \iff$ *BAD for* P_φ^l *(low frequencies)*

Sketch of the proof: localization of the roots

Lemma

The properties of the weight function φ imply that

- $\mu^l \leq 0 \Longrightarrow \mu^r \leq 0$, *i.e. BAD situation* $P^l_\varphi \Longrightarrow$ *VERY GOOD situation* P^r_φ *(low frequencies)*
- $p^s_\varphi = 0 \Longrightarrow \mu^l \leq 0$

Sketch of the proof: localization of the roots

Lemma

The properties of the weight function φ imply that

- $\mu^l \leq 0 \Longrightarrow \mu^r \leq 0$, *i.e. BAD situation* $P^l_\varphi \Longrightarrow$ *VERY GOOD situation* P^r_φ *(low frequencies)*
- $p^s_\varphi = 0 \Longrightarrow \mu^l \leq 0$

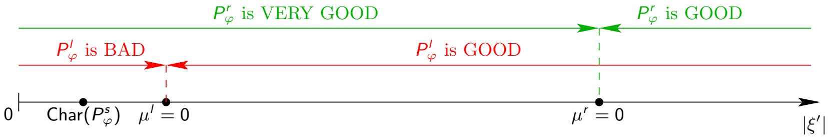


Figure: Microlocal ellipticity regions

Sketch of the proof: microlocal decomposition

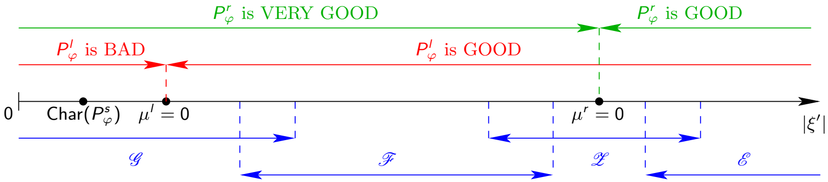


Figure: Microlocal ellipticity regions

- decompose the tangent phase-space into four regions $\mathscr{E}$, $\mathscr{Z}$, $\mathscr{F}$, $\mathscr{G}$;
- In each region $\mathscr{E}$, $\mathscr{Z}$, $\mathscr{F}$, $\mathscr{G}$: produce a microlocal Carleman estimate;
- Finally: patch together these 4 microlocal Carleman estimate.

Sketch of the proof: microlocal decomposition

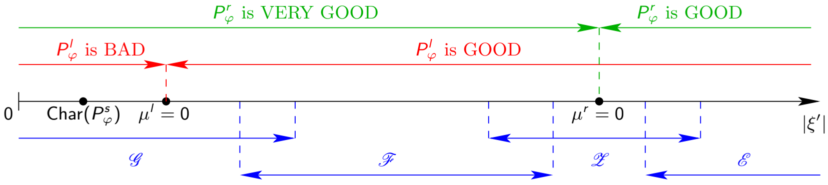


Figure: Microlocal ellipticity regions

- loss of a power $\frac{\delta^2}{h^2}$ on $\mathrm{Char}(P^s_\varphi)$; i.e. in region $\mathscr{G}$. Why?
- P^l_φ BAD $\Longrightarrow$ NO relation between $v^l_{|x_n=0}$ and $D_{x_n} v^l_{|x_n=0}$
- P^r_φ VERY GOOD $\Longrightarrow$ the two traces $v^r_{|x_n=0}$ and $D_{x_n} v^r_{|x_n=0}$ are determined.
- Hence, using

$$\begin{cases} P^s_\varphi v^s = \frac{hi}{c^s \delta}\left(c^r(D_{x_n} + i\partial_{x_n}\varphi^r)v^r_{|x_n=0} + c^l(D_{x_n} + i\partial_{x_n}\varphi^l)v^l_{|x_n=0} + \Theta^s_\varphi\right) & \text{in } S, \\ v^l_{|x_n=0} = v^r_{|x_n=0} = v^s & \text{in } S, \end{cases}$$

we can deduce $D_{x_n} v^l_{|x_n=0}$!

Sketch of the proof: loss of h^2

- loss of a power $\frac{\delta^2}{h^2}$ on $\mathrm{Char}(P^s_\varphi)$; i.e. in region $\mathscr{G}$. Why?
- P^r_φ VERY GOOD $\Longrightarrow$ the two traces $v^r_{|x_n=0}$ and $D_{x_n} v^r_{|x_n=0}$ are determined.
- Hence, using

$$\begin{cases} P^s_\varphi v^s = \frac{hi}{c^s \delta}\left(c^r(D_{x_n} + i\partial_{x_n}\varphi^r)v^r_{|x_n=0} + c^l(D_{x_n} + i\partial_{x_n}\varphi^l)v^l_{|x_n=0} + \Theta^s_\varphi\right) & \text{in } S, \\ v^l_{|x_n=0} = v^r_{|x_n=0} = v^s & \text{in } S, \end{cases}$$

  we can deduce $D_{x_n} v^l_{|x_n=0}$!
-

$$c^l D_{x_n} v^l_{|x_n=0} = \frac{c^s \delta}{hi} P^s_\varphi v^s - c^r D_{x_n} v^r_{|x_n=0} - i\partial_{x_n}\varphi^r v^r_{|x_n=0} - i\partial_{x_n}\varphi^l v^l_{|x_n=0} + \Theta^s_\varphi$$

$$|D_{x_n} v^l_{|x_n=0}|_0 \lesssim \frac{\delta}{h} \underbrace{|v^r_{|x_n=0}|_0 + |D_{x_n} v^r_{|x_n=0}|_0 + |v^r_{|x_n=0}|_0}_{\text{estimated on the right by } \|P^r_\varphi v^r\|_0} + |\Theta^s_\varphi|_0$$

- Recall that everything has to be done globally on T^*S

Other geometries

- Semiglobal estimates can be patched together $\Longrightarrow$ a global Carleman estimate (see Le Rousseau-Robbiano '11)
- Several interfaces can be considered
- Other geometries can be addressed $\rightarrow$ semiglobal nature of the Carleman estimate

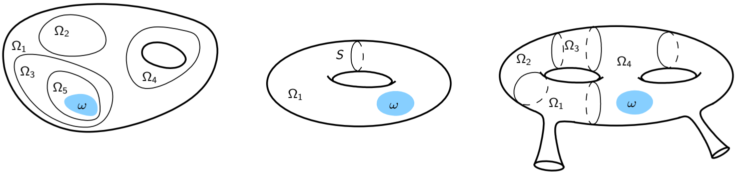


Figure: Other geometrical situations: (a) Ω is an bounded open subset of $\mathbb{R}^n$; (b) and (c) Ω is a compact manifold with boundary.

Some open problems

- Optimality of the powers of h?
- Produce a parabolic Carleman estimate (in the spirit of Fursikov-Imanuvilov/Le Rousseau-Robbiano '11)?
- Case of three non-conformal metrics at the interface (in the spirit of Le Rousseau-Lerner '11)?
- If S touches the boundary?